An Asymptotic Description of Separating Turbulent Flows

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Abstract. *The present work will shortly review some recently developed results for the description of the asymptotic structure of a separating flow. Our main purpose is to show how the single limit concept of Kaplun can be used to derive a new asymptotic structure for the flow field which is consistent with our commom knowledge of the problem. The classical structure of the velocity and temperature boundary layers is shown to develop into an one-deck structure near the separation point due to the merging of two principal equations. The work also shows that, near a separation point, the Reynolds analogy breaks down yielding a different power-law for the temperature profile. As a bonus, the work furnishes analytical expressions for the velocity and temperature near wall solutions which are shown to hold also in the reverse flow. The specified velocity profiles are based on a previous formulation of the problem by Cruz and Silva Freire(IJHMT, 41, 2097–2111, 1998; IJHMT, 45, 1459–1466, 2002). The temperature profiles near the wall are described by a newly proposed expression that reduces to a logarithmic profile in the attached region, and assumes a minus half power law profile at the separation point. In the separated region, the logarithmic profile is recovered.*

Keywords: *Kaplun limits, separation asymptotic structure.*

1. Introduction

The correct description of the asymptotic structure of the turbulent boundary layer is a problem that has been satisfactorily resolved for simple flow conditions. For the classical problem of zero-pressure-gradient boundary layers, early developments by Yajnik(1970), Mellor(1972) and Bush and Fendell(1972) have stated the boundary layer to have a two-layered structure consisting of a viscous wall layer and an outer defect region. Yajnik and Mellor do not use in their analyzes any closure hypothesis to represent the turbulent shear stress terms, relying only on asymptotic arguments. Bush and Fendell, on the other hand, use in their developments turbulence models of the mixing-length/eddy viscosity type. In all cases, the flow structure and solutions are developed in terms of an appropriate small parameter, $\epsilon = ord\ (lnR)^{-1}$. Clearly, these early analyzes had been tailored to conform the results to the then classical two-layered structure suggested by the law of the wall and the law of the wake.

Using a different asymptotic approach, Sychev and Sychev(1987) have claimed the boundary layer to have a three-deck structure. In fact, all recent discussions that led to the development of three-layered asymptotic models for the turbulent boundary layer are basically motivated by the inability of the classical two-deck model to deal with large adverse pressure gradients. When a turbulent boundary layer is subjected to a large adverse pressure gradient, the wake velocity deficit is large and the mean momentum equation is non-linear. These features make Millikan's "matchability" arguments, which result is a log-law and in a two deck structure, not valid anymore. Also the friction velocity, u_τ, used in the classical approaches as a characteristic velocity, becomes an inappropriate scaling parameter for adverse pressure gradient boundary layers since it tends to zero. All these difficulties force into the adverse pressure gradient problem a new small parameter of the order of $R^{-1/3}$, which is used to scale a power y layer that replaces Millikan's log-layer. This layer matches, on passage of the inner and outer limits, respectively, to the wall and defect layers. Thus, according to this picture, three sets of characteristic scales are needed for the asymptotic description of adverse pressure gradient turbulent boundary layers (see Durbin and Belcher(1992)).

A major difficulty of all previous theories is to establish a single scaling procedure which can naturally accommodate the far upstream boundary layer structure to a two-deck structure, and the far downstream structure to a three-deck structure. In other words, the theories are not capable of explaining, in asymptotic terms, how the logarithmic layer vanishes as separation is approached, and how the $y^{1/2}$-layer is formed. In fact, some theories (Mellor(1966), Gersten(1987), Melnik(1989)) present expressions for the intermediate layer that upon appropriate limit passages reduce to the log-law upstream and to the $y^{1/2}$-law downstream. These expressions, however, are developed in terms of inappropriate scaling parameters or conceptual frameworks that cannot explain how the logarithmic portion completely vanishes as $u_\tau \longrightarrow 0$.

In a previous work, Cruz and Silva Freire(1998) introduced a new scaling procedure which was not subject to the aforementioned prejudices. The new scaling parameter was defined through an algebraic equation, resulting in a changeable asymptotic structure for the boundary layer, different from those of other authors, but consistent with the experimental data. The major theoretical basis for the results was provided by the single limit process concept of Kaplun(1967), together with his Ansatz about domains of validity. Thus, some formal properties of the motion equations, yielded by the definition of "equivalent in the limit" of Kaplun, were used to determine the actual validity domains and overlap regions of the flow. The theory, in particular, led to a new expression for the velocity law of the wall and to a skin-friction equation that were supposed to hold up to the separation point and in the reverse flow region. Also, new expressions were proposed for the temperature law of the wall and for the Stanton number equation. All theoretical results were validated with the data of Vogel(1984).

In a subsequent work, Cruz and Silva Freire(2002) revisited the previous theory to show how some results could be improved so as to furnish more reliable expressions for the velocity and temperature laws of the wall. In relation to the original paper, the following modifications were introduced: 1) a new formulation for the reference velocity, 2) a new expression for the velocity law of the wall, and 3) a new expression for the temperature law of the wall. The new reference velocity was specified through the total shear stress, as opposed to the previous one which had to be evaluated from an algebraic transcendental equation. The single expression advanced for the velocity law of the wall replaced the three expressions (Eqs. 25, 26 and 27) of Cruz and Silva Freire(1998); this expression is supposed to hold in the whole fluid region. The temperature law of the wall was written with the help of reasonably sophisticated expressions for its angular and linear coefficients; these are a function of the turbulent Prandlt number, the pressure gradient, the reference velocity and the shear stress at the wall. The main result of all these modifications was that much better results were found for the prediction of Stanton number near the separation point.

The purpose of this work is to present the theories formerly introduced in Cruz and Silva Freire(1998, 2002) in an unified framework. Thus the work will be structured in two parts. In part one, the asymptotic structure of separating flows will be reviewed. In part two, new formulations for the law of the wall will be reviewed as well; this new formulation includes an alternative expression for the description of the near wall characteristic length which holds also in the reverse flow region. The temperature field near de wall is described by a single expression that reduces to the logarithmic profile in the attached region, and assumes a minus half power law profile at the separation point. In the separated region the logarithmic profile is recovered.

Nomenclature

A	Parameter in thermal law of the wall.
AJ	Linear coefficient of thermal law of the wall.
C_f	Skin friction coefficient.
c_p	Specific heat at constant pressure.
C_μ	Constant in κ-ϵ model (=0.09).
D	Formal limit domain.
E	Parameter in law of the wall (=9.8).
E	Denotes partial differential equation.
F, G	Generic functions defined in Ξ by a system of differential equations
H	Step height in Vogel's experiments.
L	Characteristic length.
P_r	Prandtl number.
P_{rt}	Turbulent Prandtl number.
P	Pressure.
Q	Heat flux.
R	Reynolds number.
S	Solutions of E.
S_t	Stanton number.
t, T	Temperature.
u, v	velocity components.
u_τ	Friction velocity.
u_R	Reference velocity.
x, y	Flow cartesian coordinates.

Greek symbols

α	Overlap index.
δ	Gauge function.
Δ, η	Functions defined in Ξ.
ϵ	small parameter.
ϵ	Kinetic energy dissipation.
$\varkappa$	von Karman's constant (=0.4).
κ	Turbulent kinetic energy.
θ	Error function.
μ	Viscosity.
ν	Kinematic viscosity.
ρ	Density.
τ	Shear stress.
Ξ	Space of all positive continuous functions on $(0, 1]$.

Subscripts

$1, 2$	Order of magnitude.
d	Singularity point.
i, j	Summation.
l	Local conditions.
p	First grid point.
R	Reference parameter.
t	Temperature.
w	Conditions at wall.
Δ	variable stretched according to $\Delta(\epsilon)$.
η	variable stretched according to $\eta(\epsilon)$.
∞	External flow conditions.

2. Kaplun limits

In this section, a survey of some essential ideas used in solving singular perturbation problems is made. Some of the concepts to be discussed here are those of matching of asymptotic expansions, domain of validity of such expansions, overlap, formal validity of equations and limit processes.

