On the effect of limited linear kinematic hardening on shakedown conditions

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Abstract

This paper focuses on the effect of limited linear kinematic hardening on shakedown limits for variable loadings. We briefly present the general equations of shakedown analysis [1] when internal variables are included. Then, a model for hardening plasticity due to [2] is adopted to obtain analytical solutions for the shakedown problem in a small restrained block under variable thermal and mechanical loadings.

1 Introduction

Shakedown analysis of an elastic-plastic solid or structure, submitted to variable loadings, is concerned with the conditions to guarantee that a prescribed load domain does not contain any program, or cycle, leading to failure by alternating plasticity, incremental collapse (ratcheting) or plastic collapse.

For elastic plastic bodies submitted to variable loadings, the word shakedown is traditionally understood as synonymous of elastic shakedown, elastic adaptation or elastic stabilization, all these referring to a type of response where plastic dissipation ceases after an initial stage of plastic deformation (eventually nonexistent).

All the theory of shakedown stands on the fundamental Melan-Koiter's theorem [3–5]: An elastic ideally plastic body will shakedown under a given loading history, in the sense that the total plastic work is bounded irrespective of the initial conditions, if there exist a time-independent residual stress field σ^r, a scalar $m > 1$ and an instant t_0 such that the fictitious elastic stresses σ^E, produced by the loading program acting on the unlimitedly elastic reference body, when amplified by the factor m and superposed to the fixed residual stress field σ^r render plastically admissible stresses ever after t_0.

The classic shakedown theory, in its simplest form [3], is formulated for a solid presenting ideally plastic behavior. Extensions of this basic theory, including hardening, have also been presented in terms of internal variables and most often adopting generalized standard material models; see e.g. [1, 4, 6]. This gives a framework to propose special models, and implementations of the constitutive equations, oriented to reproduce the behavior of some structures when submitted to critical cycles of

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loading.

This paper focuses on the effect of limited linear kinematic hardening on shakedown limits for variable loadings.

We briefly present the general equations of shakedown analysis [1, 2, 7–10] when internal variables are included. Then, a model for hardening plasticity due to [2, 11, 12] is adopted to obtain analytical solutions for the shakedown problem in a small restrained block under variable thermal and mechanical loadings.

The aim of obtaining this analytical solution is twofold: firstly, it allows discussion and understanding of the effects on shakedown conditions of including kinematical hardening in an ideally plastic Mises model, and secondly, this exact solution is intended to be used as benchmark for numerical procedures for shakedown analysis based on direct methods and finite element discretizations.

The notation adopted here is as follows. The stress tensor is denoted σ, the strain tensor is ε and the strain rate is d. The mean stress and deviatoric part of a stress tensor are denoted

$$\sigma_m = \frac{1}{3}\mathrm{tr}\,\sigma \qquad S := \sigma^{\mathrm{dev}} = \sigma - \sigma_m \boldsymbol{1} \tag{1}$$

where $\boldsymbol{1}$ is the identity and superscript dev denotes the deviatoric part of a tensor. The principal stresses are $\{\sigma_i,\ i = 1, 2, 3\}$.

2 Kinematical hardening following Stein's model

The plasticity model proposed by Erwin Stein and coworkers [2, 11], including limited linear kinematic hardening in the usual von Mises model, is defined by the following two plastic modes

$$f_{S1}(\sigma, A) = \frac{3}{2}\|(\sigma - A)^{\mathrm{dev}}\|^2 - \sigma_{Y0}^2 \tag{2}$$

$$f_{S2}(A) = \frac{3}{2}\|A^{\mathrm{dev}}\|^2 - (\sigma_Y - \sigma_{Y0})^2 \tag{3}$$

where the second order tensor A is the statical internal variable of the model, with the meaning of a back-stress. The material constants σ_Y and σ_{Y0} are the final and initial plastic limits of the material in the uniaxial tensile test. Further, we assume that the hardening phase is smaller than the purely elastic range, that is $\frac{1}{2}\sigma_Y \leqslant \sigma_{Y0} \leqslant \sigma_Y$. Then

