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Gain-Scheduled $\mathcal{L}_2$ Control of Discrete-Time Polytopic Time-Varying Systems

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Abstract. *The objective of this work is to present sufficient conditions that can guarantee stability of discrete-time Linear Parameter-Varying (LPV) systems. The proposed stability conditions are described in terms of linear matrix inequality (LMIs). LPV systems are systems whose dynamics changes according to a varying parameter, usually called the scheduling parameter. Practical examples of such systems are: aerospace structures that are constantly exposed to extreme variation of the temperature and robotic systems commonly used in pick-and-place applications. In this work, it is assumed that the system matrices of the LPV model belong to a polytope. The stability condition, derived using Lyapunov theory, is described by an LMI which depends on the time-varying parameter. Therefore, this LMI must be satisfied for each time instant. This is an infinite dimensional problem which is impossible to solve numerically. To avoid this issue, a finite set of sufficient LMI conditions is derived by imposing on the Lyapunov matrix a polytopic structure. The set of LMIs to be determined can take into consideration bounds on the rate of variation of the scheduling parameter. Thus, providing less conservative results than those obtained using stability conditions that allow the scheduling parameter to vary infinitely fast, as quadratic stability. Numerical simulations are performed to show the benefits of the proposed technique.*

Keywords: *discrete-time, linear time-varying system, gain-scheduled control, polytopic systems*

1. INTRODUCTION

Performance analysis and control synthesis for linear parameter varying (LPV) systems have received a lot of attention from the control community. This stems from the fact that LPV models are useful to describe the dynamics of linear systems with time-varying parameters as well as to represent nonlinear systems in terms of a family of linear models (Rugh and Shamma [2000]). In the LPV control framework, the scheduling parameters that govern the variation of the dynamics of the system are usually unknown, but supposed to be measured or estimated in real-time (Shamma and Athans [1991]). There is a continuing effort towards the design of LPV controllers, scheduled as a function of the varying parameters, to achieve higher performance while still guaranteeing stability for all possible parameter variations (Apkarian and Adams [1998], Scherer [2001]).

In the literature have been proposed several analysis and synthesis results for LPV systems have been proposed based on different types of Lyapunov functions that are able to guarantee stability and performance. The appeal of Lyapunov theory comes from the fact that it allows to recast many analysis and synthesis problems as linear matrix inequality (LMI) optimization problems (Boyd et al. [1994]). For LPV systems, the resulting parameter dependent LMI conditions need to be satisfied for the entire parameter space and, consequently, these LMI problems are infinite-dimensional. To arrive at a finite-dimensional set of LMI conditions, the choice of the parameterization (or the structure) of the Lyapunov matrix is essential.

Many of the existing Lyapunov approaches (Kaminer et al. [1993], Bernussou et al. [1989], Montagner et al. [2005b]) use the notion of quadratic stability where the Lyapunov matrix is constant. This yields a finite set of LMIs that are usually conservative for practical applications, since it allows arbitrarily fast variation of the scheduling parameters. To alleviate some of the conservatism associated with the quadratic stability-based approaches, many works have proposed the use of parameter-dependent Lyapunov functions: for time-varying systems, piecewise Lyapunov matrices are considered by, amongst others, Leite and Peres [2004] and Amato et al. [2005], affine and polytopic structures are used in, for instance, Daafouz and Bernussou [2001], Montagner et al. [2005a], Oliveira and Peres [2008], De Caigny et al. [2008b,a].

The aim of this paper is to provide linear matrix inequality (LMI) conditions that enforces an upper bound on the $\mathcal{L}_2$ gain for discrete-time linear systems with time-varying parameters belonging to a polytope with a prescribed bound on the rate of parameter variation. The proposed bound on the rate of parameter variation is more conservative than the bound used in De Caigny et al. [2008b]. However,this new bound has a more realistic interpretation from a physical point of view.

This paper is organized as follows: Section 3 presents general theoretical background regarding $\mathcal{L}_2$ gain of discrete-time LPV systems. Section 4 introduces some preliminaries with respect to the modeling of the uncertain domain, and

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then applies the results of Section 3 to the specific case of polytopic discrete-time LPV systems with known bounds on the rate of the parameter variation. Section 5 extends the analysis results and presents synthesis procedures for both gain scheduled and robust static output feedback controllers. A numerical example is presented in Section 6 that shows the benefits of the proposed approach. The conclusions and final remarks are presented in Section 7, and the Appendix presents the proof of the theorems.

1.1 Notation

The l_2^n space of square-summable sequences on the set of nonnegative integers $\mathbb{Z}_+$ is given by $l_2^n := \{f : \mathbb{Z}_+ \rightarrow \mathbb{R}_n | \sum_{k=0}^{\infty} f(k)^T f(k) < \infty\}$. The corresponding 2-norm is defined as $\|x(k)\|_2^2 = \sum_{k=0}^{\infty} x(k)^T x(k)$. The identity matrix of size $r \times r$ is denoted as I_r. The notation $0_{n,m}$ indicates an $n \times m$ matrix of zeros. The convex hull of a set X in denoted by coX. The set of positive real numbers is given by $\mathbb{R}^+$.