The purpose of perturbation methods is to try to construct approximating solutions by the study of simplified equations. For the class of problems termed singular perturbation problems, at least two expansions are needed to construct a solution which is uniformly valid in the whole interval domain. It is thus necessary to define the concept of uniform domain of validity for such approximations. This can be achieved by a direct extension of the concepts of uniform convergence on an interval and of uniform validity on an interval, to the concepts of uniform convergence on a function class and of uniform validity on a function class. The concepts of domain of validity, of overlap, of limit processes and of matching then follow immediately.

Matching is, by its nature, a comparison of two approximations in their domain of overlap. On this ground, rules and recipes can be devised for matching in which the concept of overlap does not appear explicitly. The well known technique of interchanging limit expansions can be shown to be appropriate for certain simple cases. In simpler cases, even more precise rules can be enunciated such as the matching principle of Van Dyke(1972). This leads us to the central problem in perturbation theory: How can one justify *a priori* that two approximations have an overlap domain?

Trying to overcome this difficulty, Kaplun(1967) suggested to consider some formal properties of equations, characterizing them through their domains of validity. This would be not only more basic for understanding the matching process but also essential in the construction of the asymptotic expansions. Since all techniques used for matching are based on overlap, it is clear that this can only be achieved if two approximations have a common validity domain. The formal properties of an approximation are defined through a study of limits of the original equation. Then, the concepts of formal domain, of intermediate equation, of principal equation and of formal domain of validity can be introduced. The operational details of the mathematical procedure are laid by five definitions, one Axiom and one Ansatz. These are shown below.

The formulation to be presented here is only introductory to the ample set of results presented in Kaplun(1967) and in Lagerstrom and Casten(1972). For more details on the technique, the reader is referred to these two works. Complementary material is found in Meyer(1967), in Freund(1972) and in Silva Freire and Hirata(1990).

Here we use the topology on the collection of order classes as introduced by Meyer(1967).

Let ϵ be a parameter on $(0, 1]$ and x a variable in R^n with Euclidean norm $|x|$. Let F be a function defined for ϵ and on some x-space domain with pointwise norm $\|F\|$. Our interest is to study the behaviour of F in the limit $\epsilon \to 0$. In particular, we are interested in the cases where singularities arise. For example, passage of the limit may result in the loss of the highest order derivative term in a differential equation, and hence in the impossibility of satisfying all the boundary conditions. The idea of the Kaplun limit is to study the limit as $\epsilon \to 0$ not for fixed x near a singularity point x_d, but for x tending to x_d in a definite relationship to ϵ specified by a stretching function $\eta(\epsilon)$.

Taking $x_d = 0$, we define

$$x_\eta = \frac{x}{\eta(\epsilon)}, \qquad G(x_\eta;\epsilon) = F(x;\epsilon) \tag{1}$$

with $\eta(\epsilon)$ a function defined in Ξ (= space of all positive continuous functions on $(0,1]$).

The Kaplun limit process is then defined as follows.

Definition 1 (Meyer(1967)). If the function

$$G(x_\eta;+0) = \lim_{\epsilon\to 0} G(x_\eta;\epsilon), \tag{2}$$

exists uniformly on $\{x/|x_\eta| > 0\}$; then we define $lim_\eta F(x;\epsilon) = G(x_\eta, +0)$.

If F is a function defined by a system of differential equations, then the above definition establishes to every order of η a correspondence original equation $\overset{lim_\eta}{\longrightarrow}$ associated equation on that subset of Ξ for which the associated equation exists. The passage of the η-limit process is a formal operation which results in a set of associated equations referred to by Kaplun(1967) as the "splitting" of the original differential equation; this operation establishes the basis for the definition of formal domain of validity.

Definition 2. The formal limit domain of an associated equation E is the set of orders ϵ such that the η-limit process applied to the original equation yields E.

To evaluate how close two equations are, Kaplun needed to advance a measuring procedure. This was made through the definition of equivalent in the limit.

Definition 3. Two equations E_1 and E_2 are said to be equivalent in the limit for a given limit process, lim_η, and to a given order $\delta(\epsilon)$, if,

$$\theta = \frac{E_1(x_\eta;\epsilon) - E_2(x_\eta;\epsilon)}{\delta(\epsilon)} \to 0, \quad \text{as} \quad \epsilon \to 0. \tag{3}$$

Definition 4 (of formal domain of validity). The formal domain of validity to order $\delta(\epsilon)$ of an equation E of formal limit domain D is the set $D_e = D \cup D_i's$, where $D_i's$ are the formal limit domains of all equations E_i such that E and E_i are equivalent to D_i to order $\delta(\epsilon)$.

To relate the formal domain of validity of an equation to its actual domain of validity, Kaplun(1967) advanced two assertions, the Axiom of Existence and the Ansatz about domains of validity. These assertions are primitive and unverifiable assumptions of perturbation theory. They allow one to use definitions 1 to 4 to find approximate solutions to singular perturbation problems. Because the heuristic nature of the Axiom and of the Ansatz, comparison to experiments will always be important for validation purposes. The theory, however, as implemented through the above operations, is always very helpful in understanding the matching process and in constructing the appropriate asymptotic expansions.

Axiom (of existence) (Kaplun(1967)). If equations E_1 and E_2 are equivalent in the limit to order $\delta(\epsilon)$ for a certain region, then given a solution S_1 of E_1 which lies in the region of equivalence of E_1 and E_2, there exists a solution S_2 of E_2 such that as $\epsilon \to 0$, $\mid S_1 - S_2 \mid /\delta \to 0$, in the region of equivalence of E_1 and E_2.

To this axiom, there corresponds an Ansatz to ensure that there exists a solution S_1 of E_1 which lies in the region of equivalence of E_1 and E_2.

Ansatz (about domains of validity) (Kaplun(1967)). An equation with a given formal domain of validity D has a solution whose actual domain of validity corresponds to D.

The word "corresponds to" in the Ansatz was assumed by Kaplun to actually mean "is equal to". The above formulation ceases to be valid when small terms have large integrated effects. In the example to be studied here, however, the principle is expected to work. Switchback terms, which are deduced from inspection of formally higher order terms, can always be included in the original formulation if we backtrack to the lower order terms. Large integrated effects occur when singularities occur in the approximating functions; these are not expected to occur here.

3. The equations of motion

We consider the problem of an incompressible two-dimensional turbulent flow over a smooth surface in a prescribed pressure distribution. The time-averaged motion equations; i.e., the continuity equation, the Navier-Stokes equation and the energy equation, can be cast as

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{4}$$

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_j} - \epsilon^2 \frac{\partial}{\partial x_j}\left(\overline{u_j' u_i'}\right) + \frac{1}{R}\frac{\partial^2 u_i}{\partial x_j^2}, \tag{5}$$

$$u_j \frac{\partial t}{\partial x_j} = -\epsilon^2 \frac{\partial}{\partial x_j}\left(\overline{u_j' t'}\right) + \frac{1}{P_r R}\frac{\partial^2 t}{\partial x_j^2}, \tag{6}$$

where the notation is classical. Thus $(x_1, x_2) = (x, y)$ stand for the coordinates, $(u_1, u_2) = (u, v)$ for the velocities, p for pressure, t for temperature and R and P_r for the Reynolds and the Prandtl numbers respectively. The dashes are used to indicate a fluctuating quantity. In the fluctuation terms an overbar is used to indicate a time-average.

All mean variables are referred to some characteristic quantity of the external flow. The velocity fluctuations, on the other hand, are referred to a characteristic velocity u_R, firstly introduced in Cruz and Silva Freire(1998, 2002). This characteristic velocity, fundamental for the determination of the inner layers solution is known to reduce upstream and downstream of a separation point respectively to u_τ and to $(\nu(dp/dx)/\rho)^{1/3}$. It must then be defined so as to comply with this behaviour. In the next section we will show how this can be done.