$$\zeta := \sigma_Y - \sigma_{Y0} \leqslant \sigma_{Y0} \tag{4}$$

It is worth noting that only the the deviatoric part of A appears in the constitutive equations of this material. Thus, the mean component $A_m := \frac{1}{3}\mathrm{tr}\,A$ of the back-stress is irrelevant, or indetermined. As a consequence of this, and for the sake of simplicity, we adopt: (i) a deviatoric internal variable A in general triaxial situations or (ii) a plane internal variable A when plane stress conditions apply (note that in this case the deviatoric part must be computed).

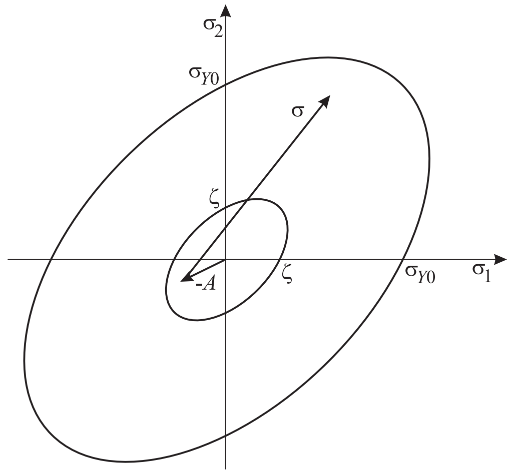

Figure 1: Plane stress representation for Stein's model of kinematical hardening. The generalized stress shown, (σ, A), is admissible with respect to both plastic modes.

The flow equations of this plasticity model are associated, thus derived by using the gradients of the plastic modes, written below

$$\nabla_\sigma f_{S1} = 3(S - A^{\text{dev}}) \qquad \nabla_A f_{S1} = -3(S - A^{\text{dev}}) \tag{5}$$

$$\nabla_\sigma f_{S2} = 0 \qquad \nabla_A f_{S2} = 3A^{\text{dev}} \tag{6}$$

In this notation, the flow equations read

$$d^p = \dot{\lambda}\nabla_\sigma f_{S1}(\sigma, A) \qquad \dot{\beta} = \dot{\lambda}\nabla_A f_{S1}(\sigma, A) + \dot{\lambda}^A \nabla_A f_{S2}(A) \tag{7}$$

where d^p is the plastic strain rate and $\dot{\beta}$ the hardening flux. The plastic multipliers $\dot{\lambda}$ and $\dot{\lambda}^A$ are constrained by the complemantarity conditions

$$\dot{\lambda} f_{S1}(\sigma, A) = 0 \qquad f_{S1}(\sigma, A) \leqslant 0 \qquad \dot{\lambda} \geqslant 0 \tag{8}$$

$$\dot{\lambda}^A f_{S2}(A) = 0 \qquad f_{S2}(A) \leqslant 0 \qquad \dot{\lambda}^A \geqslant 0 \tag{9}$$

For future use we write below the component equations obtained from the above intrinsic equations in the case of plane stress conditions. Notice that we choose $A_z = 0$ for convenience.

$$f_{S1} = (\sigma_x - A_x)^2 + (\sigma_y - A_y)^2 - (\sigma_x - A_x)(\sigma_y - A_y) + 3(\sigma_{xy} - A_{xy})^2 - \sigma_{Y0}^2 \tag{10}$$

$$f_{S2} = A_x^2 + A_y^2 - A_x A_y + 3A_{xy}^2 - (\sigma_Y - \sigma_{Y0})^2 \tag{11}$$

$$\nabla_\sigma f_{S1} = \begin{bmatrix} 2(\sigma_x - A_x) - (\sigma_y - A_y) \\ 2(\sigma_y - A_y) - (\sigma_x - A_x) \end{bmatrix} \qquad \nabla_A f_{S1} = \begin{bmatrix} -2(\sigma_x - A_x) + (\sigma_y - A_y) \\ -2(\sigma_y - A_y) + (\sigma_x - A_x) \end{bmatrix} \tag{12}$$

$$\nabla_\sigma f_{S2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \nabla_A f_{S2} = \begin{bmatrix} 2A_x - A_y \\ 2A_y - A_x \end{bmatrix} \tag{13}$$

