The boundary element method in the dynamic analysis of symmetric laminate composite plates

E. L. Albuquerque*, P. Sollero, and W. Portilho de Paiva

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Faculty of Mechanical Engineering, State University of Campinas
CP 6122, 13083-970, Campinas, SP, Brazil
*ederlima@fem.unicamp.br

Abstract

This article presents a boundary element formulation for dynamic analysis of laminate composite plates using the boundary element method. Static fundamental solutions are used and inertia effects are treated as body forces. Domain integrals that arise in the formulation due to inertia effects are transformed into boundary integrals by the radial integration method. In this method, domain loads are approximated by a sum of products between radial basis functions and unknown coefficients. Then, by an exact transformation, domain integrals are transformed into boundary integrals. Constant boundary elements are used in the boundary discretization. Integral with strong and hyper singularities are analytically integrated. Internal points are necessary to improve the approximation of domain loads. In order to assess the proposed formulation, their results are compared to analytical, finite element, and experimental solutions. An analysis of the accuracy of results and its dependence on the mesh refinement and the number of internal points is carried out.

Keywords: Boundary element method, radial integration method, dual reciprocity boundary element method, composite materials, modal analysis.

1 Introduction

Nowadays, the structural dynamic analysis of anistropic plates by the boundary element method (BEM) has been treated by two different approaches. In the first, the fundamental solution is obtained considering all terms of the equation of motion. As a consequence, this approach has no domain integrals but uses complicated fundamental solutions (Wang and Schweizerhof [1]). In the second, the static fundamental solution is used and the inertia term of the equation of motion is considered as a body force. This body force generates domain integrals that can be computed by the discretization of the domain into cells. However, this procedure eliminates, to a certain extent, one of the most interesting advantage of the BEM that is the absence of domain discretization. Many alternative procedures have been presented to treat domain integrals in the BEM as shown in books like Nowak [2] and Partridge *et al.* [3]. Among them, the most established is the dual reciprocity boundary element method (DRM), proposed by Nardini and Brebbia [4] for the analysis of dynamic problems in plane elasticity and extended by many other authors for different applications (see the book of Partridge *et al.* [3]). The DRM has been successfully used in the dynamic analysis of anisotropic structures as presented by Albuquerque *et al.* [5, 6, 7, 8] for bidimensional problems, and by Kögl and Gaul [9, 10, 11] for three-dimensional problems. Due to the complexity of governing equations of anisotropic materials, the analytical computation of particular solutions used in the DRM is restricted to some approximation functions, all of them are functions of material properties.

When body forces are known, domain integrals can be exact transformed into boundary integrals, as shown by Venturini [12] for isotropic plate bending problems, by Gao [13] for anisotropic three-dimensional problems, and by Albuquerque *et al.* [14] for anisotropic plate bending problems. This transformation, that is based on a radial integration, was named by Gao [13] as the radial integration method (RIM). More recently, Albuquerque *et al.* [15] extended the RIM method for domain integrals of unknown functions (modal analysis of anisotropic plates). In this previous work, the radial integration method was used with the same material dependent approximation function used in the dual reciprocity. It was shown that results with the same accuracy was obtained by the RIM and the DRM if the same approximation function is used in both formulations. Now, the present work will explore different approximation functions and assess their accuracy. Three approximation functions, that are well established for isotropic formulations of the DRM, are explored.