The small parameter ϵ is defined as

$$\epsilon = \frac{u_R}{U_\infty}. \tag{7}$$

The temperature fluctuation is considered to be of the order of the friction temperature t_τ, here defined as

$$t_\tau = \frac{q_w}{\rho c_p u_R}. \tag{8}$$

Note that the classical definition of t_τ based on the friction velocity breaks down near a separation point. Now, based on the adverse pressure gradient results of Orlando et alli(1974), we make

$$ord\,(u_i') = ord\,(t'). \tag{9}$$

The above assumptions concerning the relative order of magnitude of the various fluctuation terms, which are crucial for our future developments, have become a well established result in the asymptotic theories for turbulent boundary layer flow. For incompressible flow, the basic experimental support for them stems from the works of the Stanford Heat and Mass Transfer Group. The results of Kistler(1959) and of Kistler and Chen(1963) provide experimental support for compressible flows.

4. The velocity and the temperature asymptotic structures

We write the asymptotic expansions for the flow parameters as

$$u(x,y) = u_1(x,y) + \epsilon u_2(x,y), \tag{10}$$

$$v(x,y) = \frac{\eta}{\Delta}[v_1(x,y) + \epsilon v_2(x,y)], \tag{11}$$

$$p(x,y) = p_1(x,y) + \epsilon p_2(x,y), \tag{12}$$

$$t(x,y) = t_1(x,y) + \epsilon t_2(x,y), \tag{13}$$

$$u_i'(x,y) = \epsilon u_{i1}'(x,y) + \epsilon^2 u_{i2}'(x,y). \tag{14}$$

To find the asymptotic structure of the boundary layer we consider the following stretching transformation

$$x_\Delta = \frac{x}{\Delta(\epsilon)}, \quad y_\eta = \frac{y}{\eta(\epsilon)}, \quad \hat{u}_i(x_\Delta, y_\eta) = u_i(x,y). \tag{15}$$

with $\Delta(\epsilon)$ and $\eta(\epsilon)$ defined on Ξ.

Upon substitution of Eq.(15) into Eq.(5) and upon passage of the η-limit process onto the resulting equation we get (Cruz and Silva Freire(1998)):

x-momentum equation:

$$ord\ \eta = ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial x_\Delta} = 0, \tag{16}$$

$$ord\ \epsilon^2 < ord\ \eta < ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial x_\Delta} = 0, \tag{17}$$

$$ord\ \epsilon^2 = ord\ \eta : \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial x_\Delta} = -\frac{\partial \overline{\hat{u}_1' \hat{v}_1'}}{\partial y_\eta}, \tag{18}$$

$$ord\ (1/\epsilon R) < ord\ \eta < ord\ \epsilon^2 : \frac{\partial \overline{\hat{u}_1' \hat{v}_1'}}{\partial y_\eta} = 0, \tag{19}$$

$$ord\ (1/\epsilon R) = ord\ \eta : -\frac{\partial \overline{\hat{u}_1' \hat{v}_1'}}{\partial y_\eta} + \frac{\partial^2 \hat{u}_2}{\partial y_\eta^2} = 0, \tag{20}$$

$$ord\ \eta < ord\ (1/\epsilon R) : \frac{\partial^2 \hat{u}_2}{\partial y_\eta^2} = 0, \tag{21}$$

y-momentum equation:

$$ord\ \eta = ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{v}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{v}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial y_\Delta} = 0, \tag{22}$$

$$ord\ \eta < ord\ 1: \frac{\partial \hat{p}_1}{\partial y_\eta} = 0. \tag{23}$$

The term $\hat{u}_1(x_\Delta, y_\eta)$ is missing from equations (20) and (21) since from the no-slip condition $\hat{u}_1 = 0$ near the wall.

Equations (18) and (20) are distinguished in two ways: i) they are determined by specific choices of η, and ii) they are "richer" than the others in the sense that, application of the limit process to them yields some of the other equations, but neither of them can be obtained from passage of the limit process to any of the other equations. Thus, in the language of Kaplun, these equations are called principal equations. Principal equations are important since they are expected to be satisfied by the corresponding limits of the exact solution. We then make the following definition.

Definition 5 (of principal equation). An equation E of formal limit domain D, is said to be principal to order δ if:

i) one can find another equation E', of formal limit domain D', such that E and E' are equivalent in D' to order δ;

ii) E is not equivalent to order δ to any other equation in D.

A complete solution to the problem should then according to the Axiom of Existence and Kaplun's Ansatz, be obtained from the principal x-momentum and y-momentum equations located at points $ord\ \eta = 1$, $ord\ \eta = ord\ \epsilon^2$ and $ord\ \eta = ord\ (1/\epsilon R)$. The formal domains of validity of these equations cover the entire domain and overlap in a region determined according to definition 4.

To find the overlap region of equations (18) and (20), we must show these equations to have a common domain where they are equivalent; thus, we must use Kaplun's concept of equivalent in the limit, definition 4. A direct application of this definition to equations (18) and (20) yields

$$\theta = \frac{\frac{\hat{u}_1}{\Delta} \frac{\partial \hat{u}_1}{\partial x_\Delta} + \frac{\hat{v}_1}{\eta} \frac{\partial \hat{u}_1}{\partial y_\eta} + \frac{1}{\Delta} \frac{d \hat{p}_1}{d x_\Delta} - \frac{1}{R\eta^2} \frac{\partial^2 \hat{u}_1}{\partial y_\eta^2}}{\epsilon^\alpha}. \tag{24}$$

Noting that the leading order term in region $ord\ (1/\epsilon R) < ord\ \eta < ord\ \epsilon^2$ is the turbulent term, of $ord\ (\epsilon^2/\eta)$, we normalize the above equation to order unity to find

$$\bar{\theta} = \frac{\eta}{\epsilon^2}\theta. \tag{25}$$

The overlap domain is the set of orders such that the η-limit process applied to $\bar{\theta}$ tends to zero for a given α. Then since $ord\ (\partial/\partial y) = \epsilon$ and $ord\ (\partial/\partial x) = 1$, the formal overlap domain is given by

$$D = \{\eta/ \quad ord\ (\epsilon^{1+\alpha} R)^{-1} < ord\ \eta < ord\ (\epsilon^{2+\alpha}\Delta)\}. \tag{26}$$

According to Kaplun's Ansatz about domains of validity, the approximate equations, Eqs. (18) and (20), only overlap if set 26 is a non-empty set, that is, if

$$0 \leq \alpha \leq -\frac{1}{2}\left(\frac{ln\, R\Delta}{ln\, \epsilon} + 3\right). \tag{27}$$

The implication is that the two-deck turbulent boundary layer structure given by the two principal equations, equations (18) and (20), provides approximate solutions which are accurate to the order of $\epsilon^{\alpha_{max}}$, where α_{max} is the least upper bound of the interval (27). This fundamental result can only be reached through the application of Kaplun's concepts and ideas to the problem.

We conclude that the turbulent boundary layer has a two-deck structure very much like the one derived by Sychev and Sychev. This structure, however, must change as a separation point is approached. We shall see this next.

Equation (21) together with the boundary condition

$$\mu\frac{\partial u_2}{\partial y} = \tau_w. \tag{28}$$

shows that far away from the separation point $u_R = u_\tau$. Close to the separation point in the limit $\Delta \longrightarrow 0$, however, $u_\tau \longrightarrow 0$, so that an alternative value must be sought for u_R. The characteristic velocity u_R will be determined here through some order of magnitude considerations.

At the bottom of the overlap region, a balance between the turbulent and viscous stresses occurs so that we may write

$$\frac{\partial}{\partial y}(\overline{-\rho u_1' v_1'}) + \mu\frac{\partial^2 u_2}{\partial y^2} = \frac{\partial p_1}{\partial x}. \tag{29}$$

In this region, the characteristic length is given by ν/u_R. Then, considering that the turbulent fluctuations are of the order of the reference velocity, u_R, and that the viscous term can be approximated by

$$ord(\ \mu\frac{\partial u_2}{\partial y}) = ord(\tau_w), \tag{30}$$

it results from simple order of magnitude arguments that the characteristic velocity can be estimated from the algebraic equation

$$u_R^3 - \frac{\tau_w}{\rho}u_R - \frac{\nu}{\rho}\frac{\partial p}{\partial x} = 0. \tag{31}$$

In the limit $\Delta \longrightarrow 0$,

$$u_R \longrightarrow \left(\frac{\nu}{\rho}\frac{\partial p}{\partial x}\right)^{1/3}, \tag{32}$$

recovering the characteristic velocity for the near separation point region proposed by Stratford(1959) and by Townsend(1976).