3 A restrained block under thermo-mechanical loading

Consider the small block of Figure 2 that is under plane stress conditions for the plane (x, y). Additionally, deformation is restrained in the y–direction. The material is isotropic, linear elastic and obeys the limited linear kinematic hardening plasticity model described in the previous subsection. The imposed external actions are: (i) a variable traction p and (ii) a variable thermal loading $q = E c_\varepsilon \theta$ (q is a stress parameter, E is Young's modulus, c_ε is the thermal expansion coefficient and θ is the variable temperature increment).

Thus the domain of load variation Δ^0, shown in Figure 3, is represented, in the stress space, by the polygon Δ, shown in Figure 2, with vertices

$$\sigma^1 = (\sigma_x^1, \sigma_y^1) = (0, 0) \tag{14}$$

$$\sigma^2 = (\sigma_x^2, \sigma_y^2) = (0, -\overline{q}) \tag{15}$$

$$\sigma^3 = (\sigma_x^3, \sigma_y^3) = (\overline{p}, \nu\overline{p} - \overline{q}) \tag{16}$$

$$\sigma^4 = (\sigma_x^4, \sigma_y^4) = (\overline{p}, \nu\overline{p}) \tag{17}$$

4 The equations of shakedown analysis

Shakedown analysis is concerned with the computation of the critical factor μ which amplifies the load domain Δ^0, as shown in Figure 3, ensuring that elastic shakedown occurs for any load program or load cycle contained in the amplified domain $\mu\Delta^0$. Moreover, any arbitrary load domain $\mu^*\Delta^0$ with $\mu^* > \mu$ contains one program at least leading to failure by alternating plasticity, incremental collapse or plastic collapse.

Consequently, this safety factor μ ensures elastic shakedown and thus prevents against the three modes of failure described in the classical theory of shakedown [3].

The system of equations and inequalities stated below represents the problem of shakedown analysis, that is, the problem of computing the critical scalar factor μ. In the sequel, we briefly explain the meaning of this system of equations, which constitute the set of optimality conditions for the variational formulations of shakedown analysis [1, 7, 8]

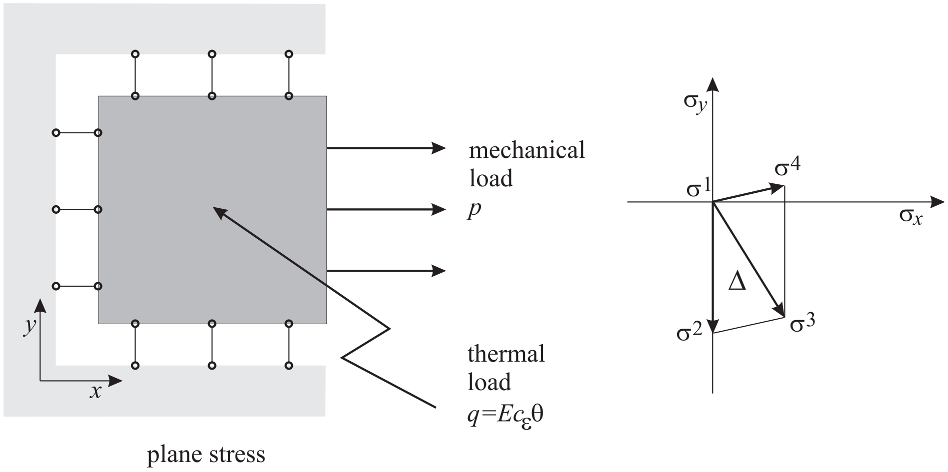

Figure 2: The domain of variable loading Δ for a restrained block under independent thermal and mechanical loadings.

