# SATURATION OF A FLOW THROUGH A RIGID POROUS MEDIUM BOUNDED BY AN IMPERMEABLE WALL: A CONSTRAINED NONLINEAR HYPERBOLIC SYSTEM

**Rogério M. Saldanha da Gama, rsggama@gmail.com**
**José Julio Pedrosa Filho, zejulio@terra.com.br**
Mechanical Engineering Graduate Program - FEN, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier, 524, 20550-013, RJ, Brazil

**Maria Laura Martins-Costa, laura@mec.uff.br**
Laboratory of Theoretical and Applied Mechanics (LMTA), Mechanical Engineering Program (TEM-PGMEC)
*Universidade Federal Fluminense, Rua Passo da Pátria, 156, 24210-240 Niterói/RJ, Brazil*

***Abstract.*** *This work proposes a mathematical model to study the filling up of an unsaturated rigid porous medium by a liquid identifying the transition from unsaturated to saturated flow and accounting for the physical upper bound of the fluid fraction that depends on the volume of the pores. The complete solution of a Riemann problem associated to the system of conservation laws satisfying the constraint given by the saturation upper bound is presented.*

***Keywords****: Flow through unsaturated porous media, transition saturated/unsaturated flow, Riemann problem, constrained unknown, shock waves.*

## 1. INTRODUCTION

This work employs a physically realistic mathematical model to represent the filling up of an unsaturated rigid porous matrix by a fluid, identifying the transition from unsaturated to saturated flow, by imposing a constraint (an upper bound) on the saturation (Saldanha da Gama et al., 2012). According to a comprehensive review by Alazmi and Vafai (2000), an adequate description of this transition remains an open subject. The mechanical modeling uses a mixture theory approach to model a porous medium bounded by an impermeable wall.

Constrained hyperbolic systems may be present in different applications, such as two-phase flows, compressible plasticity with shocks and description of traffic of vehicles or crowds, including traffic jams, presence of toll gates and prediction of traffic accidents on roads. Bouchut et al. (2000) discuss two-phase flows of gas and liquid models, for incompressible liquids, ranging from nonconservative conservation laws with relaxation to a system of pressureless gases with concentration constraint and undetermined pressure. Rossmanith (2004) employs a high-resolution finite volume method, based on a wave propagation method, using a constrained transport framework to approximate a system of non-linear hyperbolic magnetohydrodynamic equations, satisfying a divergence-free constraint on the magnetic field, subjected to shock waves and other discontinuities. Després et al. (2011) present a mathematical framework for constrained weak solutions of hyperbolic equations, to model compressible plasticity with shocks. The developed weak formalism allows accounting for both Tresca and Von Mises plasticity criteria.

Concerning traffic flow models, Daganzo (1995) observed that models like the Euler equations for gas dynamics may lead to absurd behavior like vehicles going backwards. Aw and Rascle (2000) proposed a model – the Aw-Rascle model – correcting these problems and ensuring that both density and velocity remain nonnegative. Berthelin et al. (2008) propose a traffic flow model describing the formation and the dynamics of traffic jams, consisting of a pressureless gas dynamics system under a maximal constraint on the density; in other words, the density constraint is preserved at any time. Herty and Schleper (2011), aiming at predicting traffic accidents, discuss predictive mathematical models based on a macroscopic description of traffic flow. Essentially they consider mathematical properties of a coupled macroscopic second-order traffic model with different pressure laws on the connected roads. Aiming at modeling of a tollgate along a highway, Colombo and Goatin (2007) consider a single hyperbolic equation (Cauchy problem) subjected to a local variable unilateral constraint on the flux, while in the present work a system of two partial differential equations is considered. The total number of vehicles is conserved and the traffic speed is assumed to be a function of traffic density.

