COUPLED PROBLEMS OF THERMOELASTICITY AND FRACTURE MECHANICS IN DURABILITY ESTIMATES

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Abstract. *Considering durability of structures with cracks it is necessary to take into account a possibility of crack surfaces contact. Indeed, under arbitrary loading of structures there is no guarantee, that the cracks will be completely opened. The complete or partial closure can take place if the material is under compression [1]. The crack surfaces contacting leads to changing the stress and deformations within the structure and influences on the conditions of the crack growth and on life-time of structures.*

Problem of partial opening of a penny-shaped crack due to the heat sources is considered. Analytical results are obtained by means of Hankel transforms and corresponding dual integral equations. The closed form solutions for heat flux across the crack surfaces and for the aperture of the crack are obtained. The solution is illustrated by several numerical results. The crack openings as functions of the distance between the heat sources and the crack for different initial openings of the crack are shown.

1. INTRODUCTION

Let an infinite elastic body contain a penny-shaped crack occupying the region $r \le a$ in the plane $z = 0$ referred to the cylindrical coordinates system (r, φ, z). It is assumed that the crack surfaces are free of load and the arbitrary initial crack opening $\overline{\omega}(r)$, as well as the corresponding stresses $\overline{\sigma}_{ij}$, within the body are given. Let two concentrated heat sources of a constant intensity W are located on z-axis symmetrically with respect to the crack plane at the points $P = \pm\,\xi$ produce stresses σ_{ij}^{*} and displacements (u^{*}, ω^{*}). In addition, we assume that the crack surfaces are kept at the constant temperature $T^{*} = 0$. From the superposition principle, the total stresses and displacements can be described as

$$\overline{\overline{\sigma}}_{ij} = \overline{\sigma}_{ij} + \sigma_{ij}^{*}, \quad \overline{\overline{u}} = \overline{u} + u^{*}, \quad \overline{\overline{\omega}} = \overline{\omega} + \omega^{*} \tag{1}$$

With increase of compressive thermoelastic stresses, "overlapping" of crack surfaces can take place and a zone of "negative" displacement can occur, e. g. $\overline{\overline{\omega}} \le 0$. To avoid this phenomenon, the restrictions on the crack surfaces displacements have to be applied, namely $\overline{\overline{\omega}} \ge 0$ for $0 \le r \le a$.

If the crack surfaces come into contact the boundary conditions for the problem on the partially closed crack are as follows

$$\begin{aligned} &T^{*} = 0, \quad \overline{\overline{\sigma}}_{zz} = 0, \quad \overline{\overline{\omega}} \ge 0 \quad for \quad x \le r \le a, \quad z = 0, \\ &\frac{\partial T^{*}}{\partial z} = 0, \quad \overline{\overline{\sigma}}_{zz} = 0, \quad \overline{\overline{\omega}} = 0 \quad for \quad 0 \le r \le x, \quad z = 0, \end{aligned} \tag{2}$$

where T^{*} describes the temperature within the body while the regions $x \le r \le a$ and $0 \le r \le x$ represent the opening and contact zone, respectively.

It was proved by Ju and Rowlands [2] and by Goldstein and Spector [3] that such a boundary problem can be solved iteratively and the process of iterations converges to the exact solution for the contact domain. To determine stresses σ_{ij}^{*} and displacements (u^{*}, ω^{*}) caused by two heat sources located symmetrically with respect to the crack plane the following boundary value problem has to be solved [4]:

$$\frac{\partial^2 T^{*}}{\partial r^2} + \frac{1}{r}\frac{\partial T^{*}}{\partial r} + \frac{\partial^2 T^{*}}{\partial z^2} = -\frac{W}{\lambda}[\delta(z-\xi) + \delta(z+\xi)], \quad T^{*}(r,0) = 0 \;\; for \;\; 0 \le r \le a, \quad \frac{\partial T^{*}(r,0)}{\partial z} = 0 \;\; for \quad r \ge a, \tag{3}$$

$$\begin{aligned} &2(1-\nu)\left(\frac{\partial^2 u^{*}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{*}}{\partial r} - \frac{u^{*}}{r}\right) + (1-2\nu)\frac{\partial^2 u^{*}}{\partial z^2} + \frac{\partial^2 \omega^{*}}{\partial r \partial z} = 2(1+\nu)\alpha\frac{\partial T^{*}}{\partial r}, \\ &(1-2\nu)\left(\frac{\partial^2 \omega^{*}}{\partial r^2} + \frac{1}{r}\frac{\partial \omega^{*}}{\partial r}\right) + 2(1-\nu)\frac{\partial^2 \omega^{*}}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial u^{*}}{\partial r} + \frac{u^{*}}{r}\right) = 2(1+\nu)\alpha\frac{\partial T^{*}}{\partial z}, \\ &\sigma_{zz}^{*}(r,0) = 0 \quad for \quad 0 \le r \le a, \quad \sigma_{rz}^{*}(r,0) = 0 \quad for \quad r \ge 0, \quad \omega^{*}(r,0) = 0 \quad for \quad r \ge a, \end{aligned} \tag{4}$$

where α, λ, ν are the coefficients of thermal expansion, heat conductivity and Poisson's ratio, respectively.