For given $\{\sigma^k,\ k = 1:4\}$, find μ, σ^r, A and v such that

$$B^T \sigma^r = 0 \tag{18}$$

$$\sum d^k = Bv \tag{19}$$

$$\sum \dot{\beta}^k + \dot{\beta}^A = 0 \tag{20}$$

$$\sum \sigma^k \cdot d^k = 1 \tag{21}$$

$$d^k = \dot{\lambda}^k \nabla_\sigma f^k \qquad k = 1:4 \tag{22}$$

$$\dot{\beta}^k = \dot{\lambda}^k \nabla_A f^k \qquad k = 1:4 \tag{23}$$

$$\dot{\beta}^A = \dot{\lambda}^A \nabla_A f^A \tag{24}$$

$$\dot{\lambda}^k f^k = 0 \qquad k = 1:4 \tag{25}$$

$$\dot{\lambda}^A f^A = 0 \tag{26}$$

$$f^k := f_{S1}(\mu\sigma^k + \sigma^r, A) \leq 0 \qquad k = 1:4 \tag{27}$$

$$f^A := f_{S2}(A) \leq 0 \tag{28}$$

$$\dot{\lambda}^k \geq 0 \qquad k = 1:4 \tag{29}$$

$$\dot{\lambda}^A \geq 0 \tag{30}$$

Equation (18) imposes that σ^r is a residual stress (i.e. a self-equilibrated stress field). Indeed, we consider here the following discrete form of compatibility and equilibrium

$$d = Bv \qquad B^T\sigma^r = 0 \tag{31}$$

where B is the strain-displacement matrix and v is the vector of velocities after imposing displacement constraints. This is exact for the case of the example treated in following sections and, in the case of a continuum, it is created by a finite element discretization.

Equation (22) defines a plastic strain rate d^k as the flow produced by the compound stress $\mu\sigma^k + \sigma^r$ and the thermodynamic force A. Whenever this state, associated to the load vertex k, is active (i.e. at the boundary of $f^k = 0$), it may effectively act during the critical load cycle. In this case, this load may produce a non-vanishing plastic flow contribution, d^k, adding to the admissible plastic strain rate cycle Bv, according to (19). Likewise, (23) and (24) define hardening fluxes that must sum zero, according to (20), because there is no change in hardening variables during a critical mechanism of failure under variable loadings and thus the total hardening flux $\dot{\beta}$ is zero.

The complementarity conditions required by Melan's theorem complete the system.

The system is reduced by substituting (22), (23) e (24) in the remaining conditions:

$$B^T\sigma^r = 0 \tag{32}$$

$$\sum \dot{\lambda}^k \nabla_\sigma f^k = Bv \tag{33}$$

$$\sum \dot{\lambda}^k \nabla_A f^k + \dot{\lambda}^A \nabla_A f^A = 0 \tag{34}$$

$$\sum \sigma^k \cdot \dot{\lambda}^k \nabla_\sigma f^k = 1 \tag{35}$$

$$\dot{\lambda}^A f^A = 0, \qquad \dot{\lambda}^k f^k = 0 \qquad k = 1:4 \tag{36}$$

$$f^A := f_{S2}(A) \leq 0, \qquad f^k := f_{S1}(\mu\sigma^k + \sigma^r, A) \leq 0 \qquad k = 1:4 \tag{37}$$

$$\dot{\lambda}^A \geq 0, \qquad \dot{\lambda}^k \geq 0 \qquad k = 1:4 \tag{38}$$