2. $\mathcal{L}_2$ GAIN OF DISCRETE-TIME LTV SYSTEMS

Consider the following discrete-time linear time-varying (LTV) system

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_u(k)u(k) + B_w(k)w(k) \\ y(k) &= C_y(k)x(k) + D_u(k)u(k) + D_w(k)w(k) \end{aligned} \tag{1}$$

where the state vector $x(k) \in \mathbb{R}^n$, the exogenous input $w(k) \in \mathbb{R}^r$, the control input $u(k) \in \mathbb{R}^m$ and the system output $y(k) \in \mathbb{R}^p$. The system matrices $A(k) \in \mathbb{R}^{n\times n}$, $B_w(k) \in \mathbb{R}^{n\times r}$, $B_u(k) \in \mathbb{R}^{n\times m}$, $C_y(k) \in \mathbb{R}^{p\times n}$, $D_w(k) \in \mathbb{R}_{p\times r}$ and $D_u(k) \in \mathbb{R}^{p\times m}$.

The $\mathcal{L}_2$ gain γ^* of system (1) is defined by the quantity $\gamma^* = \sup_{\|w(k)\|_2 \neq 0} \|y(k)\|_2 / \|w(k)\|_2$, with $w(k) \in l_2^m$ and $y(k) \in l_2^p$. The next theorem provide an upper bound γ on the $\mathcal{L}_2$ gain.

Theorem 1: If there exist symmetric positive-definite matrix $P(k)$, such that $V(x(k), k) = x(k)^T P(k) x(k) > 0$ for all $k \geq 0$ and

$$\Delta V(x(k), k) + y(k)^T y(k) - \gamma^2 w(k)^T w(k) \leq 0 \tag{2}$$

for all $x(k)$ and $y(k)$ satisfying system (1), then the $\mathcal{L}_2$ gain is less than γ. The proof of the theorem 1 can be found in Appendix 7.. An upper bound on the $\mathcal{L}_2$ gain of system (1) can be characterized using a parameter-dependent LMI as described in the next theorem.

Theorem 2: If there exist symmetric positive-definite matrix $P(k)$, such that

$$\begin{bmatrix} P(k+1) & * & * & * \\ A(k)^T P(k+1) & P(k) & * & * \\ B_w(k)^T P(k+1) & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_y(k) & D_w(k) & I_p \end{bmatrix} \geq 0. \tag{3}$$

then γ is an upper bound on the $\mathcal{L}_2$ gain of system (1).

The LMI condition in theorem 2 can be derivation from Theorem 1, the derivation can be found in Appendix (7).

3. $\mathcal{L}_2$ GAIN OF DISCRETE-TIME POLYTOPIC LPV SYSTEMS

In this section, Theorem 2 is particularized for the specific case of polytopic LPV systems. For this class of systems, it is provided a finite set of LMIs, defined in the vertices of the polytope, that guarantees an upper bound on the performance of a polytopic LPV system. Bounds on the rate of variation of the scheduling parameter are also considered. The modeling of the uncertainty domain is first presented, afterwards, the finite sets of LMIs that guarantee an $\mathcal{L}_2$ upper bound on the system gain are introduced.

3.1 Modeling of the Uncertainty Domain

Consider the following polytopic time-varying discrete-time linear system

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B_u(\alpha(k))u(k) + B_w(\alpha(k))w(k) \\ y(k) &= C_y(\alpha(k))x(k) + D_u(\alpha(k))u(k) + D_w(\alpha(k))w(k) \end{aligned} \tag{4}$$

where the state vector $x(k) \in \mathbb{R}^n$, the exogenous input $w(k) \in \mathbb{R}^r$, the control input $u(k) \in \mathbb{R}^m$ and the system output $y(k) \in \mathbb{R}^p$. The system matrices $A\alpha((k)) \in \mathbb{R}^{n\times n}$, $B_w(\alpha(k)) \in \mathbb{R}^{n\times r}$, $B_u(\alpha(k)) \in \mathbb{R}^{n\times m}$, $C_y(\alpha(k)) \in \mathbb{R}^{p\times n}$,

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$D_w(\alpha(k)) \in \mathbb{R}^{p\times r}$ and $D_u(\alpha(k)) \in \mathbb{R}^{pxm}$ belong to the polytope

$$\begin{aligned}\wp = \{&(A, B_u, B_w, C_y, D_u, D_w)(\alpha(k)) : (A, B_u, B_w, C_y, D_u, D_w)(\alpha(k)) \\ &= \sum_{i=1}^{N} \alpha_i(k)(A, B_u, B_w, C_y, D_u, D_w)_i,\ \alpha(k) \in \Lambda_N\},\end{aligned}$$

this model depends on $\alpha(k) \in \mathbb{R}^n$, a vector of time-varying parameters lying in the unit simplex Λ_N for all $0 \leq k \in \mathbb{N}$, where

$$\Lambda_N = \{\alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \ldots, N\}. \tag{5}$$