The characteristic velocity u_R is determined by the highest real root of (31).

The implication is that, close to the separation point, $ord\ (\epsilon^2) = ord\ (1/\epsilon R)$, and the two "rich" equations merge giving origin to a new structure. This merging provokes the disappearance of the log-region, reducing the flow structure to a wake region and a viscous region.

The flow structure then becomes:

x-momentum equation:

$$ord\ \Delta = ord\ 1: \quad \hat{u}_2 \frac{\partial \hat{u}_2}{\partial x_\Delta} + \hat{v}_2 \frac{\partial \hat{u}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = 0, \tag{33}$$

$$ord\ \epsilon^2 < ord\ \Delta < ord\ 1: \quad \hat{u}_2 \frac{\partial \hat{u}_2}{\partial x_\Delta} + \hat{v}_2 \frac{\partial \hat{u}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = 0, \tag{34}$$

$$ord\ \epsilon^2 = ord\ \Delta: \hat{u}_2 \frac{\partial \hat{u}_2}{\partial x_\Delta} + \hat{v}_2 \frac{\partial \hat{u}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = -\frac{\partial \overline{\hat{u}_1'^2}}{\partial x_\Delta} - \frac{\partial \overline{\hat{u}_1' \hat{v}_1'}}{\partial y_\eta} + \frac{\partial^2 \hat{u}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{u}_2}{\partial y_\eta^2}, \tag{35}$$

$$ord\ \Delta < ord\ \epsilon^2: \quad \frac{\partial^2 \hat{u}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{u}_2}{\partial y_\eta^2} = 0; \tag{36}$$

y-momentum equation:

$$ord\ \Delta = ord\ 1: \quad \hat{u}_2 \frac{\partial \hat{v}_2}{\partial x_\Delta} + \hat{v}_2 \frac{\partial \hat{v}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial y_\eta} = 0, \tag{37}$$

$$ord\ \epsilon^2 < ord\ 1 < ord\ \Delta: \quad \hat{u}_2 \frac{\partial \hat{v}_2}{\partial x_\Delta} + \hat{v}_2 \frac{\partial \hat{v}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial y_\eta} = 0, \tag{38}$$

$$ord\ \epsilon^2 = ord\ \Delta: \hat{u}_2 \frac{\partial \hat{v}_2}{\partial x_\Delta} + \hat{v}_2 \frac{\partial \hat{v}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = -\frac{\partial \overline{\hat{v}_1'^2}}{\partial x_\Delta} - \frac{\partial \overline{\hat{u}_1' \hat{v}_1'}}{\partial y_\eta} + \frac{\partial^2 \hat{v}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{v}_2}{\partial y_\eta^2}, \tag{39}$$

$$ord\ \Delta < ord\ \epsilon^2: \quad \frac{\partial^2 \hat{v}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{v}_2}{\partial y_\eta^2} = 0. \tag{40}$$

At this point it is of interest to note that in the region $(\Delta, \eta) = (\epsilon^2, \epsilon^2)$ the full Navier-Stokes averaged equation is recovered. The leading order equations for $\hat{u}_1$ together with the no-slip condition at the wall gives $\hat{u}_1 = 0$.

According to the above results, a global solution for the problem can only be obtained through equations (35) and (39). Unfortunately, these non-linear equations are of difficult solution, having the turbulent term yet to be defined.

To study the temperature boundary layer asymptotic structure we will use the same procedure as before. Application of the stretching transformation defined by equations (15) to the energy equation followed by passage of the η-limit process, yields:

$$ord\ \eta = ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{t}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{t}_1}{\partial y_\eta} = 0, \tag{41}$$

$$ord\ \epsilon^2 < ord\ \eta < ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{t}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{t}_1}{\partial y_\eta} = 0, \tag{42}$$

$$ord\ \eta = ord\ \epsilon^2: \quad \hat{u}_1 \frac{\partial \hat{t}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{t}_1}{\partial y_\eta} = -\frac{\partial \overline{\hat{v}_1' \hat{t}_1'}}{\partial y_\eta}, \tag{43}$$

$$ord\ (1/\epsilon R P_r) < ord\ \eta < ord\ \epsilon^2: \quad \frac{\partial \overline{\hat{v}_1' \hat{t}_1'}}{\partial y_\eta} = 0, \tag{44}$$

$$ord\ \eta = ord\ (1/\epsilon RP_r): \quad -\frac{\partial \overline{\hat{v}_1'\hat{t}_1'}}{\partial y_\eta} - \frac{\partial^2 \hat{t}_2}{\partial y_\eta^2} = 0, \tag{45}$$

$$ord\ \eta < ord\ (1/\epsilon RP_r): \quad \frac{\partial^2 \hat{t}_2}{\partial y_\eta^2} = 0. \tag{46}$$

The above equations imply that the temperature turbulent boundary layer has a two-layered structure, the two "rich" equations being located at points $ord\ \eta = ord\ \epsilon^2$ and $ord\ \eta = ord\ (1/\epsilon RP_r)$ of the Ξ space. The consequence is that the outer "rich" equations for both the velocity and the temperature boundary layers are always located at the same point of Ξ. The location of the inner "rich" equations, however, differs by a scale factor, the Prandtl number. Thus if $P_r = 1$, the asymptotic structures of both the velocity and the temperature boundary layers will be identical, and logarithmic solutions will arise.

The overlap domain of equations (43) and (45) can be calculated just in the same way as the velocity field overlap domain was calculated (set 26). For this reason, the procedure will not be repeated here. We just point out to the reader that for the temperature case the overlap domain will depend on the Prandtl number

Near to a separation point, however, the temperature boundary layer structure must change, much in the same way as the velocity boundary layer structure changes. To study this change we pass the Δ-limit process onto equations (41) to (46) to obtain:

$$ord\ \epsilon_\Delta = ord\ 1: \quad \hat{u}_2\frac{\partial \hat{t}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{t}_2}{\partial y_\eta} = 0, \tag{47}$$

$$ord\ \epsilon^2 < ord\ \Delta < ord\ 1: \quad \hat{u}_2\frac{\partial \hat{t}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{t}_2}{\partial y_\eta} = 0 \tag{48}$$

$$ord\ \epsilon^2 < ord\ \Delta < ord\ 1: \quad \hat{u}_2\frac{\partial \hat{t}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{t}_2}{\partial y_\eta} = -\frac{\partial \overline{\hat{v}_1'\hat{t}_1'}}{\partial y_\eta} - \frac{\partial \overline{\hat{u}_1'\hat{t}_1'}}{\partial x_\Delta}, \tag{49}$$

$$ord\ (1/\epsilon RP_r) < ord\ \Delta < ord\ \epsilon^2: \quad \frac{\partial \overline{\hat{v}_1'\hat{t}_1'}}{\partial y_\eta} + \frac{\partial \overline{\hat{u}_1'\hat{t}_1'}}{\partial x_\Delta} = 0, \tag{50}$$

$$ord\ (1/\epsilon RP_r) = ord\ \Delta: \quad \frac{\partial^2 \hat{t}_2}{\partial y_\eta^2} + \frac{\partial^2 \hat{t}_2}{\partial x_\Delta^2} - \frac{\partial \overline{\hat{v}_1'\hat{t}_1'}}{\partial y_\eta} - \frac{\partial \overline{\hat{u}_1'\hat{t}_1'}}{\partial x_\Delta} = 0, \tag{51}$$

$$ord\ \Delta < ord\ (1/\epsilon R): \quad \frac{\partial^2 \hat{t}_2}{\partial y_\eta^2} + \frac{\partial^2 \hat{t}_2}{\partial x_\Delta^2} = 0. \tag{52}$$

The temperature leading order equation together with the boundary condition gives the solution $t_1 = T_w$, where T_w stands for wall temperature. According to the above equations, the temperature boundary layer will have three different asymptotic structures depending on the order of magnitude of the Prandtl number. If the Prandtl number is order unity, $ord\ (\epsilon^2) = ord\ (1/\epsilon RP_r)$ near a separation point and the two "rich" equations merge yielding a structure identical to the velocity boundary layer structure. If $ord\ (P_r) > ord\ (1)$, the two "rich" equations will remain defined in different points of the Ξ space and the temperature two-layered structure will be preserved. In the third possibility, $ord\ (P_r) < ord\ (1)$, the merging of the two temperature "rich" equations will occur prior to the merging of the velocity "rich" equations. This "premature" merging occurs at point $(\eta, \Delta) = ((RP_r)^{-2/3}, (RP_r)^{-2/3})$. The flow structure for both cases $P_r < 1$ and $P_r > 1$ are shown in Figure 1.