Saldanha da Gama (1986) proposed a constitutive relation for the partial pressure accounting for a geometrical bound, which arises from the rigidity of the porous matrix and the incompressibility of the fluid, avoiding solutions without physical meaning. Martins-Costa and Saldanha da Gama (2011) proposed an improvement to this constitutive in which the unilateral geometrical constraint for the fluid fraction, instead of being assumed in the whole domain, is considered only in a convenient neighborhood of the porosity (provided that the fluid fraction is smaller than the porosity), besides assuring continuity for the pressure and for its first derivative, thus allowing the analytical computation of the Riemann invariants associated to the problem. In this work the porous medium can actually be saturated by the fluid, while the equation proposed by Saldanha da Gama (1986) imposes a physical behavior for the fluid preventing it to saturate the porous matrix.

The unsaturated porous medium is modeled as a mixture approach (Atkin and Craine, 1976; Rajagopal and Tao, 1995) of three overlapping continuous constituents: a solid (a rigid, homogeneous and isotropic porous matrix), a liquid (an incompressible fluid) and an inert gas, assumed with very low mass density; which was included to account for the compressibility of the system as a whole.

## 2. PROBLEM FORMULATION AND SOLUTION

Figure 1 presents the scheme of a porous medium bounded by an impermeable wall. This problem is built by considering at $t<0$ two distinct bodies: the former is a porous slab of thickness $L$ (represented by the region $0<x<L$) and porosity $\varepsilon$ containing a liquid with constant fluid fraction $\phi_1$. The latter is a semi-infinite porous medium with porosity $\varepsilon$, constant fluid fraction $\phi_2$ and velocity $v_2$, such that at $t<0$ both the liquid and the porous matrix (occupying the second body) have velocity $v_2$. At the time $t=0$ the semi-infinite porous medium reaches the porous slab, so the semi-infinite porous medium (the second body) stops and remains at rest for all $t>0$. So, for $t\geq 0$ the liquid flows towards the porous slab. The following problem, sketched in figure 1, is to be solved: Find $\phi$ and $v$ as functions of the position and time assuming that $\varepsilon>\phi_2>\phi_1$.

The mathematical model for the above-described phenomenon is given by

$$\left.\begin{array}{l}\dfrac{\partial\phi}{\partial t}+\dfrac{\partial}{\partial x}\left(\phi v\right)=0\\ \dfrac{\partial}{\partial t}\left(\phi v\right)+\dfrac{\partial}{\partial x}\left(c^2\phi+\phi v^2\right)=0\\ \phi\leq\varepsilon\end{array}\right\}\quad\text{with}\quad\left\langle\begin{array}{l}\left(\phi,v\right)=\begin{cases}\left(\phi_1,0\right) & \text{for}\quad t=0,\quad 0<x<L\\ \left(\phi_2,v_2\right) & \text{for}\quad t=0,\quad L<x<\infty\end{cases}\\ v=0\quad\text{at}\quad x=0\quad\text{for all}\quad t>0\end{array}\right. \tag{1}$$

in which $\phi$ is the fluid fraction, $v$ the velocity $c$ is a positive constant and $\varepsilon$ is the porous matrix porosity. The first two equations in the left hand side of Eq. (1) represent the mechanical model obtained by considering a mixture theory approach for an unsaturated flow through a porous matrix (see Martins-Costa and Saldanha da Gama, 2011 and references therein), using the following relationship $p=c^2\phi$ (Saldanha da Gama et al., 2012) provided $0\leq\phi<\varepsilon$. It is important to note that the following relations must hold for a rigid and homogeneous porous medium: $p=\hat{p}(\phi)$ for $0<\phi<\varepsilon$, characterizing unsaturated flow and $\hat{p}(\varepsilon)\leq p<\infty$ for $\phi=\varepsilon$, characterizing saturated flow (Saldanha da Gama et al., 2012).