The rate of the parameter variation is given by

$$\Delta\alpha_i(k) = \alpha(k+1) - \alpha(k),\ i = 1, \ldots, N \tag{6}$$

observe that due to (5) and (6), we have that

$$\sum_{i=1}^{N} \Delta\alpha_i(k) = 0, \forall\, k \geq 0 \tag{7}$$

the rate of parameter variation $\Delta\alpha k$ is assumed to be limited by an *a priori* known bound b such that

$$-b \leq \Delta\alpha_i(k) \leq b,\ i = 1, \ldots, N \tag{8}$$

with $b \in [0, 1]$. In Oliveira and Peres [2008], the geometric aspects of the domain of the time-varying parameters have been exploited to derive a less conservative model for the space where the vector $\Delta\alpha(k)$ can lie. We now briefly present this model. The vector $\Delta\alpha$ is assumed to belong, for all $k \geq 0$, to the compact set

$$\Gamma_b = \{\delta \in \mathbb{R}^N : \delta \in co\{h^1, \ldots, h^M\}, \sum_{i=1}^{N} h_i^j = 0,\ j = 1, \ldots, M\}.$$

From (7) and (8) the columns $h^j, j = 1, \ldots, M$ of the set Γ_b can be constructed as

$$\begin{bmatrix} h^1 & h^2 & \ldots & h^M \end{bmatrix} = b \begin{bmatrix} 1 & 1 & 1 & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -1 \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

by construction, the number of columns M is given by $M = N(N-1)$. Now, taking a convex combination of these M columns, we obtain

$$\Delta\alpha = b\Gamma_b\beta, \quad \text{where} \quad \beta \in \Lambda_M.$$

Let $\Gamma^{(i)}$ be the i-th row of the matrix Γ_b, then $\Delta\alpha_i$ is given by

$$\Delta\alpha_i = b\Gamma^{(i)}\beta = \sum_{k=1}^{M} b\Gamma_k^{(i)}\beta_k.$$

3.2 $\mathcal{L}_2$ Gain of Discrete-Time Polytopic LPV Systems

The LMI condition in Theorem 2 is now particularized for the polytopic systems (4). This LMI condition follows directly from (3) in Theorem 2 by considering the specific time dependency of system (1) on the time-varying parameter $\alpha(k)$, we this have:

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Theorem 3: If there exist parameter-dependent symmetric positive-definite matrix $P(\alpha(k))$, for all $\alpha(k) \in \Lambda_N$, such that

$$\Phi(\alpha) = \begin{bmatrix} P(\alpha(k+1)) & * & * & * \\ A(\alpha(k))^T P(\alpha(k+1)) & P(\alpha(k)) & * & * \\ B_w(\alpha(k))^T P(\alpha(k+1)) & 0_{m,n} & \gamma I_m & * \\ 0_{p,m} & C_y(\alpha(k)) & D_w(\alpha(k)) & I_p \end{bmatrix} \geq 0. \quad (9)$$

then γ is an upper bound on the $\mathcal{L}_2$ gain of system (4).

The conditions of Theorem 3, which consist of evaluating the parameter-dependent LMI for all $\alpha(k)$ in the unit simplex Λ_N, lead to an infinite dimensional problem. However, by imposing on the Lyapunov matrix $P(\alpha(k))$ the following affine parameter-dependent polytopic structure $P(k) = \sum_{i=1}^{N} \alpha_i(k) P_i$, a finite-dimensional set of LMIs in terms of the vertices of polytope $\wp$ can be obtained, as shown in the next theorem.

Theorem 4: If there exist symmetric positive-definite matrices $P_i \in \mathbb{R}^{nxn}$, $i = 1, \ldots, N$ such that the following LMIs hold

$$\Phi_{il} = \begin{bmatrix} P_i + b\bar{P}_l & * & * & * \\ A_i^T (P_i + b\bar{P}_l) & P_i & * & * \\ B_{wi}^T (P_i + b\bar{P}_l) & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_{yi} & D_{wi} & I_p \end{bmatrix} \geq 0 \quad (10)$$

with $\bar{P}_l = \sum_{k=1}^{N} \Gamma_l^{(k)} P_k$ for $l = 1, \ldots, M,\ i = 1, \ldots, N$, and

$$\Phi_{ijl} = \begin{bmatrix} P_i + P_j + 2b\bar{P}_l & * & * & * \\ A_i^T (P_j + b\bar{P}_l) + A_j^T (P_i + b\bar{P}_l) & P_i + P_j & * & * \\ B_{wi}^T (P_j + b\bar{P}_l) + B_{wj}^T (P_i + b\bar{P}_l) & 0_{m,n} & 2\gamma^2 I_m & * \\ 0_{p,n} & C_{yi} + C_{yj} & D_{wi} + D_{wj} & 2I_p \end{bmatrix} \geq 0 \quad (11)$$

with $\bar{P}_l = \sum_{k=1}^{N} \Gamma_l^{(k)} P_k$ for $l = 1, \ldots, M,\ i = 1, \ldots, N-1,\ j = i+1, \ldots, N$, then γ is an upper bound on the $\mathcal{L}_2$ gain of system (4).