2 Boundary integral equations

The boundary element formulation for anisotropic thin plate problems uses two integral equations, for displacement and rotation (see [16, 14]). The transversal displacement equation is given by:

$$Kw(Q) + \int_{\Gamma} \left[V_n^*(Q,P)w(P) - m_n^*(Q,P)\frac{\partial w(P)}{\partial n} \right] d\Gamma(P) +$$

$$\sum_{i=1}^{N_c} R_{c_i}^*(Q,P)w_{c_i}(P) = \sum_{i=1}^{N_c} R_{c_i}(P)w_{c_i}^*(Q,P) + \int_{\Omega_g} b(P)w^*(Q,P)d\Omega +$$

$$\int_{\Gamma} \left[V_n(P)w^*(Q,P) - m_n(P)\frac{\partial w^*}{\partial n}(Q,P) \right] d\Gamma(P), \qquad (1)$$

where P is the field point; Q is the source point; Γ is the boundary of the domain Ω of the plate; Ω_g is the part of the domain Ω where the body force b is applied; the constant K is introduced in order to consider that the source point Q can be placed in the domain, on the boundary, or outside the domain (if the point Q is on a smooth boundary, then $K = 1/2$); $\frac{\partial()}{\partial n}$ is the derivative to the outward unity vector $\mathbf{n}$ that is normal to the boundary Γ at the field point P; m_n and V_n are, respectively, the normal bending moment and the Kirchhoff's equivalent shear force on the boundary Γ; R_c is the thin plate reaction of corners; w_c is the transversal displacement of corners; N_c is the number of corners; and the symbol * stands for fundamental solutions that can be found in Shi and Bezine [16].

The rotation equation is given by:

$$\frac{1}{2}\frac{\partial w(Q)}{\partial n_1} + \int_{\Gamma} \left[\frac{\partial V^*}{\partial n_1}(Q,P)w(P) - \frac{\partial m_n^*}{\partial n_1}(Q,P)\frac{\partial w}{\partial n}(P) \right] d\Gamma(P) +$$

$$\sum_{i=1}^{N_c} \frac{\partial R_{c_i}^*}{\partial n_1}(Q,P)w_{c_i}(P) = \sum_{i=1}^{N_c} R_{c_i}(P)\frac{\partial w_{c_i}^*}{\partial n_1}(Q,P) + \int_{\Omega_g} b(P)\frac{\partial w^*}{\partial n_1}(Q,P)d\Omega +$$

$$\int_{\Gamma} \left\{ V_n(P)\frac{\partial w^*}{\partial n_1}(Q,P) - m_n(P)\frac{\partial}{\partial n_1}\left[\frac{\partial w^*}{\partial n}(Q,P) \right] \right\} d\Gamma(P), \qquad (2)$$

where $\frac{\partial()}{\partial n_1}$ is the derivative to the outward unity vector $\mathbf{n}_1$ that is normal to the boundary Γ at the source point Q.

As can be seen, domain integrals arise in the formulation owing to the presence of the body force b. In order to transform these integrals into boundary integrals, consider, as in the DRM, that the body force b is approximated over the domain Ω_g as a sum of M products between radial basis functions f^m and unknown coefficients γ^m, that is:

$$b(P) = \sum_{m=1}^{M} \gamma^m f^m. \tag{3}$$

Thus, the domain integrals of equations (1) and (2) can be written, respectively, as:

$$\int_{\Omega_g} b(P) w^*(Q,P) d\Omega = \sum_{m=1}^{M} \gamma^m \int_{\Omega_g} f^m w^*(Q) d\Omega. \tag{4}$$

and

$$\int_{\Omega_g} b(P) \frac{\partial w^*(Q,P)}{\partial n_1} d\Omega = \sum_{m=1}^{M} \gamma^m \int_{\Omega_g} f^m \frac{\partial w^*(Q)}{\partial n_1} d\Omega. \tag{5}$$

Following the development presented by Albuquerque *et al.* [15], domain integrals (4) and (5) are transformed, respectively, in the boundary integrals:

$$\int_{\Omega_g} b(P) w^*(Q,P) d\Omega = \sum_{m=1}^{M} \gamma^m \int_{\Gamma_g} \frac{F^m(Q)}{r} \mathbf{n}.\mathbf{r} d\Gamma. \tag{6}$$

and

$$\int_{\Omega_g} b(P) \frac{\partial w^*(Q,P)}{\partial n_1} d\Omega = \sum_{m=1}^{M} \gamma^m \int_{\Gamma_g} \frac{G^m(Q)}{r} \mathbf{n}.\mathbf{r} d\Gamma, \tag{7}$$

where

$$F^m(Q) = \int_0^r f^m w^*(Q,P) \rho d\rho, \tag{8}$$

and

$$G^m(Q) = \int_0^r f^m \frac{\partial w^*(Q,P)}{\partial n_1} \rho d\rho. \tag{9}$$