Problem (1) may be rewritten considering $\bar{x}=x-L$ so that $v=0$ at $\bar{x}=-L$ for all $t>0$, giving rise to the following associated Riemann problem

$$\left.\begin{array}{l}\dfrac{\partial\phi}{\partial t}+\dfrac{\partial}{\partial x}\left(\phi v\right)=0\\ \dfrac{\partial}{\partial t}\left(\phi v\right)+\dfrac{\partial}{\partial x}\left(c^2\phi+\phi v^2\right)=0\\ \phi\leq\varepsilon\end{array}\right\}\ \text{with}\ \left(\phi,v\right)=\begin{cases}\left(\phi_1,0\right) & \text{for}\quad t=0,\ -\infty<\bar{x}<0\\ \left(\phi_2,v_2\right) & \text{for}\quad t=0,\quad 0<\bar{x}<\infty\end{cases} \tag{2}$$

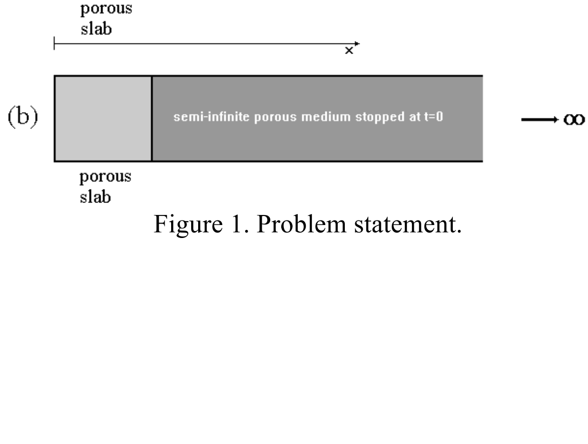


Figure 1. Problem statement.

Since $v_2 < 0$ (as shown in figure 1), $\phi_2 > \phi_1$ and the solution of the system (2) is either 1-shock/2-rarefaction or 1-shock/2-shock (Saldanha da Gama et al., 2012). If $v_2\sqrt{\phi_2\phi_1} > c(\phi_1 - \phi_2)$, the solution is 1-shock/2-rarefaction, being given by

$$(\phi,v) = \begin{cases} (\phi_1,0) & \text{if} \quad -\infty < \bar{x}/t < s_1 \\ (\phi_A,v_A) & \text{if} \quad s_1 < \bar{x}/t < v_A + c \\ (f_2,g_2) & \text{if} \quad v_A + c \le \bar{x}/t \le c + v_2 \\ (\phi_2,v_2) & \text{if} \quad c + v_2 < \bar{x}/t < \infty \end{cases} \quad \text{with} \begin{cases} g_2 = \dfrac{\bar{x}}{t} - c \\ f_2 = \phi_2 \exp\left[\dfrac{\bar{x}}{ct} - \dfrac{v_2}{c} - 1\right] \end{cases} \tag{3}$$

while $\phi_A$ and $v_A$ are given by $\ln\dfrac{\phi_2}{\phi_A} - \sqrt{\dfrac{\phi_A}{\phi_1}} + \sqrt{\dfrac{\phi_1}{\phi_A}} = v_2$ and $v_A = -c\ln\dfrac{\phi_2}{\phi_A} + v_2$, with the shock speed $s_1$ given by $s_1 = (\phi_A v_A)/(\phi_A - \phi_1)$.

The solution of problem (1) is given by system (3) until the 1-shock, with speed $s_1$, reaches the impermeable surface $\bar{x} = -L$. In other words, the solution of Eq. (1) is given by Eq. (3) while $t \le -L/s_1$. When $t = -L/s_1$, the 1-shock reaches the surface $\bar{x} = -L$ and the boundary condition must be imposed.

In order to satisfy the boundary condition, a new Riemann problem is constructed, centered at the impermeable wall, symmetric with respect to $x = 0$, starting from the time $\bar{t} = t + L/s_1$. This Riemann problem is given by

$$\left.\begin{array}{l} \dfrac{\partial\phi}{\partial t} + \dfrac{\partial}{\partial x}(\phi v) = 0 \\ \dfrac{\partial}{\partial t}(\phi v) + \dfrac{\partial}{\partial x}\left(c^2\phi + \phi v^2\right) = 0 \\ \phi \le \varepsilon \end{array}\right\} \text{with} \quad (\phi,v) = \begin{cases} (\phi_A,-v_A) & \text{for} \quad \bar{t} = 0, \quad -\infty < x < 0 \\ (\phi_A,v_A) & \text{for} \quad \bar{t} = 0, \quad 0 < x < \infty \end{cases} \tag{4}$$

