ENHANCED LUMPED-DIFFERENTIAL FORMULATIONS FOR THERMALLY DEVELOPING FLOW WITH ISOTHERMAL WALL

L. S. de Barros, lsbarros@id.uff.br
L. A. Sphaier, lasphaier@id.uff.br
Department of Mechanical Engineering – TEM/PGMEC, Universidade Federal Fluminense, Rua Passo da Pátria 156, bloco E, sala 216, Niterói, Rio de Janeiro, 24210-240, Brazil

Abstract. *This paper shows how enhanced lumping approximation techniques can be employed for calculating the mean stream temperature in thermally developing fluid flow. The adopted methodology consists in transforming the original convection-diffusion partial-differential equations into a simpler one-dimensional form, using approximation rules provided by the Coupled Integral Equations Approach (CIEA). The simpler transformed system can then be directly integrated and analytical solutions for the mean stream temperature can be readily obtained, which implies in a significant reduction the required computational effort. The results calculated with the simplified formulations are then compared with solutions for the original partial-differential problem and very reasonable agreement is seen.*

Keywords: *laminar flow, forced convection, duct flow, lumped-capacitance analysis*

1. INTRODUCTION

Approximating an integral by a linear combination of the integrand values and its derivatives at the integration limits, was an idea originally developed by Hermite (1878) and first presented by Menning *et al.* (1983), the first ones to use this two-point approach, deriving it in a fully differential form called $H_{\alpha,\beta}$. Using the Hermite formulas for obtained improved-lumped formulations, known as the Coupled Integral Equaitons Approach (CIEA), can be found in a variety of heat transfer studies, among which recent applications include ablation (Ruperti *et al.*, 2004), drying (Dantas *et al.*, 2007), heat conduction with temperature-dependent conductivity (Su *et al.*, 2009) and adsorbed gas storage (Sphaier and Jurumenha, Available online on May 2012) In this study, the CIEA is employed for the problem of thermally developing fluid flow within a parallel-plates duct. With this approach enhanced lumped-differential formulations for representing the problem are obtained. The formulations are naturally simpler the the original equation since it consists of simple ODEs for determining the mean stream temperature, while the original problem was a PDE for calculating the temperature field and from this result calculating the same averaged temperature.

2. PROBLEM FORMULATION AND HERMITE APPROXIMATION

In order to illustrate the proposed methodology, a general problem of flow within parallel plates is considered, which written in dimensionless form is given by:

$$u^* \frac{\partial\theta}{\partial\xi} = \mathrm{Pe}^{-2}\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2}, \qquad \theta(0,\eta) = 0, \qquad \left|\frac{\partial\theta}{\partial\xi}\right|_{\xi\to\infty} < \infty, \tag{1a}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = 0, \qquad \theta(\xi,1) = 1, \tag{1b}$$

where the employed dimensionless parameters and variables are defined as:

$$\mathrm{Pe} = \frac{\bar{u}\,H/2}{\alpha} \qquad \eta = \frac{y}{H/2}, \qquad \xi = \frac{x}{L}, \qquad L = \frac{H}{2}\,\mathrm{Pe}, \qquad \theta = \frac{T - T_{\min}}{T_{\max} - T_{\min}}, \tag{2}$$

The basis for the Coupled Integral Equations Approach (CIEA) is the Hermite approximation of an integral, denoted,

$H_{\alpha,\beta}$, which is given by the general expression:

$$\int_{x_{i-1}}^{x_i} f(x)dx = \sum_{\nu=0}^{\alpha} c_\nu(\alpha,\beta) h_i^{\nu+1} f^{(\nu)}(x_{i-1}) + \sum_{\nu=0}^{\beta} c_\nu(\beta,\alpha)(-1)^\nu h_i^{\nu+1} f^{(\nu)}(x_i) + E_{\alpha,\beta} \tag{3a}$$

where,

$$h_i = x_i - x_{i-1}, \qquad c_\nu(\alpha,\beta) = \frac{(\alpha+1)!(\alpha+\beta-\nu+1)!}{(\nu+1)!(\alpha-\nu)!(\alpha+\beta+2)!} \tag{3b}$$

and $f(x)$ and its derivatives $f^{(\nu)}(x)$ are defined for all $x \in [x_{i-1}, x_i]$. $E_{\alpha,\beta}$ is the error in the approximation. It is assumed that $f^{(\nu)}(x_{i-1}) = f_{i-1}^{(\nu)}$ for $\nu = 0, 1, 2, \ldots, \alpha$ and $f^{(\nu)}(x_i) = f_i^{(\nu)}$ for $\nu = 0, 1, 2, \ldots, \beta$.