These figures confirm all results that were expected to occur beforehand. For $P_r < 1$, the disappearance of the temperature log-region takes place before the disappearance of the velocity log-region. The converse is true for $P_r > 1$. The case $P_r < 1$ should then be more amenable to analytical treatment.

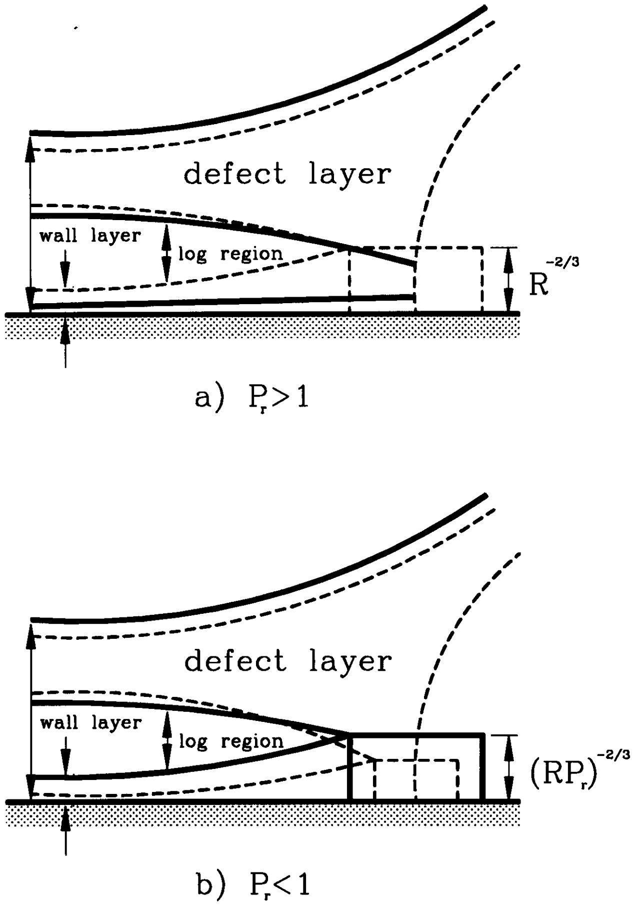

Figure 1. Flow asymptotic structure

5. Velocity law of the wall

Since a second concern of this work is to provide the means for a good numerical simulation of the flow near a separation point, we will show next the implications of the above findings on the specification of near wall analytical expressions for the velocity and the temperature fields.

Following the procedure of Cruz and Silva Freire(2002), the law of the wall for a separating flow can be written as

$$u = \frac{\tau_w}{|\tau_w|}\frac{2}{\varkappa}\sqrt{\frac{\tau_w}{\rho} + \frac{1}{\rho}\frac{dP_w}{dx}y} + \frac{\tau_w}{|\tau_w|}\frac{u_\tau}{\varkappa}\ln\left(\frac{y}{L_c}\right), \tag{53}$$

where

$$L_c = \frac{\sqrt{\left(\frac{\tau_w}{\rho}\right)^2 + 2\frac{\nu}{\rho}\frac{dP_w}{dx}u_R} - \frac{\tau_w}{\rho}}{\frac{1}{\rho}\frac{dP_w}{dx}}, \tag{54}$$

and all symbols have their classical meaning; $\varkappa$ is the von Kármán constant (=0.4), u_τ is the friction velocity, and u_R ($= \sqrt{\tau/\rho}$, τ = total shear stress) is a reference velocity (which will be fully defined in the following through Eqs. (58) and (59)).

Equation (53) is a generalisation of the classical law of the wall and replaces the three expressions advanced in Cruz and Silva Freire(1998), Eqs. (25, 26, 27). Equation (54) is an expression for the near wall region characteristic length, which is assumed to be valid in the attached and in the reverse flow regions. Far away from the separation point, where the shear stress is positive and $y(dP_w/dx) << \tau_w$, Eq. (53) reduces to

$$u = \frac{2}{\varkappa}u_\tau + \frac{u_\tau}{\varkappa}\ln\left(\frac{y}{L_c}\right), \quad L_c = \nu/u_\tau, \tag{55}$$

that is, to the classical law of the wall.

Close to the separation point where τ_w=0, equation (53)reduces to

$$u = \frac{2}{\varkappa}\sqrt{\frac{y}{\rho}\frac{dP_w}{dx}}, \tag{56}$$

an equation similar to Stratford's equation (see Stratford(1959)).

In the reverse flow region where $y(dP_w/dx) >> \tau_w$, equation (53) can be written as

$$u = -\frac{2}{\varkappa}u_\tau - \frac{u_\tau}{\varkappa}\ln(\frac{y}{L_c}), \quad L_c = 2\left|\frac{\tau_w}{dP_w/dx}\right|. \tag{57}$$

Some comments seem now in order. The form of Eq. (53) was entirely inspired by Eqs. (25), (26), (27) and (36) of Cruz and Silva Freire(1998). In fact, the equations remain the same but for a major simplification achieved by changing the arguments of the logarithmic terms by y/L_c. The reference length L_c defined by equation (54) is not new; in Cruz and Silva Freire(1998) it had been previously introduced through Eq.(36). The generalization provided by Eq. (53), however, implied that the friction velocity, u_τ, used in the definition of L_c had to be replaced by the reference velocity u_R. Finally, note that the characteristic length in the reverse flow region is different from the classical characteristic length given by Eq. (55). Equation (57) is in agreement with Simpson et al.(1981) which suggested that a characteristic length for the backflow region should be directly proportional to the absolute value of the wall shear stress.

We will now describe how the wall shear stress can be evaluated from the above equations and through the use of a κ-ϵ model.

A clear difficulty with the implementation of Eq. (53) as a boundary condition in a numerical code is that the wall shear stress cannot be obtained in an explicit form. The numerical solution of Eq. (53) for the wall shear stress is not a stable process which can therefore affect code robustness. To avoid these problems a linearisation procedure for Eq. (53) was developed.

The total shear stress can be evaluated from

$$\tau_p = C_\mu^{1/2}\rho\kappa_p + \mu\left|\frac{\partial u}{\partial y}\right|_p \tag{58}$$

where the subscript p denotes the first grid point.

The reference velocity u_R can then be directly determined from

$$u_R = \sqrt{\frac{\tau_p}{\rho}}. \tag{59}$$

Please note that the above procedure replaces Eq. 31. Equation (58) was obtained from a momentum balance in the near wall region; it is similar to a relation usually employed by other authors to relate the wall shear stress to the turbulent kinetic energy in a κ-ϵ formulation (see, e.g., Launder and Spalding (1974)), the only difference here is the inclusion of the viscous term to improve calculations when the first node near the wall is located at a distance shorter than $y/L_c \leq 30$.

This equation can be used as a first estimate for the wall shear stress if we consider

$$\tau_{wo} = \frac{u_p C_\nu^{1/4} {\tau_p}^{1/2} \rho^{1/2} \varkappa}{\ln\left(Ey\frac{(\tau_p/\rho)^{1/2}}{\nu}\right)}. \tag{60}$$

with $E = 9.8$.

In order to maintain code stability, the pressure gradient at the wall was obtained using Eqs. (58) and (60) to furnish the following equation.

$$\frac{dP_w}{dx} = \frac{\tau_p - \tau_{wo}}{y_p}. \tag{61}$$

This equation was obtained directly from the inner layer approximated equations; it represents the balance of forces in that layer.