ρ is a local coordinate that defines the integration around the radial direction (see Albuquerque *et al.* [15]).

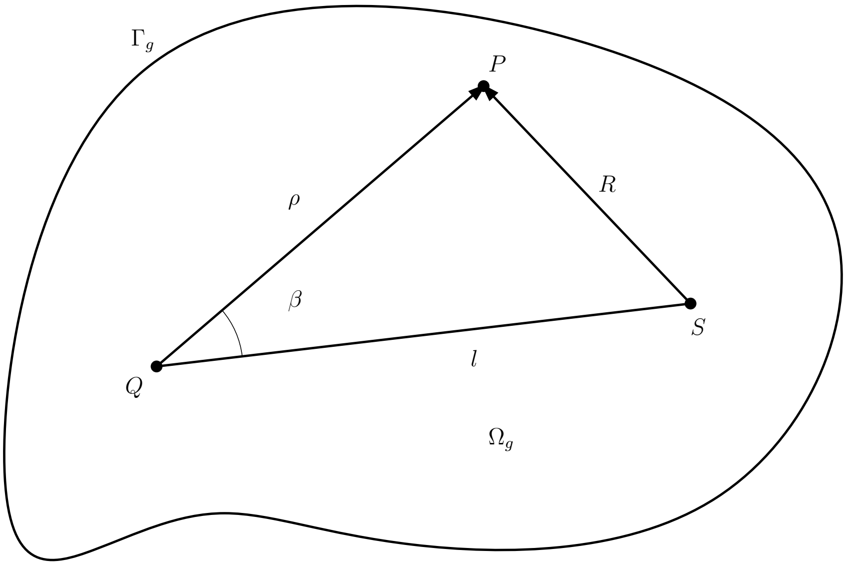

Figure 1: Positions of points in the domain.

In this work, the approximation functions f^m are radial basis function written in terms of R, where R is the distance between the centre S of the radial basis function and the integration point P (Figure 1).

Three approximation functions are used in this work:

$$f_1^m = 1 + R, \tag{10}$$

$$f_2^m = 1 + R + R^3, \tag{11}$$

and

$$f_3^m = R\log(R). \tag{12}$$

Equation (3) can be written in a matrix form, considering all source points, as:

$$\mathbf{b} = \mathbf{F}\boldsymbol{\gamma} \tag{13}$$

Thus, $\boldsymbol{\gamma}$ can be computed as:

$$\gamma = \mathbf{F}^{-1}\mathbf{b} \tag{14}$$

For free vibration dynamic problems, the body force vector is given by:

$$\mathbf{b} = \varrho h \omega^2 \mathbf{w} \tag{15}$$

where ϱ is the material density, h is the plate thickness, ω is the circular frequency of vibration, and $\mathbf{w}$ are transversal displacement of the nodes where displacements are free.

As shown by Albuquerque *et al.* [15], the boundary integral equations (1) and 2 can be discretized into boundary elements and transformed into a matrix equation that, for free vibration, can be written as an eigen problem given by:

$$\mathbf{A}\mathbf{w} = \lambda \mathbf{w}, \tag{16}$$

where

$$\mathbf{A} = \hat{\mathbf{H}}^{-\mathbf{1}}\hat{\mathbf{M}}, \tag{17}$$

λ is the eigenvalue that can be written as:

$$\lambda = \frac{1}{\omega^2}, \tag{18}$$

and $\mathbf{H}$ and $\mathbf{M}$ are non nosymmetric matrix obtained by the integration of terms of equations (1) and (2) over the boundary elements.