The solution of the Riemann problem (4) is given by

$$(\phi,v) = \begin{cases} (\phi_A,-v_A) & \text{if} \quad -\infty < x/\bar{t} < \dfrac{\phi_A v_A}{\phi_* - \phi_A} \\ (\phi_*,0) & \text{if} \quad \dfrac{\phi_A v_A}{\phi_* - \phi_A} < x/\bar{t} < -\dfrac{\phi_A v_A}{\phi_* - \phi_A} \\ (\phi_A,v_A) & \text{if} \quad -\dfrac{\phi_A v_A}{\phi_* - \phi_A} < x/\bar{t} < \infty \end{cases} \tag{5}$$

in which the intermediate fluid fraction $\phi_*$ is obtained from

$$\phi_* = \frac{1}{2}\left\{\left[\phi_A\left(\frac{-v_A}{2c} + \sqrt{\left(\frac{v_A}{2c}\right)^2 + 1}\right)^2 + \varepsilon\right] - \left|\phi_A\left(\frac{-v_A}{2c} + \sqrt{\left(\frac{v_A}{2c}\right)^2 + 1}\right)^2 - \varepsilon\right|\right\} \tag{6}$$

Combining equations (3) and (5) the following solution of problem (1) is reached

$$(\phi,\nu)=\begin{cases}(\phi_1,0) \text{ if } x>0;\ -\infty<\dfrac{x-L}{t}<\dfrac{\phi_A\nu_A}{\phi_A-\phi_1}\\ (\phi_A,\nu_A) \text{ if } \ x>0;\ \dfrac{\phi_A\nu_A}{\phi_A-\phi_1}<\dfrac{x-L}{t}<\nu_A+c\\ \left(\phi_2 e^{\alpha},\dfrac{x-L}{t}-c\right) \text{ if } \ x>0;\ \nu_A+c\le\dfrac{x-L}{t}\le c+\nu_2 \text{ ; for } \alpha=\dfrac{x-L}{ct}-\dfrac{\nu_2}{c}-1\\ (\phi_2,\nu_2) \text{ if } \ x>0;\ c+\nu_2<\dfrac{x-L}{t}<\infty\\ (\phi_*,0) \text{ if } \ t>-L/s;\ -\dfrac{\phi_A\nu_A}{\phi_*-\phi_A}<\dfrac{x}{t+L/s_1}<\infty\\ (\phi_A,\nu_A) \text{ if } \ t>-L/s_1;\ -\dfrac{\phi_A\nu_A}{\phi_*-\phi_A}<\dfrac{x}{t+L/s_1}<\infty\end{cases} \tag{7}$$

until $t<t_+=L\left(1-\dfrac{\phi_*-\phi_A}{\phi_A-\phi_1}\right)/\left(\dfrac{-\phi_*\nu_A}{\phi_*-\phi_A}-c\right)$.

Figure 2 illustrates the solution in the semi-plane $x-t$ presented by equations (7), for the particular case in which $\varepsilon=0.505$, $\phi_1=0.1$, $\phi_2=0.5$, $\nu_2=-0.1$, $L=1$ and $c=1$. In this case, $\phi_A=0.2321$, $\nu_A=-0.867$ and $\phi_*=0.505=\varepsilon$. Without the constraint $\phi\le\varepsilon$ the fluid fraction $\phi_*$ would be 0.5389 – a value greater than the porosity, thus physically unrealistic, emphasizing the importance of the constrained model proposed in this work.

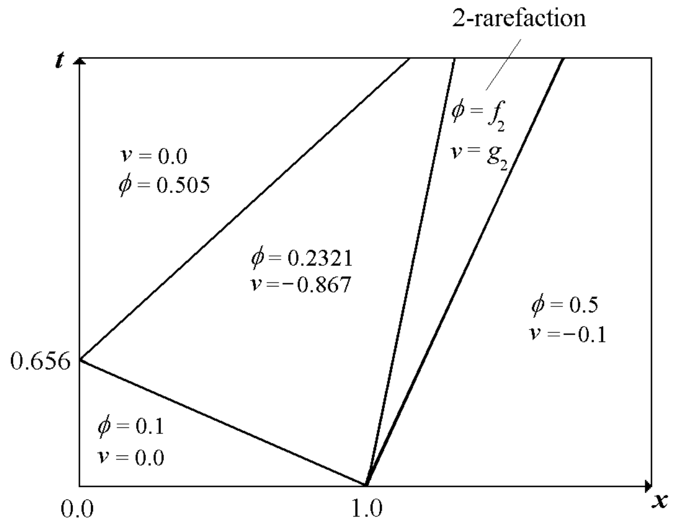


Figure 2. Solution of problem 1 in the semi-plane *x*-*t*.