The Hermite integration formula can provide different approximation levels, starting from the classical lumped system analysis towards improved lumped-differential formulations. A detailed error analysis of the application of the CIEA to diffusion problems using $H_{0,0}$, $H_{0,1}$, $H_{1,0}$, and $H_{1,1}$ Hermite approximations was carried out in (de B. Alves *et al.*, 2000). Since approximations of order higher than $H_{1,1}$ involve derivatives of order higher than one, these are avoided for the sake of simplicity of the methodology. Hence, only the two different approximations below are considered:

$$H_{0,0} \Rightarrow \int_0^h f(x)\,\mathrm{d}x \approx \frac{1}{2}h(f(0)+f(h)), \tag{4a}$$

$$H_{1,1} \Rightarrow \int_0^h f(x)\,\mathrm{d}x \approx \frac{1}{2}h(f(0)+f(h)) + \frac{1}{12}h^2(f'(0)-f'(h)), \tag{4b}$$

which correspond to the well-known trapezoidal and corrected trapezoidal integration rules, respectively.

3. PLUG-FLOW ANALYSIS

For the simplified plug-flow case, $u^* = 1$ and the mean stream temperature equals the average temperature definition. Integrating equations (1a) and applying the average definition leads to the following ODE system:

$$\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}, \qquad \overline{\theta}(0) = 0, \qquad \left|\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi}\right|_{\xi\to\infty} < \infty. \tag{5}$$

The Classical Lumped-System Analysis (CLSA) consists in approximating the averages directly by boundary values, which corresponds to applying the rectangular integration approximation rule. In order to avoid a constant mean stream temperature, and in order to obtain a relation that leads to $\theta(\xi,0)$ different than $\theta(\xi,1)$, the following integral approximations are used:

$$\overline{\theta}(\xi) = \int_0^1 \theta\,\mathrm{d}\eta \,\approx\, \theta(\xi,0), \qquad \int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \,\approx\, \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \tag{6}$$

which leads to the following relation for the wall derivative:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx\, \theta(\xi,1) \,-\, \overline{\theta}(\xi) \,=\, 1 \,-\, \overline{\theta}(\xi) \tag{7}$$

such that the solution of the averaged system (5) is given by:

$$\overline{\theta}(\xi) \,=\, 1 \,-\, \exp\left(\mathrm{Pe}^2\,\xi/2 - \frac{\mathrm{Pe}}{2}\sqrt{4+\mathrm{Pe}^2}\,\xi\right) \tag{8}$$

3.1 Improved Lumped-System Analysis

3.1.1 $H_{0,0}/H_{0,0}$ *formulation*

Using the $H_{0,0}$ scheme for approximating the integrals of θ and its derivative yields:

$$\int_0^1 \theta \,\mathrm{d}\eta \approx \frac{1}{2}\Big(\theta(\xi,0) + \theta(\xi,1)\Big), \qquad \int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \approx \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right). \tag{9}$$

Using the boundary conditions and solving for the wall derivative gives:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx 4\left(1 - \overline{\theta}(\xi)\right) \tag{10}$$

such that the solution of the averaged system (5) is given by:

$$\overline{\theta}(\xi) = 1 - \exp\left(\mathrm{Pe}^2\,\xi/2 - \frac{\mathrm{Pe}}{2}\sqrt{16+\mathrm{Pe}^2}\,\xi\right) \tag{11}$$