Furthermore, the equations for the shakedown analysis of the hardening block are now written in component form as follows

$$\sigma_x^r = 0 \tag{39}$$

$$\sum \dot{\lambda}^k [\mu(2\sigma_x^k - \sigma_y^k) - \sigma_y^r - 2A_x + A_y] = d_x \tag{40}$$

$$\sum \dot{\lambda}^k [\mu(2\sigma_y^k - \sigma_x^k) + 2\sigma_y^r - 2A_y + A_x] = 0 \tag{41}$$

$$\sum \dot{\lambda}^k [-\mu(2\sigma_x^k - \sigma_y^k) + \sigma_y^r + 2A_x - A_y] + \dot{\lambda}^A (2A_x - A_y) = 0 \tag{42}$$

$$\sum \dot{\lambda}^k [-\mu(2\sigma_y^k - \sigma_x^k) - 2\sigma_y^r + 2A_y - A_x] + \dot{\lambda}^A (2A_y - A_x) = 0 \tag{43}$$

$$\sum \dot{\lambda}^k \{\sigma_x^k[\mu(2\sigma_x^k - \sigma_y^k) - \sigma_y^r - 2A_x + A_y] + \sigma_y^k[\mu(2\sigma_y^k - \sigma_x^k) + 2\sigma_y^r - 2A_y + A_x]\} = 1 \tag{44}$$

$$\dot{\lambda}^k[(\mu\sigma_x^k - A_x)^2 + (\mu\sigma_y^k + \sigma_y^r - A_y)^2 - (\mu\sigma_x^k - A_x)(\mu\sigma_y^k + \sigma_y^r - A_y) - \sigma_{Y0}^2] = 0 \quad k = 1:4 \tag{45}$$

$$\dot{\lambda}^A[A_x^2 + A_y^2 - A_xA_y - (\sigma_Y - \sigma_{Y0})^2] = 0 \tag{46}$$

$$(\mu\sigma_x^k - A_x)^2 + (\mu\sigma_y^k + \sigma_y^r - A_y)^2 - (\mu\sigma_x^k - A_x)(\mu\sigma_y^k + \sigma_y^r - A_y) - \sigma_{Y0}^2 \leq 0 \qquad k = 1:4 \tag{47}$$

$$A_x^2 + A_y^2 - A_xA_y - (\sigma_Y - \sigma_{Y0})^2 \leq 0 \tag{48}$$

$$\dot{\lambda}^A \geq 0 \qquad \dot{\lambda}^k \geq 0 \qquad k = 1:4 \tag{49}$$

5 Finding shakedown limits for the hardening block

This section is devoted to find closed form solutions for the shakedown problem of the proposed example of a hardening block under variable thermal and mechanical loadings.

Our procedure is to identify different ranges of loading parameters where we assume some hypothesis on the response of the body, then use a subset of equations to compute explicit solutions and finally verify our initial guess by checking the entire system of equations given in the previous section.

We begin by observing that (40) and (42) imply that

$$\dot{\lambda}^A(2A_x - A_y) = d_x \tag{50}$$

Likewise (41) and (43) give

$$\dot{\lambda}^A(2A_y - A_x) = 0 \tag{51}$$

It follows from (50) and (51) that:

1. Equation (42) can be substituted in the system above by the simpler one (50). Likewise, (43) can be substituted by (51).
2. If we assume that the critical mechanism is incremental collapse or plastic collapse, i.e. $d \neq 0$, then

$$\dot{\lambda}^A > 0 \tag{52}$$

   because otherwise (50) enforces $d = 0$. This, due to (51), implies that

$$A_x = 2A_y \tag{53}$$

   Consequently

$$f_{S2}(A) = 0 \tag{54}$$

   and also

$$\sqrt{3}A_x = \pm 2\zeta \qquad \sqrt{3}A_y = \pm\zeta \qquad d_x = \pm\sqrt{3}\zeta\dot{\lambda}^A \tag{55}$$