If $\nu_2\sqrt{\phi_2\phi_1}<c(\phi_1-\phi_2)$, the solution is 1-shock/2-shock, being given by

$$(\phi,\nu)=\begin{cases}(\phi_1,0) & \text{if} \quad -\infty<\bar{x}/t<s_1\\ (\phi_A,\nu_A) & \text{if} \quad s_1<\bar{x}/t<s_2\\ (\phi_2,\nu_2) & \text{if} \quad s_2<\bar{x}/t<\infty\end{cases} \tag{8}$$

with $\nu_A$, $\phi_A$, and the shock speeds $s_1$ and $s_2$ being given by (Saldanha da Gama et al., 2012)

$$v_A = -c\left(\sqrt{\frac{\phi_A}{\phi_1}} - \sqrt{\frac{\phi_1}{\phi_A}}\right) \qquad \phi_A = \frac{\phi_2\phi_1}{\left(\sqrt{\phi_2}+\sqrt{\phi_1}\right)^2}\left\{\frac{-v_2}{2c} + \sqrt{\left(\frac{-v_2}{2c}\right)^2 + 2 + \sqrt{\frac{\phi_2}{\phi_1}} + \sqrt{\frac{\phi_1}{\phi_2}}}\right\}^2$$
$$s_1 = \frac{\phi_A v_A}{\phi_A - \phi_1} \qquad s_2 = \frac{\phi_2 v_2 - \phi_A v_A}{\phi_2 - \phi_A} \tag{9}$$

The solution of Eq. (1) is given by Eq. (8) until the 1-shock, with speed $s_1$, reaches the impermeable surface $\bar{x} = -L$. In other words, the solution of Eq. (1) is given by Eq. (7) while $t \le -L/s_1$. When $t = -L/s_1$ the 1-shock reaches the surface $\bar{x} = -L$ and the boundary condition must be imposed.

In order to satisfy the boundary condition, a new Riemann problem is built, centered at the impermeable wall, symmetric with respect to $x = 0$, starting from the time $\bar{t} = t + L/s_1$. This Riemann problem is also given by Eq. (4), its previously known solution being given by

$$(\phi, v) = \begin{cases} (\phi_A, -v_A) & \text{if} \quad -\infty < x/\bar{t} < \dfrac{\phi_A v_A}{\phi_* - \phi_A} \\ (\phi_*, 0) & \text{if} \quad \dfrac{\phi_A v_A}{\phi_* - \phi_A} < x/\bar{t} < -\dfrac{\phi_A v_A}{\phi_* - \phi_A} \\ (\phi_A, v_A) & \text{if} \quad -\dfrac{\phi_A v_A}{\phi_* - \phi_A} < x/\bar{t} < \infty \end{cases} \tag{10}$$

in which the intermediate fluid fraction $\phi_*$ is, again, obtained from Eq. (6).

So, when $v_2\sqrt{\phi_2\phi_1} < c(\phi_1 - \phi_2)$, the solution of problem (1) is represented by