3.1.2 $H_{1,1}/H_{0,0}$ *formulation*

This scheme is based on using $H_{1,1}$ approximation for the temperature integral:

$$\int_0^1 \theta \,\mathrm{d}\eta \approx \frac{1}{2}\Big(\theta(\xi,0) + \theta(\xi,1)\Big) + \frac{1}{12}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} - \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) \tag{12}$$

and the same $H_{0,0}$ approximation for its derivative integral. Applying boundary conditions and solving for the wall derivative yields:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx 3\left(1 - \overline{\theta}(\xi)\right) \tag{13}$$

such that the solution of the averaged system (5) is given by:

$$\overline{\theta}(\xi) = 1 - \exp\left(\mathrm{Pe}^2\,\xi/2 - \frac{\mathrm{Pe}}{2}\sqrt{12+\mathrm{Pe}^2}\,\xi\right) \tag{14}$$

3.1.3 $H_{1,1}/H_{1,1}$ *formulation*

This approximation scheme relies on using the $H_{1,1}$ for approximating the integral of θ and its derivative, the latter being given by:

$$\int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \approx \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) + \frac{1}{12}\left(\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0} - \left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=1}\right) \tag{15}$$

Substituting boundary conditions leads to:

$$1 - \theta(\xi,0) \approx \frac{1}{2}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} + \frac{1}{12}\left(\left(\frac{\partial\theta}{\partial\xi}\right)_{\eta=0} - \mathrm{Pe}^{-2}\left(\frac{\partial^2\theta}{\partial\xi^2}\right)_{\eta=0}\right) \tag{16}$$

Eliminating the wall derivative from the previous equation and the $H_{1,1}$ temperature integral, and substituting in the integrated energy balance leads to:

$$\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi} = 6 + 6\,\theta_0(\xi) - 12\,\overline{\theta}(\xi) + \mathrm{Pe}^{-2}\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2}, \qquad 24 + 48\,\theta_0(\xi) - 72\,\overline{\theta}(\xi) + \frac{\mathrm{d}\theta_0}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\theta_0}{\mathrm{d}\xi^2}, \tag{17}$$

where $\theta_0 = \theta(\xi, 0)$. This coupled ODE system can be solved directly for $\overline{\theta}$; however the solution is not presented due to space limitations. For large Péclet numbers a simple form can be obtained:

$$\overline{\theta}(\xi) = 1 - \frac{1}{21}\exp(-30\,\xi)\,\left(21\cosh(6\sqrt{21}\,\xi) + 4\sqrt{21}\sinh(6\sqrt{21}\,\xi)\right) \tag{18}$$

4. LAMINAR FLOW ANALYSIS

This section presents the methodology for laminar flow (Hagen-Poiseuille profile), $u^* = \bar{u}\,\frac{3}{2}\,\left(1-\eta^2\right)$, in which the average is given by $\overline{\theta}(\xi) = \int_0^1 \theta(\xi,\eta)\,\mathrm{d}\eta$ and the meam stream temperature is defined by $\theta_m(\xi) = \int_0^1 u^*\theta(\xi,\eta)\,\mathrm{d}\eta$. Average and mean stream temperature definition and boundary condition substitution:

$$\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}, \tag{19}$$

For isothermal wall different levels of approximation can lead to different lumped formulations, as described next.

4.1 Improved Lumped-System Analysis

4.1.1 $H_{0,0}/H_{0,0}/H_{0,0}$ *formulation*

This scheme is based on using $H_{0,0}$ approximation for the misture temperature integral:

$$\int_0^1 u^*\theta\,\mathrm{d}\eta \approx \frac{1}{2}\Big(u^*(0)\,\theta(\xi,0) + u^*(1)\,\theta(\xi,1)\Big) \tag{20}$$

and the same $H_{0,0}$ approximation for the temperature and for its derivative integral. Applying boundary conditions and solving for the wall derivatives yields:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx 1 - \frac{2}{3}\,\theta_m(\xi), \qquad \frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} \approx \frac{2}{3}\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2} \tag{21}$$

such that the solution of the averaged system (19) is given by:

$$\theta_m(\xi) = \frac{3}{4} - \frac{3}{4}\exp\left(3\,\mathrm{Pe}^2\,\xi/4 - \frac{\mathrm{Pe}}{4}\,\sqrt{64 + 9\,\mathrm{Pe}^2}\xi\right) \tag{22}$$