Proof Multiply (10) by $\alpha_i^2 \beta_l$ and sum for $l = 1, \ldots, M,\ i = 1, \ldots, N$. Multiply (11) by $\alpha_i \alpha_j \beta_l$ and sum for $l = 1, \ldots, M,\ i = 1, \ldots, N-1,\ j = i+1, \ldots, N$. Summing the results yields

$$\Phi(\alpha(k)) = \sum_{l=1}^{M} \sum_{i=1}^{N} \alpha_i^2 \beta_l \Phi_{il} + \sum_{l=1}^{M} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \beta_l \Phi_{ijl}$$

The set of LMIs(10)-(11) guarantees that $\Phi(\alpha(k))$ is positive semidefinite. ■

3.3 Extended $\mathcal{L}_2$ Gain of Discrete-Time Polytopic LPV Systems

Extended characterization for the $\mathcal{L}_2$ gain in derived using some additional variables. These extra variables will prove themselves valuable in the control design. First, multiply (9) pre and post with

$$T = \begin{bmatrix} P(\alpha(k+1))^{-1} & 0 & 0 & 0 \\ 0 & P(\alpha(k))^{-1} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}$$

to arrive at the following LMI, where $Q(k) = P(k)^{-1}$

$$\begin{bmatrix} Q(\alpha(k+1)) & * & * & * \\ Q(\alpha(k))A(\alpha(k))^T & Q(k) & * & * \\ B_w(\alpha(k))^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_y(\alpha(k))Q(\alpha(k)) & D_w(k) & I_p \end{bmatrix} \geq 0.$$

Theorem 5 The $\mathcal{L}_2$ gain of system (4) has as is upper bound γ if the following LMI is feasible

$$\Psi = \begin{bmatrix} Q(\alpha(k+1)) & * & * & * \\ G(\alpha(k))^T A(\alpha(k))^T & G(\alpha(k)) + G(\alpha(k))^T - Q(\alpha(k)) & * & * \\ B_w(\alpha(k))^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_y(\alpha(k))G(\alpha(k)) & D_w(\alpha(k)) & I_p \end{bmatrix} \geq 0. \quad (12)$$

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The proof the theorem 5 can be found in Appendix (7).

Theorem 6 If there exist symmetric matrices $Q_i \in \mathbb{R}_{nxn}$ and matrices $G_i \in \mathbb{R}_{nxn}, i = 1, \ldots, N$ such that the following LMIs hold

$$\Psi_{il} = \begin{bmatrix} Q_i + b\bar{Q}_l & * & * & * \\ G_i^T A_i^T & G_i + G_i^T - Q_i & * & * \\ B_{wi}^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_{yi} G_j & D_{wi} & I_p \end{bmatrix} \geq 0$$

with $\bar{Q}_l = \sum_{k=1}^{N} \Gamma_l^{(k)} Q_k$ for $l = 1, \ldots, M,\ i = 1, \ldots, N$, and

$$\Psi_{il} = \begin{bmatrix} Q_i + Q_j + 2b\bar{Q}_l & * & * & * \\ G_i^T A_j^T + G_j^T A_i^T & G_i + G_j + G_i^T + G_j^T - Q_i - Q_j & * & * \\ B_{wi}^T + B_{wj}^T & 0_{m,n} & 2\gamma^2 I_m & * \\ 0_{p,n} & C_{yi} G_j + C_{yj} G_i & D_{wi} + D_{wj} & 2I_p \end{bmatrix} \geq 0$$

with $\bar{Q}_l = \sum_{k=1}^{N} \Gamma_l^{(k)} Q_k$ for $l = 1, \ldots, M,\ i = 1, \ldots, N-1,\ j = i+1, \ldots, N$, then γ is an upper bound on the $\mathcal{L}_2$ gain of system (4). The proof of theorem 6 is similar to the proof of theorem 4.

4. $\mathcal{L}_2$ GAIN STATIC OUTPUT FEEDBACK

In this section, the analysis result presented in Theorem 6 is extended to provide a finite set of LMI conditions for the synthesis of robust and gain scheduled static output feedback controller that guarantee an upper bound on the closed-loop $\mathcal{L}_2$ gain for the discrete-time polytopic linear time-varying system (4).

4.1 Gain Scheduled Case

We assume that the first q states of the system can be measured in real-time for feedback without corruption by the exogenous input $w(k)$ or the control input $u(k)$, that is, $y(k) = C_y x(k)$ where $y(k) \in \mathbb{R}^q$ is the measured output. The matrix C_y is dependent of the time-varying parameters and is assumed to have the structure

$$C_y = \begin{bmatrix} I_q & O_{q,n-q} \end{bmatrix}. \tag{13}$$

If this is not the case, one can use a similarity transformation as proposed in Geromel et al. [1996], whenever the output matrix is not affected by the time-varying parameter.