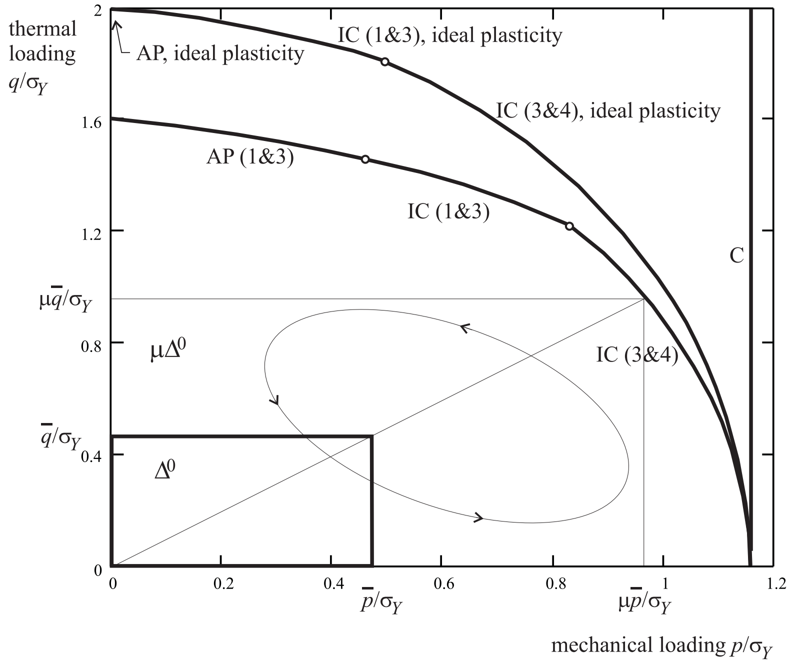

Figure 3: Interaction (Bree) diagram for a restrained block under independent variations of thermal and mechanical loadings. Hardening material, according to Stein's model with $\nu = 0.2$ and $\sigma_{Y0} = 0.8\sigma_Y$, compared to ideally plastic Mises material. The limiting curves are labeled: AP for alternating plasticity, given by (59), IC(1&3) for incremental collapse acted by loads 1 and 3, given by (68), IC(3&4) given by (74)) and C for plastic collapse.

5.1 Alternating plasticity acted by loads 1 and 3

We assume now that the failure mode is alternating plasticity produced by loads 1 and 3 and that the inequality constraints corresponding to loads 2 and 4 are inactive. In this case (34) reads

$$\dot{\lambda}^1 \nabla_\sigma f_{S1}(\mu\sigma^1 + \sigma^r, A) + \dot{\lambda}^3 \nabla_\sigma f_{S1}(\mu\sigma^3 + \sigma^r, A) = 0 \tag{56}$$

This condition can be interpreted geometrically considering the ellipse representing the first plastic mode, $f_{S1} = 0$, in the plane with coordinates σ_x and σ_y. In fact, in this representation we have $(\mu\sigma^1 + \sigma^r, A) \equiv (-A_x, \sigma_y^r - A_y)$ and $(\mu\sigma^3 + \sigma^r, A) \equiv (p - A_x, \nu p - q + \sigma_y^r - A_y)$ and the condition above means that these points determine a diameter because the normals are opposite. Hence, the coordinates of these points are respectively equal in absolute value and opposite in sign. This gives the following relations

$$2A_x = p \qquad 2(\sigma_y^r - A_y) = q - \nu p \tag{57}$$

Now, we use $f_{S1}(\mu\sigma^1 + \sigma^r, A) = 0$ to obtain

$$A_x^2 + (\sigma_y^r - A_y)^2 + A_x(\sigma_y^r - A_y) = \sigma_{Y0}^2 \tag{58}$$

The following interaction relation for the critical loading parameters $p = \mu\overline{p}$ and $q = \mu\overline{q}$ associated to the mechanism of alternating plasticity comes from (57) and (58)

$$(1 - \nu + \nu^2)p^2 + q^2 + (1 - 2\nu)pq = 4\sigma_{Y0}^2 \tag{59}$$

Moreover, the solution for the critical amplifying factor is

$$\mu = \frac{2\sigma_{Y0}}{(1 - \nu + \nu^2)\overline{p}^2 + \overline{q}^2 + (1 - 2\nu)\overline{pq}} \tag{60}$$

The alternating plasticity (AP) limit given by (59) is represented by the curve ab in Figure 3.