$$(\phi, v) = \begin{cases} (\phi_1, 0) & \text{if } x > 0;\ -\infty < \dfrac{x-L}{t} < \dfrac{\phi_A v_A}{\phi_A - \phi_1} \\ (\phi_A, v_A) & \text{if } x > 0;\ \dfrac{\phi_A v_A}{\phi_A - \phi_1} < \dfrac{x-L}{t} < \dfrac{\phi_2 v_2 - \phi_A v_A}{\phi_2 - \phi_A} \\ (\phi_2, v_2) & \text{if } x > 0;\ \dfrac{\phi_2 v_2 - \phi_A v_A}{\phi_2 - \phi_A} < \dfrac{x-L}{t} < \infty \\ (\phi_*, 0) & \text{if } t > -L(\phi_A - \phi_1)/\phi_A v_A;\ 0 < \dfrac{x}{t + L(\phi_A - \phi_1)/\phi_A v_A} < -\dfrac{\phi_A v_A}{\phi_* - \phi_A} \\ (\phi_A, v_A) & \text{if } t > -L(\phi_A - \phi_1)/\phi_A v_A;\ -\dfrac{\phi_A v_A}{\phi_* - \phi_A} < \dfrac{x}{t + L(\phi_A - \phi_1)/\phi_A v_A} < \infty \end{cases} \tag{11}$$

## 3. FINAL REMARKS

In this work a mathematical model for flows through unsaturated porous media, identifying the transition from unsaturated to saturated flow, was proposed by including a constraint that must be satisfied to build physically realistic generalized solutions for any initial data. The complete solution of a constrained nonlinear hyperbolic problem with shock waves – an associated Riemann problem containing a restriction (an upper bound for the fluid fraction, represented by the porosity), was presented as well as its application to flows through porous media, emphasizing a problem involving boundary conditions (impermeable wall).

## 4. ACKNOWLEDGEMENTS

The authors R.M. Saldanha da Gama and M. L. Martins-Costa gratefully acknowledge the financial support provided by the Brazilian agency CNPq.

## 5. REFERENCES

Alazmi, B. and Vafai, K, 2000, "Analysis of variants within the porous media transport models", *J. Heat Transfer*, vol. 122, pp. 303-326.

Atkin, R. J. and Craine, R. E., 1976, "Continuum theories of mixtures. Basic theory and historical development", *Quart. J. Mech. Appl. Math.*, vol. 29, pp. 209-244

Aw A. and Rascle, M., 2000, "Resurrection of "second order" models of traffic flow", *SIAM J. Appl. Math.,* vol. 60 (3), pp. 916–938.

Berthelin, F., Degond, P., Delitala, M. and M. Rascle, M. , 2008, "A model for the formation and evolution of traffic jams", *Arch. Rational Mech. Anal.*, vol. 187, pp. 185-220.

Bouchut, F., Brenier, Y., Cortes, J. and Ripoll, J.-F., 2000, "A hierarchy of models for two-phase flows", *J. Nonlinear Sci*., vol. 10, pp. 639-660.

Colombo, R. M. and Goatin, P., 2007, "A well posed conservation law with a variable unilateral constraint", J. *Differential Equations*, vol. 234, pp. 654–675.

Daganzo, C. 1995, "Requiem for second order fluid approximations of traffic flow", *Transp. Res. B*, vol. 29B, pp. 277–286.

Herty, M. and Schleper, V., 2011, "Traffic flow with unobservant drivers", *Z. Angew. Math. Mech*., vol. 91-10, pp. 763-776.

Martins-Costa, M.L. and Saldanha da Gama, R.M. 2011, "A new constitutive equation for unsaturated flows of incompressible liquids through rigid porous media", *J. Porous Media*, vol. 14, pp.205-217.

Rajagopal, K. R. and L. Tao, 1995, *Mechanics of Mixtures*, Series Advances in Math. Appl. Sci., vol 35, World Scientific, Singapore

Rossmanith, J. A., 2004, "A high-resolution constrained transport method with adaptive mesh refinement for ideal MHD", *Computer Physics Comm.,* vol. 164, 128–133

Saldanha da Gama, R. M., 1986, "A new mathematical modelling for describing the unsaturated flow of an incompressible liquid through a rigid porous medium*", Int. J. Non-Linear Mech.*, vol. 40, pp. 59-68.

Saldanha da Gama, R.M., Pedrosa Filho, J.J. and Martins-Costa, M.L., 2012, "Modeling the saturation process of flows through rigid porous media by the solution of a nonlinear hyperbolic system with one constrained unknown", ZAMM, *DOI 10.1002/zamm.201100031*.

## 6. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.