The aim is to provide a parameter-dependent control law $u(k) = K(\alpha(k))y(k)$ with $K(\alpha(k)) \in \mathbb{R}^{mxq}$ such that the closed-loop system

$$\begin{aligned} x(k+1) &= A_{cl}(\alpha(k))x(k) + B_w(\alpha(k))w(k) \\ y(k) &= C_{cl}(\alpha(k))x(k) + D_w(\alpha(k))w(k), \end{aligned} \tag{14}$$

with

$$\begin{aligned} A_{cl}(\alpha(k)) &= A(\alpha(k)) + B_u(\alpha(k))K(\alpha(k))C_y \\ C_{cl}(\alpha(k)) &= C_y(\alpha(k)) + D_u(\alpha(k))K(\alpha(k))C_y \end{aligned}$$

is asymptotically stable with a bound γ on the closed-loop $\mathcal{L}_2$ gain, from $W(k)$ to $y(k)$ guaranteed for all possible variation of the parameter $\alpha(k) \in \Lambda_N$.

By replacing the state-space matrices in (12) with those from the closed-loop (14), we obtain

$$\Theta((\alpha(k)) = \begin{bmatrix} Q(\alpha(k+1)) & * & * & * \\ \Theta_{21} & G(\alpha(k)) + G(\alpha(k))^T - Q(\alpha(k)) & * & * \\ B_w(\alpha(k))^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & (C_y(\alpha(k)) + D_u(\alpha(k))K(\alpha(k)))G(\alpha(k)) & D_w(\alpha(k)) & I_p \end{bmatrix} \geq 0$$

where $\Theta_{21} = G(\alpha(k))^T(A(\alpha(k)) + B_u(\alpha(k))K(\alpha(k)))^T$ and by substituting $Z(k) = K(k)G(k)$, we obtain

$$\Omega(\alpha(k)) = \begin{bmatrix} Q(\alpha(k+1)) & * & * & * \\ \Omega_{21} & G(\alpha(k)) + G(\alpha(k))^T - Q(\alpha(k)) & * & * \\ B_w(\alpha(k))^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_y(\alpha(k))G(\alpha(k)) + D_u(\alpha(k))Z(\alpha(k)) & D_w(\alpha(k)) & I_p \end{bmatrix} \geq 0.$$

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where $\Omega_{21} = G(\alpha(k))^T A(\alpha(k))^T + Z(\alpha(k))^T B_u(\alpha(k))^T$.

Theorem 7 If there exist symmetric matrices $Q_i \in \mathbb{R}_{nxn}$, $G_{i,1} \in \mathbb{R}_{qxq}$, $G_{i,2} \in \mathbb{R}_{n-qxq}$, $G_{i,3} \in \mathbb{R}_{n-qxn-q}$ and $Z_{i,1} \in \mathbb{R}_{mxq}, i = 1, \ldots, N$ such that the following LMIs meets

$$\Omega_{il} = \begin{bmatrix} Q_i + b\bar{Q}_l & * & * & * \\ G_i^T A_i^T + Z_i^T B_{ui}^T & G_i + G_i^T - Q_i & * & * \\ B_{wi}^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_{yi}G_i + D_{ui}Z_i & D_{wi} & I_p \end{bmatrix} \geq 0 \tag{15}$$

with $\bar{Q}_l = \sum_{k=1}^{N} \Gamma_l^{(k)} Q_k$ for $l = 1, \ldots, M,\ i = 1, \ldots, N$, and

$$\Omega_{il} = \begin{bmatrix} Q_i + Q_j + 2b\bar{Q}_l & * & * & * \\ G_i^T A_j^T + G_j^T A_i^T + Z_i^T B_{uj} + Z_j^T B_{ui}^T & G_i + G_j + G_i^T + G_j^T - Q_i - Q_j & * & * \\ B_{wi}^T + B_{wj}^T & 0_{m,n} & 2\gamma^2 I_m & * \\ 0_{p,n} & C_{yi}G_j + C_{yj}G_i + D_{ui}Z_j + D_{uj}Z_i & D_{wi} + D_{wj} & 2I_p \end{bmatrix} \geq 0 \tag{16}$$

with $\bar{Q}_l = \sum_{k=1}^{N} \Gamma_l^{(k)} Q_k$ for $l = 1, \ldots, M,\ i = 1, \ldots, N-1,\ j = i+1, \ldots, N$, with

$$G_i = \begin{bmatrix} G_{i,1} & 0_{q,n-q} \\ G_{i,2} & G_{i,1} \end{bmatrix} \quad \text{and} \quad Z_i \begin{bmatrix} Z_{i,1} & 0_{m,n-q} \end{bmatrix} \tag{17}$$

are feasible, then the parameter-dependent controller

$$K(k) = Z(k)G(k)^{-1} \tag{18}$$

stabilizes the open loop with γ is an upper bound on the $\mathcal{L}_2$ gain of the closed-loop (14).