4.1.2 $H_{1,1}/H_{0,0}/H_{1,1}$ *formulation*

This scheme is based on using $H_{1,1}$ approximation for the misture temperature integral and for the averaged temperature integral:

$$\int_0^1 u^*\theta\,\mathrm{d}\eta \approx \frac{1}{2}\Big(u^*(0)\,\theta(\xi,0) + u^*(1)\,\theta(\xi,1)\Big) + \frac{1}{12}\left(\left(\frac{\partial(u^*\theta)}{\partial\eta}\right)_{\eta=0} - \left(\frac{\partial(u^*\theta)}{\partial\eta}\right)_{\eta=1}\right) \tag{23}$$

$$\int_0^1 \theta\,\mathrm{d}\eta \approx \Big(\theta(\xi,0) + \theta(\xi,1)\Big) + \frac{1}{12}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} - \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) \tag{24}$$

and the same $H_{0,0}$approximation for its derivative integral. Applying boundary conditions and solving for the wall derivatives yields:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx \frac{8}{3}\Big(1 - \theta_m(\xi)\Big), \qquad \frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} \approx \frac{8}{9}\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2} \tag{25}$$

such that the solution of the averaged system (19) is given by:

$$\theta_m(\xi) = 1 - \exp\left(9\,\mathrm{Pe}^2\,\xi/16 - \frac{\mathrm{Pe}}{16}\sqrt{768 + 81\,\mathrm{Pe}^2}\xi\right) \tag{26}$$

4.2 $H_{1,1}/H_{1,1}/H_{1,1} formulation$

This scheme is based on using $H_{1,1}$ approximation for the derivative temperature integral:

$$\int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \approx \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) + \frac{1}{12}\left(\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0} - \left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=1}\right) \tag{27}$$

and the same $H_{1,1}$approximation for the averaged and misture temperature integral. Substituting boundary conditions leads to:

$$1 - \theta(\xi,0) \approx \frac{1}{2}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} + \frac{1}{12}\Big(\frac{3}{4}\left(\frac{\partial\theta}{\partial\xi}\right)_{\eta=0} - \mathrm{Pe}^{-2}\left(\frac{\partial^2\theta}{\partial\xi^2}\right)_{\eta=0}\Big) \tag{28}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx \frac{1}{9}\Big(24 - 24\,\theta_m(\xi) - 3\,\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} + 2\,\mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2}\Big) \tag{29}$$

substituting in equation (5), leads to:

$$\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + \frac{1}{9}\Big(24 - 24\,\theta_m(\xi) - 3\,\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} + 2\,\mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2}\Big) \tag{30}$$

For no axial diffusion (large Péclet number):

$$\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} = 2\left(1 - \theta_m\right) \tag{31}$$

which provides the following solution:

$$\theta_m(\xi) = 1 - \exp(-2\,\xi) \tag{32}$$

The solution with axial diffusion is not presented due to space limitations.

5. RESULTS AND DISCUSSION

The previous solutions are compared with the exact solution of the Graetz problem with a plug-flow profile, given by:

$$\theta_m(\xi) = 1 + \sum_{n=1}^{\infty} b_n\,\exp(-\beta_n\,\xi), \qquad b_n = \frac{8}{(2\,n-1)^2\,\pi^2}, \qquad \beta_n = \frac{1}{2}\left(\mathrm{Pe}^2 - \sqrt{\mathrm{Pe}^4 + 4\,\mathrm{Pe}^2\,\mu_n^2}\right) \tag{33}$$

where $\mu_n = (n - 1/2)\,\pi$.

Figure 1 Presents comparative results of all different approximation schemes, including the previous exact solution, for different Péclect values. As can be seen, the CLSA solution underestimates the mean temperature by a significant amount, while the $H_{0,0}/H_{0,0}$ scheme generally overestimates it by a smaller amount. In spite of this, The $H_{0,0}/H_{1,1}$ and $H_{1,1}/H_{1,1}$ show a very good agreement with the exact solution.