Next, the characteristic length can be calculated from

$$L_c = \frac{\sqrt{\left(\frac{\tau_{wo}}{\rho}\right)^2 + 2\frac{\nu}{\rho}\frac{dP_w}{dx}u_R} - \frac{\tau_{wo}}{\rho}}{\frac{1}{\rho}\frac{dP_w}{dx}}. \tag{62}$$

Table 1: The experimental data of Vogel(1984).

$Author$	$U[m/s]$	R	$Q_w[W/m^2]$
Vogel(1984)	11.3	28000	270

Finally, the wall shear stress is calculated from

$$\tau_w = \frac{u_p {\tau_p}^{1/2} \rho^{1/2} \varkappa}{2\sqrt{\left|\frac{\tau_p}{\tau_{wo}}\right| + \ln(\frac{y_p}{L_c})}}. \tag{63}$$

Using some production-dissipation equilibrium assumptions and Eq. (53) the kinetic energy dissipation and the production terms can be cast respectively as follows:

$$\epsilon = C_\mu{}^{\frac{1}{2}} \kappa_p \Big(\frac{(\tau_p/\rho)^{1/2}}{\varkappa y} + \frac{\frac{1}{\rho}\frac{dP_w}{dx}}{\varkappa (\tau_p/\rho)^{1/2}} \Big), \tag{64}$$

$$Production = \frac{C_\mu{}^{\frac{1}{2}} \kappa_p \rho}{y} \Big(\frac{2(\tau_p/\rho)^{\frac{1}{2}}}{\varkappa} + \frac{\mid (\tau_{wo}/\rho) \mid^{\frac{1}{2}}}{\varkappa} \ln(\frac{y}{L_c}) \Big). \tag{65}$$

6. Temperature law of the wall

Because the temperature profiles can be written in terms of reference parameters already determined for the velocity profiles, the three Eqs. (28), (29) and (30) of Cruz and Silva Freire(1998) can be re-written here in a simplified form. Thus, the temperature law of the law can be cast as

$$\frac{T_w - T}{Q_w} = \frac{P_{rt}}{\varkappa_t \rho c_p u_\tau} \ln \frac{\sqrt{\tau_w/\rho + \frac{1}{\rho}\frac{dP_w}{dx} y} - \sqrt{\tau_w/\rho}}{\sqrt{\tau_w/\rho + \frac{1}{\rho}\frac{dP_w}{dx} y} + \sqrt{\tau_w/\rho}} + C_q, \tag{66}$$

where

$$C_q = \frac{P_{rt}}{\varkappa_t \rho c_p u_R} \ln \frac{4 E u_R^3}{\nu \left|\frac{dP_w}{dx}\right|} + AJ, \tag{67}$$

$$AJ = 1.11 P_{rt} \sqrt{\frac{A}{\varkappa}} \big(\frac{P_r}{P_{rt}} - 1\big)\big(\frac{P_r}{P_{rt}}\big)^{0.25}, \tag{68}$$

$$A = 26 \frac{\left|\tau_w/\rho\right|^{1/2}}{u_R}, \quad P_{rt} = 0.9, \tag{69}$$

and all symbols have their classical meaning.

To improve the performance of Eq. (66) its linear coefficient was replaced by a more sophisticated equation. C_q was basically developed so that Eq. (66) reduces to the classical law of the wall far away for a separation point. Equation (68) was first proposed by Launder and Spalding(1974). Equation (69) has been modified from the original formulation ($A = 26$) in order to perform better in the separated flow region. In Cruz and Silva Freire(1998) the predicted values of S_t were well below the experimental values so that Eqs. (67) to (69) had to be introduced to rectify that.

7. Results

The results found with the present formulation will now be compared with the standard κ-ϵ model and the data of Vogel(1984) for the backward facing step flow. The flow conditions of Vogel are shown in Table 1.

The governing equations are discretized using a finite volume formulation coupled with an hybrid scheme for the treatment of the convective and diffusive terms simultaneously. The set of finite difference equations is solved using a very robust and intensively validated version of TEACH-2E (Teach Elliptic Axi-symmetrical

Characteristics Heuristically) code which incorporates the SIMPLE algorithm specific for pressure velocity handing in incompressible flow. The grid had 146x102 points. The computational domain is shown in Figure 2.

To perform the calculations the limit of the viscous region was taken as $y^+ = 5$. For flows subjected to a zero pressure gradient, one normally considers this region to be defined by $y^+ = 11$. For the present experimental conditions, however, we found $y^+ = 5$ to be the appropriate value. Please note that we have taken the data from Vogel's doctoral thesis which are different from the data presented in Vogel and Eaton(1985). The result is that the curves and experimental points to be shown here do not coincide with the corresponding ones in Cruz and Silva Freire(1998).

The next figures show the results for the computed velocity and temperature profiles, for several stations. All figures present three curves: one with the experimental data of Vogel(1984), one with the present computations, and a final one with computations made with the standard κ- ϵ model. The velocity profiles show that the present formulation only alters the results for the very near wall region. Indeed, for most of the velocity profile, both numerical appproaches give very close data. Through the velocity profiles, the position of the point of flow reattachment can also be estimated. Here, our solution starts to depart from the classical κ-ϵ solution. The results are shown in Table 2.

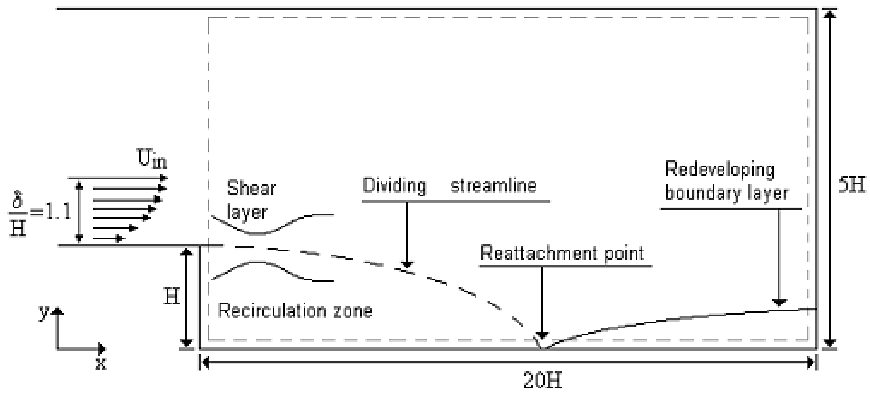

Figure 2. Flow domain.

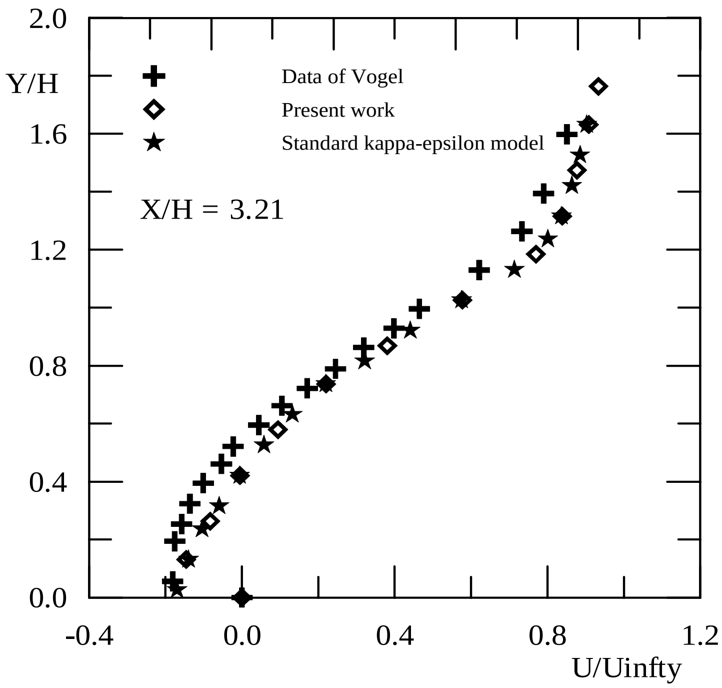

Figure 3: Velocity profiles. X/H = 3.21. +, data of Vogel(1984); $\diamond$, present work; $\star$, standard κ-ϵ model. H = 20 cm (step height); X = distance from step.