In the particular case when the loading is solely due to temperature variations, with no mechanical load, the critical parameter, used to prevent alternating plasticity, is

$$\mu = \frac{2\sigma_{Y0}}{\overline{q}} \tag{61}$$

The above solution, given by (59) and (60), is valid while the alternating plasticity mechanism is the critical failure mode that determines the elastic shakedown limit. It can be shown that this is the case for the following range of loading parameters

$$\overline{q} \geqslant m\overline{p} \tag{62}$$

with

$$m = \frac{1}{2}\left[\sqrt{(1 - 2\nu)^2 - 4\left[1 - \nu + \nu^2 - 3\left(\frac{\sigma_{Y0}}{2\zeta}\right)^2\right]} - 1 + 2\nu\right] \tag{63}$$

This determines the limit point of the alternating plasticity (AP) part in the interaction diagram of Figure 3.

To conclude the analysis of alternating plasticity let us consider the particular case of a block made of an ideally plastic material. For this system $\sigma_{Y0} = \sigma_Y$ and the mechanism of alternating plasticity can only occur without mechanical loading and with the following shakedown factor

$$\mu = \frac{2\sigma_Y}{\overline{q}} \tag{64}$$

5.2 Incremental collapse acted by loads 1 and 3

According to (55) we assume

$$\sqrt{3}A_x = 2\zeta \qquad \sqrt{3}A_y = \zeta \tag{65}$$

Then, we introduce these relations in $f_{S1}(\mu\sigma^1 + \sigma^r, A) = 0$ and $f_{S1}(\mu\sigma^3 + \sigma^r, A) = 0$ to obtain

$$\sigma_y^r = \sqrt{\sigma_{Y0}^2 - \zeta^2} \tag{66}$$

$$p^2 + (\nu p - q + \sigma_y^r)^2 - p(\nu p - q + \sigma_y^r) - \sqrt{3}\zeta p + \zeta^2 - \sigma_{Y0}^2 = 0 \tag{67}$$

The following interaction relation for the critical loading parameters $p = \mu\overline{p}$ and $q = \mu\overline{q}$ associated to the mechanism of incremental collapse involving loads 1 and 3 comes from (66) and (67)

$$(1 - \nu + \nu^2)p^2 + q^2 + (1 - 2\nu)pq - [(1 - 2\nu)p + 2q)]\sqrt{\sigma_{Y0}^2 - \zeta^2} - \sqrt{3}\zeta p = 0 \tag{68}$$

Furthermore, the critical factor is

$$\mu = \frac{[(1 - 2\nu)\overline{p} + 2\overline{q})]\sqrt{\sigma_{Y0}^2 - \zeta^2} + \sqrt{3}\zeta\overline{p}}{(1 - \nu + \nu^2)\overline{p}^2 + \overline{q}^2 + (1 - 2\nu)\overline{pq}} \tag{69}$$

The incremental collapse limit given by (68) is represented by the curve bc, labeled IC 1&3, in Figure 3. The analogous limit for the case of ideal plasticity is also given in the figure and corresponds to the equation below

$$(1 - \nu + \nu^2)p^2 + q^2 + (1 - 2\nu)pq - (1 - 2\nu)\sigma_Y p - 2\sigma_Y q = 0 \tag{70}$$

5.3 Incremental collapse acted by loads 3 e 4

According to (55) we assume

$$\sqrt{3}A_x = 2\zeta \qquad \sqrt{3}A_y = \zeta \tag{71}$$

Then, we introduce these relations in $f_{S1}(\mu\sigma^3 + \sigma^r, A) = 0$ and $f_{S1}(\mu\sigma^4 + \sigma^r, A) = 0$ to obtain

$$p^2 + (\nu p - q + \sigma_y^r)^2 - p(\nu p - q + \sigma_y^r) - \sqrt{3}\zeta p + \zeta^2 - \sigma_{Y0}^2 = 0 \tag{72}$$

$$p^2 + (\nu p + \sigma_y^r)^2 - p(\nu p + \sigma_y^r) - \sqrt{3}\zeta p + \zeta^2 - \sigma_{Y0}^2 = 0 \tag{73}$$