Proof Multiply (15) by $\alpha_i^2 \beta_l$ and sum for $l = 1, \ldots, M,\ i = 1, \ldots, N$,. Multiply (16) by $\alpha_i \alpha_j \beta_l$ and sum for $l = 1, \ldots, M,\ i = 1, \ldots, N-1,\ j = 1, \ldots, N$. Summing the results yields

$$\Omega(\alpha(k)) = \sum_{l=1}^{M} \sum_{i=1}^{N} \alpha_i^2 \beta_l \Omega_{il} + \sum_{l=1}^{M} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \beta_l \Omega_{ijl}$$

using (17) and (18) and considering the specific structure (13) for C_y, the LMI $\Omega(\alpha(k))$ can be written as

$$\begin{bmatrix} Q(\alpha(k+1)) & * & * & * \\ G(\alpha(k))^T (A_{cl}(\alpha(k)))^T & G(\alpha(k)) + G(\alpha(k))^T - Q(\alpha(k)) & * & * \\ B_w(\alpha(k))^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & (C_{cl}(\alpha(k)))G(\alpha(k)) & D_w(\alpha(k)) & I_p \end{bmatrix} \geq 0$$

Therefore, as a result of Theorem 5, feasibility of the LMIs (15) and (16) ensures that the closed-loop (14) is asymptotically stable with an upper bound γ on its $\mathcal{L}_2$ gain. ■

4.2 Robust Static Case

Robust $\mathcal{L}_2$ gain static output feedback controller $u(k) = KC_y x(k)$ is a particular case of gain-scheduled and can be calculated using theorem 7 by fixing $K = ZG^{-1}$ for $Z_i = Z$ and $G_i = G$.

5. NUMERICAL EXAMPLE

Consider the polytopic time-varying (4) for $n = 3$ and $N = 2$ with the following system matrices:

$$\begin{bmatrix} A_1 \,\vdots\, A_2 \end{bmatrix} = \mu \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 2 & -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -2 & -1 \end{array}\right], B_{w,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$B_{w,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_{u,i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_{z,i} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T, D_{u,i} = D_{w,i} = 0,$$

with $i = 1, 2$ and $\mu \in \mathbb{R}^+$. The system matrices are borrowed from Oliveira and Peres [2008]. The aim in this example is to determine the maximum bound b_{max} on the rate of parameter variation as a function of the scalar μ such that the system can be stabilized by an $\mathcal{L}_2$ gain static output feedback controller presented in this paper and compare with results shown in De Caigny et al. [2008b]. Both gain-scheduled and robust output feedback controllers are designed for case the measurement equation $y(t) = C_y x(k)$, with all states are available for feedback ($C_y = I_n$).

Figure 1 shows B_{max} as a function of μ. The curves R_1 and R_2 are robust controllers, G_1 and G_2 are gain-scheduled controllers, being the R_1 and G_1 controllers presented in this paper and R_2 and G_2 controllers presented in De Caigny et al. [2008b].

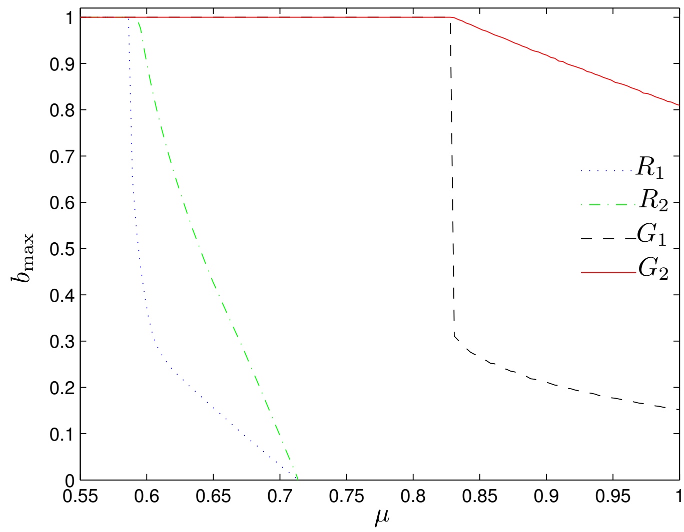

Figure 1. Maximal bound $b_{\max}$ on the rate of parameter variation as a function of the scalar μ