For the temperature profiles, the differences between the present approach and the results given by the classical κ-ϵ model are significant at the wall. This can be clearly seen from Figures 5 to 9; note in particular that the first grid point presents very different values for both numerical predictions. Equation (66) clearly

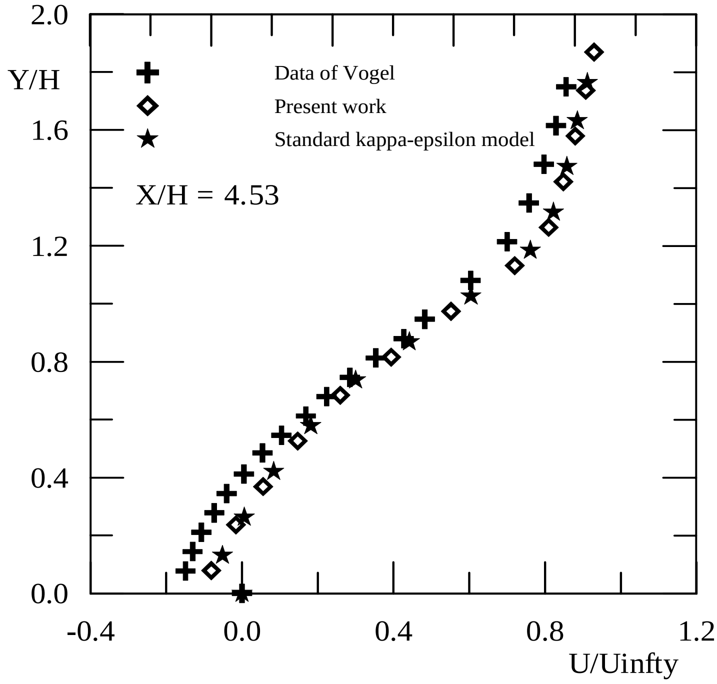

Figure 4: Velocity profiles. X/H = 4.53. +, data of Vogel(1984); ⋄, present work; ⋆, standard κ-ϵ model. H = 20 cm (step height); X = distance from step.

Table 2: Prediction of flow reattachment point.

Reattachment point	X/H
Experiments	6.6
Present Work	6.0
Standard κ-ϵ model	5.5

provides results which are much closer to the experimental data than the standard formulation and which will result in a much better prediction of the Stanton number. This fact is, indeed, directly connected with the estimation of Stanton number.

Results for the skin-friction coefficient and the Stanton number are shown next. The improvement in the predictions, thanks to the use of the new formulations for the law of the wall proposed here, is remarkable in the flow separation region. It is important to note that these results were obtained with no additional computational cost or loss of code robustness, in comparison with the original code which used the standard $\kappa - \epsilon$ model.

8. Conclusion

The present work had a very distinct goal at its beginning, to provide an alternative method for the calculation of flows subjected to separation. Specially, we wanted to improve the calculation methods for the skin-friction coefficient and the Stanton number which were developed in the past to use the law of the wall.

Apparently, the goal has been achieved with the specification of expressions (53) and (66). These expressions were shown to stand very well against the data of Vogel(1984), giving very good results for the velocity and temperature fields and the skin-friction coefficient and the Stanton number. We have chosen the data of Vogel(1984) as our refence data for they represent the most detailed account of the problem we have so far encountered. Through his thesis we had access to a complete set of tabulated data which could be used in detail for validation of the present approach.

Of course, a limitation of the present simulation is its inability no capture any unsteadiness occuring in the flow. As such the location of the re-attachment point is fixed, so is the location of the point of zero C_f and maximum S_t.

Presently, the authors are subjecting those expressions to further scrutiny. This will be reported in another occasion.

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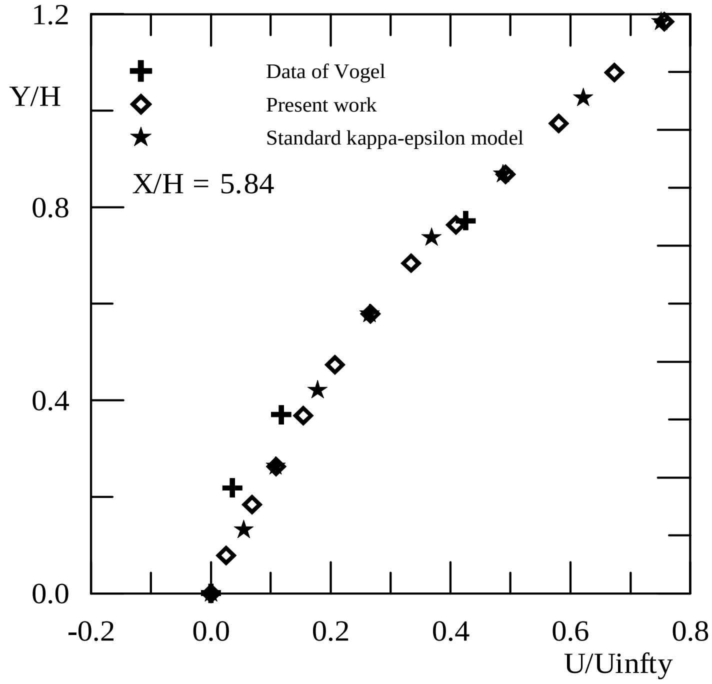

Figure 5: Velocity profiles. X/H = 5.84. +, data of Vogel(1984); ⋄, present work; ⋆, standard κ-ϵ model. H = 20 cm (step height); X = distance from step.

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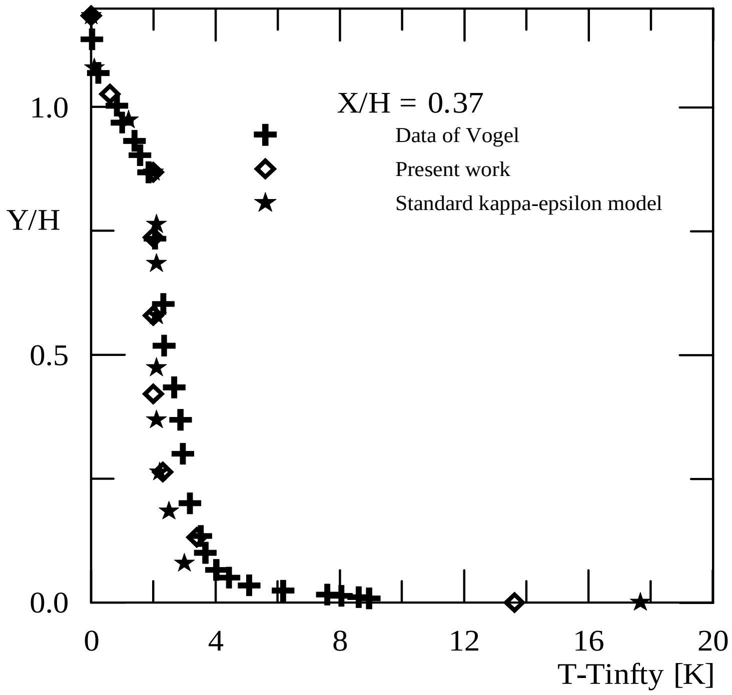

Figure 6: Temperature profiles. X/H = 0.37. +, data of Vogel(1984); $\diamond$, present work; $\star$, standard κ-ϵ model.

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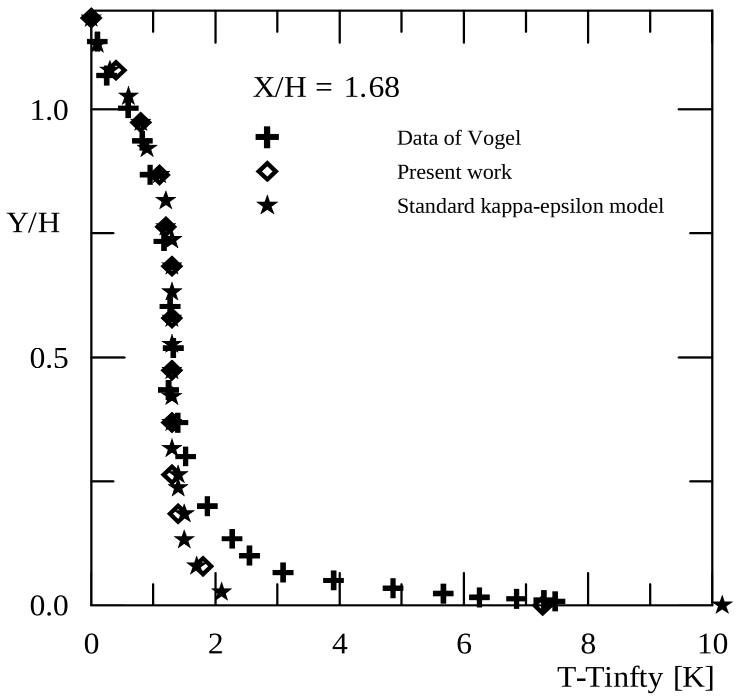

Figure 7: Temperature profiles. X/H = 1.68. +, data of Vogel(1984); $\diamond$, present work; $\star$, standard κ-ϵ model.