3 Numerical results

In order to compare the accuracy of the different approximation functions, the method is applied to the free vibration problem proposed by Albuquerque *et al.* [15]. Consider a rectangular plate with width $b = 350$ mm, length $a = 450$ mm, and thickness $h = 2.1$ mm (Figure 2). The plate is orthotropic with the following material properties: $E_x = 120$ GPa, $E_y = 10$ GPa, $G_{xy} = 4.8$ GPa, $\nu_{12} = 0.3$ and $\varrho = 1510$ kg/m^3.

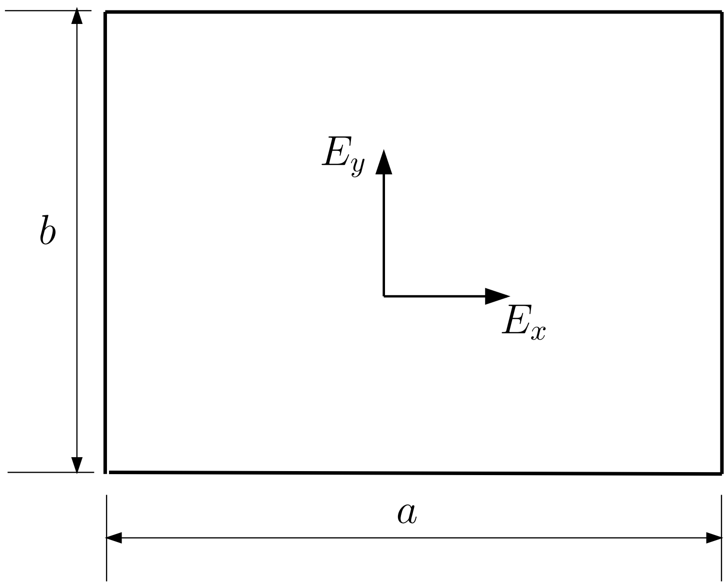

Figure 2: Orthotropic rectangular plate.

Table 1: Natural frequencies computed by the FEM and the RIM using the material dependent approximation function of the previous work and the three approximation functions presented here.

Mode shape	FEM	Reference [15]		f_1^m		f_2^m		f_3^m	
	$\omega/(2\pi)$ (Hz)	$\omega/(2\pi)$ (Hz)	Diff. (%)	$\omega/(2\pi)$ (Hz)	Diff. (%)	$\omega/(2\pi)$ (Hz)	Diff. (%)	$\omega/(2\pi)$ (Hz)	Diff. (%)
1	25.50	24.39	4.34	24.41	4.29	24.37	4.42	24.38	4.40
2	45.35	45.57	0.48	45.18	0.38	45.15	0.43	45.34	0.03
3	68.78	67.37	2.05	67.15	2.38	67.16	2.36	67.49	1.87
4	95.25	94.60	0.68	95.22	0.04	95.21	0.04	95.51	0.27
5	107.61	106.34	1.18	106.42	1.11	106.41	1.12	106.62	0.92

The free vibration of this plate was analysed using the three radial basis functions, given by equations (10), (11), and (12), with a mesh of 66 constant boundary elements and 63 internal points as shown in Figure 3.

Table 1 shows the first five natural frequencies computed by the RIM using the three approximation functions, by RIM as presented in the previous work of the authors (Albuquerque *et al.*[15]), where were used a material dependent approximation function, and by the finite element method (FEM) using a very refined mesh. The finite element mesh has 150 (15×10) quadrilateral finite elements (8 nodes per element). As it can be seen, the agreement between the results of the RIM using different approximation function with FEM is very similar.

Figures 4, 5, 6, 7, and 8 show the first five vibration modes of the orthotropic free edge plate.