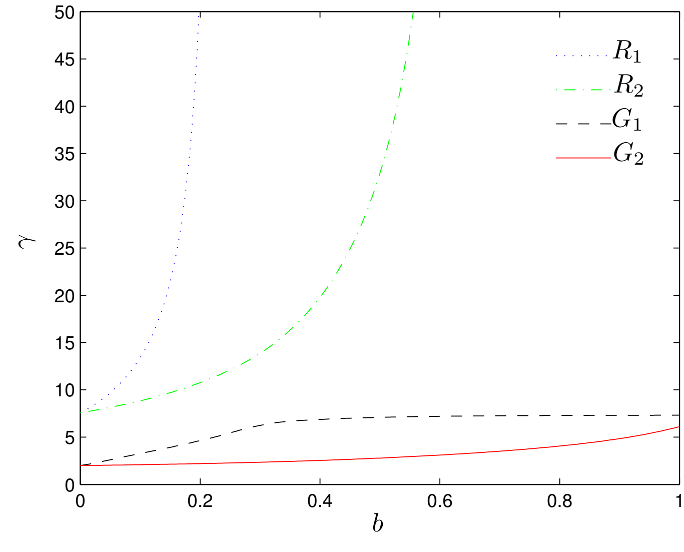

Figure 2. Guaranteed upper bound γ on the $\mathcal{L}_2$ gain as a functino of the scalar b

For low values of μ, all control designs result in controllers that allow the parameters to vary arbitrarily fast in the unit simplex since $b_{max} = 1$. However, as μ increases, the maximal allowed bound b_{max} becomes smaller. Obviously, this occurs first for the robust case R_1, since it is the most restrictive control design. Note also that since the gain-scheduled controllers are less restrictive than the robust controllers, the curves associated with the gain-scheduled controllers are always on the right of the curves associated with the corresponding (in terms of output measurements) robust controllers.

To check the achieved performance, μ is now fixed to be $\mu = 0.6185$. Figure 2 shows the achieved upper bound γ on the closed-loop $\mathcal{L}_2$ gain as a function of the allowed bound $0 \leq b \leq 1$ on the rate of variation.

For all control designs, it is clear from Figure 2, that as the bound b increases, the performance becomes worse since the upper bound γ increases. In the robust case R_1 and R_2, the upper bound γ increases very fast as the value of the bound b increases. This can be expected since Figure 1 shows that for the robust case R_1 with $\mu = 0.6185$ the LMI conditions become infeasible for $b > 0.2422$ and for the the robust case R_2 the LMIs become infeasible for $b > 0.6836$. In the gain scheduled case G_1 and G_2, the conditions are feasible for all values of b.

For the specific case $b = 1$, where the parameters can vary arbitrarily fast in the unit simplex Λ_N, the gain-scheduled case G_1 yields the gain $\gamma = 7.2569$, in case G_2 the gain is $\gamma = 6.0642$. As seen in Figure 2, the LMI conditions in Theorem 7, by explicitly considering the bound b on the rate of variation, can provide a value very near $\mathcal{L}_2$ gain bound γ for the gain-scheduled case G_1 as compared to the results of the G_2, same more conservative. For the case b= 0, the robust cases R_1 and R_2 presented same yields $\gamma = 7.4563$ and gain-scheduled cases G_1 and G_2 presented yields $\gamma = 1,9935$.

As shown in Figure 2, the results of the controllers G_1 and G_2 are near, but the results presented this paper with controller G_1 are more reliable, by be more conservative.

6. CONCLUSION

In this work, new LMI conditions are presented for the synthesis of robust and gain-scheduled $\mathcal{L}_2$ gain static output feedback controllers for discrete-time polytopic linear time-varying systems based on parameter dependent Lyapunov functions. The synthesis procedures explicitly take an a priori known bound on the rate of parameter variation into account, thus reducing the conservatism generally associated with methods that allow arbitrarily fast parameter variation.

Compared to the conditions of De Caigny et al. [2008b], the proposed approach yields similar results. They have different modeling for the rate of the parameter variation which has a more realistic physical interpretation.

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Proof of Theorem 1

Summing (1) for $k = 0, \ldots, T$, with $x(0) = 0$, leads to

$$V(x(T+1), T+1) + \sum_{k=0}^{T} (y(k)^T y(k) - \gamma^2 w(k)^T w(k)) \leq 0.$$

Since $V(x(T+1), T+1) > 0$, this implies

$$\sum_{k=0}^{T} y(k)^T y(k) \leq \gamma^2 \sum_{k=0}^{T} w(k)^T w(k)$$

and consequently

$$\|z(k)\|_2 / \|w(k)\|_2 \leq \gamma.$$

Since (1) holds for all $w(k)$ and $y(k)$, we conclude that

$$\sup \|z(k)\|_2 / \|w(k)\|_2 \leq \gamma.$$

■

Proof do Theorem 2

From Theorem 1, we now that

$$\Delta V(x(k), k) + y(k)^T y(k) - \gamma^2 w(k)^T w(k) \leq 0$$

This inequality is equivalent to

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A(k)^T P(k+1) A(k) - P(k) + C_y(k)^T C_y(k) & A(k)^T P(k+1) B_w(k) + C_y(k)^T D_w(k) \\ B_w(k)^T P(k+1) A(k) + D_w(k)^T C_y(k) & B_w(k)^T P(k+1) B_w(k) + D_w(k)^T D_w(k) - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0$$