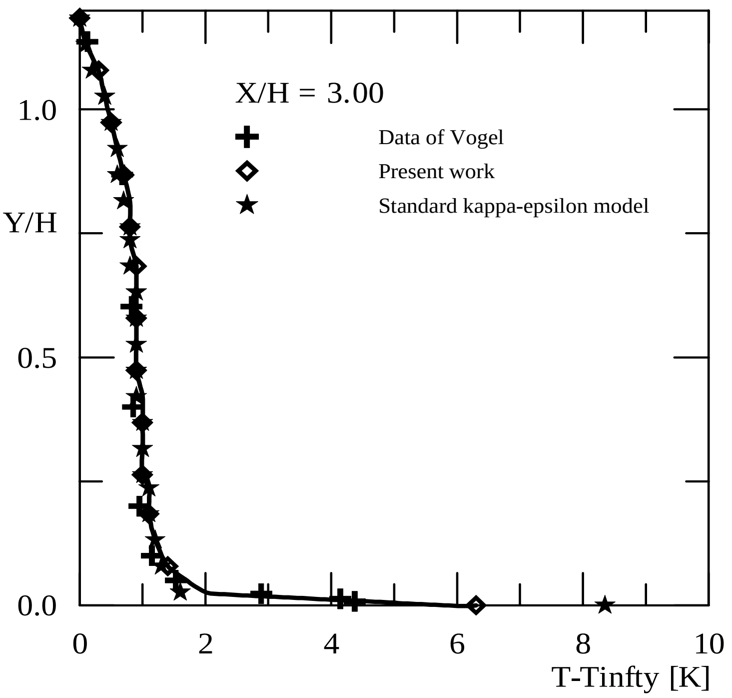

Figure 8: Temperature profiles. X/H = 3.00. +, data of Vogel(1984); $\diamond$, present work; $\star$, standard κ-ϵ model.

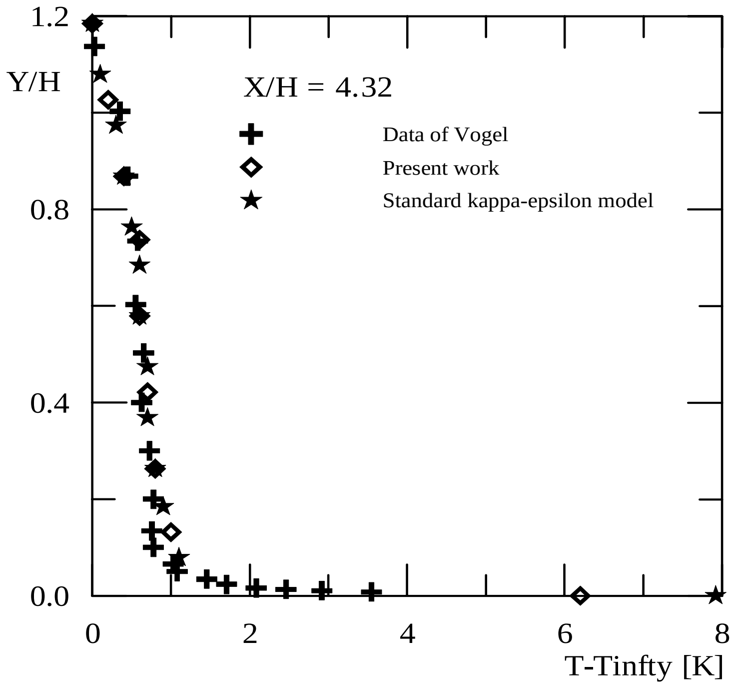

Figure 9: Temperature profiles. X/H = 4.32. +, data of Vogel(1984); $\diamond$, present work; $\star$, standard κ-ϵ model.

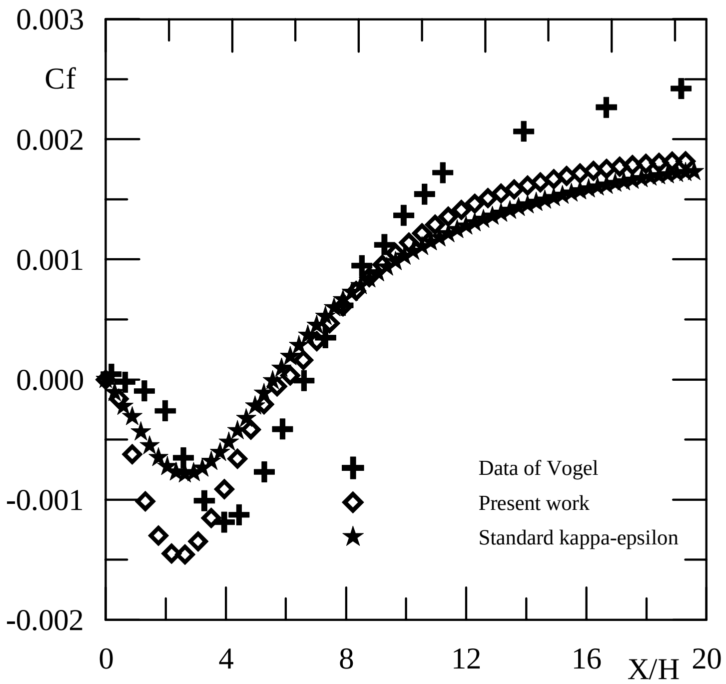

Figure 10: Skin-friction results. +, data of Vogel(1984); $\diamond$, present work; $\star$, standard κ-ϵ model.

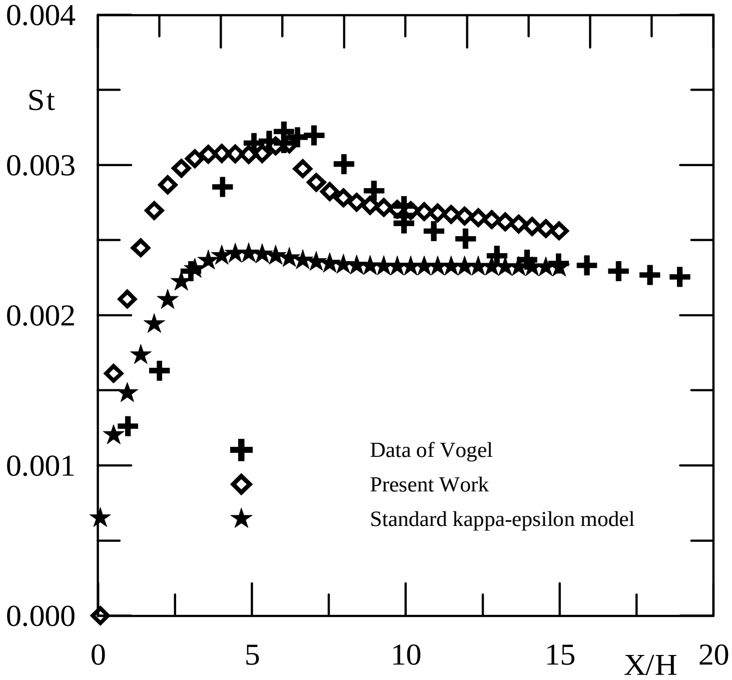

Figure 11: Stanton number results. $+$, data of Vogel(1984); $\diamond$, present work; $\star$, standard κ-ϵ model.