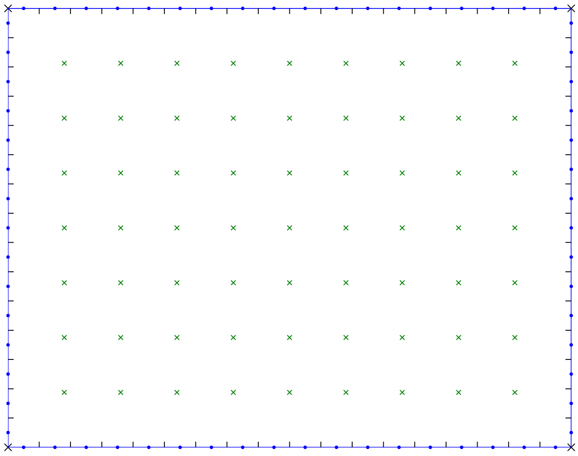

Figure 3: Boundary elements and internal points for the rectangular plate.

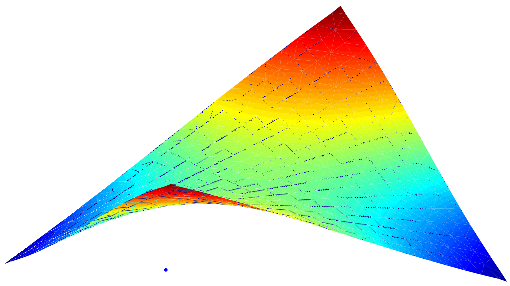

Figure 4: Mode 1.

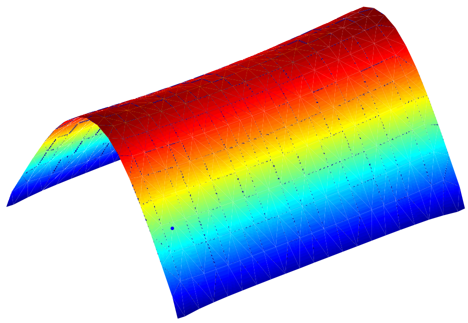

Figure 5: Mode 2.

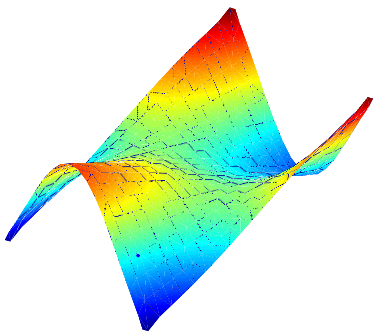

Figure 6: Mode 3.

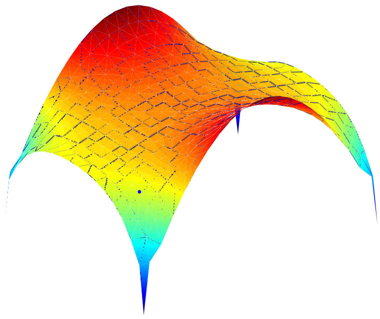

Figure 7: Mode 4.

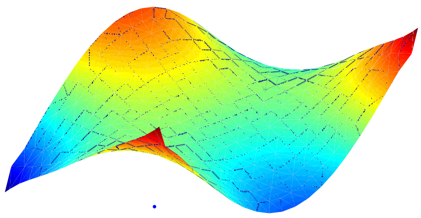

Figure 8: Mode 5.

4 Conclusions

This paper analysed the use of different approximation functions in the radial integration method applied to modal analysis of anistotropic plates. Three approximation functions that are well established in the dual reciprocity boundary element method for isotropic material problems were used and there results compared to results obtained by a material dependent approximation function, previously presented by the authors, and the finite element method. The agreement between all results are very good. It can be concluded that the radial integration method is a very suitable method for the transformation of domain integrals to boundary integrals in anisotropic material problems. It can give good results with those simple approximation functions used in the dual reciprocity boundary element method for isotropic material structural problems.

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