Now, this condition is satisfied if the linear matrix inequality

$$\begin{bmatrix} P(k) - A(k)^T P(k+1) A(k) - C_y(k)^T C_y(k) & -A(k)^T P(k+1) B_w(k) - C_y(k)^T D_w(k) \\ -B_w(k)^T P(k+1) A(k) - D_w(k)^T C_y(k) & -B_w(k)^T P(k+1) B_w(k) - D_w(k)^T D_w(k) + \gamma^2 I \end{bmatrix} \geq 0$$

holds. Using Schur complement we can write the following LMI

$$\begin{bmatrix} P(k) - A(k)^T P(k+1) A(k) - P(k) & -A(k)^T P(k+1) B_w(k) - C_y(k)^T D_w(k) & C_y(k)^T \\ -B_w(k)^T P(k+1) A(k) & -B_w(k)^T P(k+1) B_w(k) + \gamma^2 I & D_w(k)^T \\ C_y(k) & D_w(k) & I \end{bmatrix} \geq 0$$

which is equivalent to equation (3) using Schur complement again . ■

Derivation of the Theorem 4

First note that if $P(\alpha(k))$ is given by $P(k) = \sum_{i=1}^{N} \alpha_i(k) P_i$. Then $P(\alpha(k+1))$ can be written as

$$\begin{aligned} P(\alpha(k+1)) &= \sum_{i=1}^{N} \alpha_i(k+1) P_i = \sum_{i=1}^{N} (\alpha_i(k) + \Delta\alpha_i(k)) P_i \\ &= \sum_{i=1}^{N} \alpha_i(k) P_i + \sum_{k=1}^{N} \Delta\alpha_k(k) P_k = \sum_{i=1}^{N} \alpha_i(k) P_i + \sum_{j=1}^{M} \sum_{k=1}^{N} b\Gamma_j^{(k)} \beta_j P_k \end{aligned}$$

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Define $\bar{P}_j = \sum_{k=1}^{N} \Gamma_j^{(k)} P_k$, we obtain

$$P(\alpha(k+1)) = \sum_{i=1}^{N} \alpha_i(k) P_i + \sum_{j=1}^{M} b\beta_j \bar{P}_j.$$

We will make the derivation of the theorem for the element (2,1) of $\Phi(\alpha(k))$:

$$\Phi(\alpha(k))_{(2,1)} = A(\alpha(k))^T P(\alpha(k+1))$$

using the formula for $P(\alpha(k+1))$, we obtain

$$\Phi(\alpha(k))_{(2,1)} = (\sum_{i=1}^{N} \alpha_i A_i^T)(\sum_{j=1}^{N} \alpha_j P_j + \sum_{l=1}^{M} b\beta_l \bar{P}_l)$$

can be write

$$\Phi(\alpha(k))_{(2,1)} = (\sum_{i=1}^{N} \alpha_i A_i^T)(\sum_{j=1}^{N} \alpha_j P_j) + (\sum_{i=1}^{N} \alpha_i A_i^T)(\sum_{l=1}^{M} b\beta_l \bar{P}_l).$$

Multiplying the first term by $\sum_{l=1}^{M} \beta_l = 1$ and the second term by $\sum_{j=1}^{N} \alpha_j = 1$ we obtain

$$\Phi(\alpha(k))_{(2,1)} = (\sum_{i=1}^{N} \alpha_i A_i^T)(\sum_{j=1}^{N} \alpha_j P_j)(\sum_{l=1}^{M} \beta_l) + (\sum_{i=1}^{N} \alpha_i A_i^T)(\sum_{l=1}^{M} b\beta_l \bar{P}_l)(\sum_{j=1}^{N} \alpha_j).$$

The first term can be worked out as

$$(\sum_{i=1}^{N} \alpha_i^2 A_i^T P_i + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j (A_i^T P_j + A_j^T P_i))(\sum_{l=1}^{M} \beta_l)$$

and the second term can be worked out as

$$(\sum_{i=1}^{N} \alpha_i^2 A_i^T + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j (A_i^T + A_j^T))(\sum_{l=1}^{M} b\beta_l \bar{P}_l).$$

Summing these two up and rearranging the terms, we obtain

$$\Phi(\alpha(k))_{(2,1)} = \sum_{l=1}^{M} \sum_{i=1}^{N} \alpha_i^2 \beta_l A_i^T (P_i + b\bar{P}_l) + \sum_{l=1}^{M} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \beta_l (A_i^T (P_i + b\bar{P}_l) + A_i^T (P_i + 2b\bar{P}_l))$$

or equivalently as

$$\Phi(\alpha(k))_{(2,1)} = \sum_{l=1}^{M} \sum_{i=1}^{N} \alpha_i^2 \beta_l \Phi_{il(2,1)} + \sum_{l=1}^{M} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \beta_l \Phi_{ijl(2,1)}.$$

In this case $\Phi(\alpha(k))$ can be written as

$$\Phi(\alpha(k)) = \sum_{l=1}^{M} \sum_{i=1}^{N} \alpha_i^2 \beta_l \Phi_{il} + \sum_{l=1}^{M} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \beta_l \Phi_{ijl}.$$