Combining the above relations we eliminate the unknown σ_y^r in order to obtain the following interaction relation for the critical loading parameters $p = \mu\overline{p}$ and $q = \mu\overline{q}$ associated to the mechanism of incremental collapse involving loads 3 and 4.

$$3p^2 + q^2 - 4\sqrt{3}\zeta p = 4\left(\sigma_{Y0}^2 - \zeta^2\right) \tag{74}$$

with the corresponding amplifying factor

$$\mu = \frac{2\left[\sqrt{3}\zeta\overline{p} + \sqrt{\left(3\overline{p}^2 + \overline{q}^2\right)\sigma_{Y0}^2 - \overline{q}^2\zeta^2}\right]}{3\overline{p}^2 + \overline{q}^2} \tag{75}$$

Moreover, the solution for the residual stress is

$$\sigma_y^r = \frac{[(1-2\nu)\overline{p}+\overline{q}]\left[\sqrt{3}\zeta\overline{p}+\sqrt{\left(3\overline{p}^2+\overline{q}^2\right)\sigma_{Y0}^2-\overline{q}^2\zeta^2}\right]}{3\overline{p}^2+\overline{q}^2} \tag{76}$$

The incremental collapse limit given by (74) is represented by the curve cd, labeled IC 3&4, in Figure 3. The analogous limit for the case of ideal plasticity is also given in the figure and corresponds to the equation below

$$3p^2+q^2=4\sigma_Y \tag{77}$$

6 Conclusions

The effects of including limited linear kinematic hardening in a Mises material, with respect to shakedown limits, are demonstrated in this paper by considering explicit solutions for a block submitted to independent variations of thermal and mechanical loadings.

The example is chosen because it is simple and presents the essential characteristic of the classical Bree's problem of a tube under combined temperature and pressure fluctuation [13].

The general picture of the analytical results for the shakedown analysis of the block of hardening material are depicted in Figure 3, together with analogous results for a Mises ideally plastic material. It is important for the interpretation of this comparison and the discussion in the sequel to point out that we compare a hardening material and an ideally plastic one that *share the same maximum yield stress* with the real material. Indeed, we never use in this paper the point of view that hardening adds an extra amount of plastic strength to the ideally plastic model. Instead, we consider that ideal plasticity models neglect any strain hardening *below the maximum yielding stress* of the real material.

The hardening block of the example presents some expected features in its behavior. For instance: (i) the collapse load is unaffected compared to ideal plasticity and (ii) the critical cycle characterizing failure under pure temperature variation reduces in amplitude the same amount than the reduction in the initial yielding parameter.

The hardening material may fail by alternating plasticity, i.e. low cycle fatigue, under thermal loads superposed to small mechanical loads (see curve AP in Figure 3), even before incremental collapse becomes feasible, while in ideal plasticity alternating plasticity is only critical with no mechanical load.

It is worth to recall now that the class of failure mechanisms called incremental collapse, that is those presenting cumulative plastic deformation per cycle, can be split in simple mechanisms of incremental collapse (SMIC) and combined mechanisms of incremental collapse (CMIC). In the latter there is one point in the body, at least, undergoing alternating plastic deformations, besides the global increase in the plastic deformation per cycle of the body. In the example adopted in this paper both regimes of incremental collapse of the block, IC (1&3) and IC (3&4) in Figure 3, are combined mechanisms of incremental collapse [7, 8]. Consequently, the reduction of the limits determined by incremental collapse failure, with respect to the ideally plastic material, can be explained by the previously observed reduction in the capacity to resist low cycle fatigue introduced by considering hardening before yielding.

The analytical solutions accomplished in the present study are intended also to serve as benchmark for algorithms and numerical procedures pertaining to the class of direct methods of shakedown analysis using finite element discretizations [14].

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