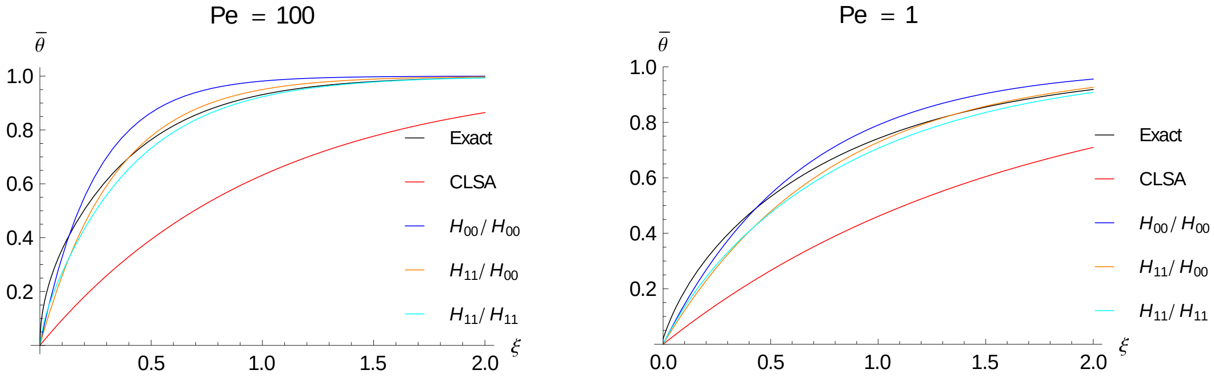

Figure 1. Comparison between different lumped approximation schemes and exact solution for different Péclet numbers.

Figure 2 Presents comparative results of all different approximation schemes, including the exact solution, for different Péclet values. As can be seen, the CLSA solution is not a good approximation to the mean temperature by a significant amount, and the $H_{0,0}/H_{0,0}/H_{0,0}$ scheme does not approximate very well also. In spite of this, The $H_{1,1}/H_{0,0}/H_{1,1}$ and $H_{1,1}/H_{1,1}/H_{1,1}$ show a very good agreement with the exact solution.

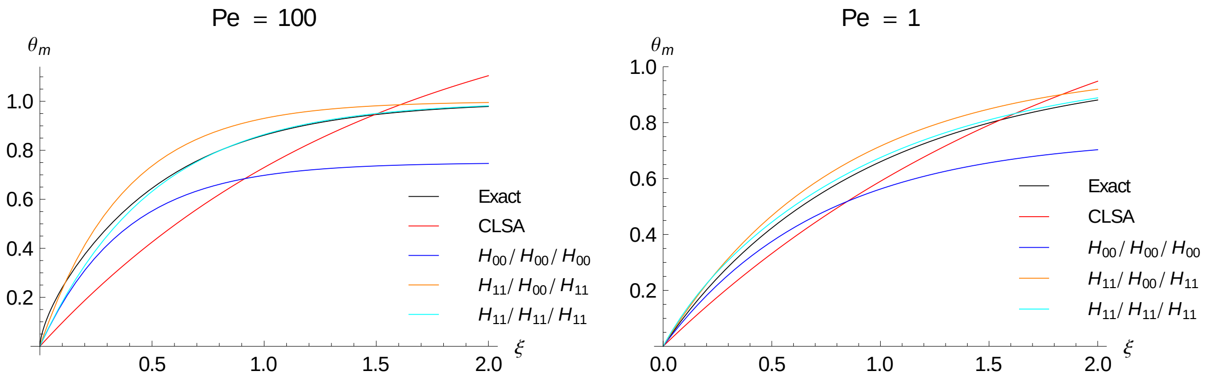

Figure 2. Comparison between different lumped approximation schemes and exact solution for different Péclet numbers for a laminar flow.

6. CONCLUSIONS

This paper presented an alternative approach for calculating the mean stream temperature for dynamically-developed thermally-developing flow. The simplified case of plug-flow was presented for illustrating the methodology and comparing the results with a fully analytical solution. An approximate analytical methodology, based on the CIEA was utilized, and the results showed very good agreement with the two-dimensional Graetz solution. A laminar flow case was presented and the results showed very good aggrement.

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