2. TEMPERATURE DISTRIBUTION

Distribution of temperature $T^*(r,z)$ within the body containing a crack is represented as

$$T^*(r,z) = T^0(r,z) + T(r,z) \tag{5}$$

where T^0 describes the temperature in the body without crack and caused by the heat sources located at the points $P(\pm\xi)$ while T stands for the temperature caused by the application of the prescribed temperature on the crack surfaces and according to Nowacki [4]

$$T^0(r,z) = \frac{W}{4\pi\lambda}\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$
$$R_1 = \sqrt{r^2 + (z-\xi)^2}, R_2 = \sqrt{r^2 + (z+\xi)^2}. \tag{6}$$

To define the temperature $T(r, z)$, the following boundary value problem has to be solved:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0,$$
$$T(r,0) = -T^0(r,0) = -\frac{1}{2\pi} g(r) \quad for \quad 0 \le r \le a, \qquad g(r) = \frac{W}{\lambda}\frac{1}{\sqrt{r^2+\xi^2}}. \tag{7-8}$$
$$\frac{\partial T(r,0)}{\partial z} = 0 \quad for \;\; r \ge a,$$

The boundary value problem (7-8) is reduced to the following dual integral equations [4]:

$$\int_0^\infty \Phi(u) J_0(ur)du = \frac{1}{2\pi} g(r) \quad for \;\; 0 \le r \le a,$$
$$\int_0^\infty u\Phi(u) J_0(ur)du = 0 \quad for \;\; r \ge a, \tag{9}$$

and the heat flux across the crack surfaces is given by

$$q(r) = \int_0^\infty u\Phi(u) J_0(ur)du \quad for \;\; 0 \le r \le a, \tag{10}$$

where $J_0(r)$ is the Bessel function of the first kind. Solution of the dual integral equations (9) can be represented in the form [4]:

$$\Phi(u) = \frac{1}{\pi^2}\int_0^a \cos(ut)\left(\frac{d}{dt}\int_0^t \frac{sg(s)}{\sqrt{t^2-s^2}}ds\right)dt. \tag{11}$$

From (8), (10) and (11) the heat flux across the crack surfaces for $0 \le r \le a$ is finally given by

$$q(r) = \frac{W}{\lambda\pi^2}\frac{\xi}{\xi^2+r^2} \times \left[\frac{1}{\sqrt{a^2-r^2}} + \frac{1}{\sqrt{\xi^2+r^2}}\left(\pi - \arctan\sqrt{\frac{a^2-r^2}{r^2+\xi^2}}\right)\right] \tag{12}$$

3. OPENING OF THE CRACK

From the superposition principle, stresses σ^*_{ij} and displacements (u^*, ω^*) corresponding to problem (4) can be represented

$$\sigma^*_{ij} = \sigma^0_{ij} + \sigma^{(1)}_{ij} + \sigma_{ij}$$
$$u^* = u^0 + u^{(1)} + u, \omega^* = \omega^0 + \omega^{(1)} \tag{13}$$

Here, σ^0_{ij}, and (u^0, ω^0) describe stresses and displacements caused by the temperature T^0 in the body without crack; $\sigma^{(1)}_{ij}$, and $u^{(1)}, \omega^{(1)}$ stand for stresses and displacements within the body containing the crack, which is loaded by stresses equal to evaluated from the previous step but with opposite sign. Notations σ_{ij}, and (u, ω) denote stresses and displacements in the body with the crack with the surfaces, which are free of mechanical load but heated with the prescribed temperature given by Eq. $(7)_2$.

Stresses σ_{ij}^0 can be read as [4]

$$\frac{\sigma_{zz}^0(r,z)}{2\mu} = A\left[\left(r^2 R_1^{-2} - 2\right)R_1^{-1} + \left(r^2 R_2^{-2} - 2\right)R_2^{-1}\right]$$
$$\frac{\sigma_{rz}^0(r,z)}{2\mu} = Ar\left[(\xi - z)R_1^{-3} - (\xi + z)R_2^{-3}\right], \qquad A = \frac{mW}{8\pi\lambda}, m = \frac{1+\nu}{1-\nu}\alpha. \tag{14}$$

where functions R_1 and R_2 are described by Eqs. $(6)_{2\text{-}3}$, and μ is a Lame constant. According to the solution given by Sneddon [5], the opening of the crack caused by the vertical load applied to its surfaces is defined for $0 \le \rho \le 1$ by

$$\omega^{(1)}(\rho) = -\frac{1-\nu}{\pi\mu}\int_\rho^1 \frac{1}{\sqrt{t^2-\rho^2}}\left(\int_0^t \frac{s\,f(s)}{\sqrt{t^2-s^2}}\,ds\right)dt, \tag{15}$$

where according to Eq. $(14)_1$

$$f(\rho) = -\sigma_{zz}^0(\rho,0) = \frac{4\mu A}{a}\frac{1}{\sqrt{\rho^2+c^2}}\left(1+\frac{c^2}{\rho^2+c^2}\right), \qquad \rho = \frac{r}{a}, \quad c = \frac{\xi}{a}. \tag{16}$$

From Eq.$(16)_1$ and Eq. (15), the crack opening $\omega^{(1)}(\rho)$ can be derived as follows

$$\omega^{(1)}(\rho) = -A_0\left[\begin{array}{l}\left(\frac{1}{4}+\frac{1}{\pi}\arcsin\frac{1-c^2}{1+c^2}\right)\times\ln(1+\sqrt{1-\rho^2}) - \left(\frac{1}{4}+\frac{1}{2\pi}\arcsin\frac{\rho^2-c^2}{\rho^2+c^2}\right)\ln\rho + \\ \frac{2c}{\sqrt{\rho^2+c^2}}\arctan\sqrt{\frac{1-\rho^2}{c^2+\rho^2}} - F(\rho,c)\end{array}\right] \tag{17}$$

where

$$A_0 = 4A(1-\nu), \qquad F(\rho,c) = \frac{c}{2\pi}\int_\rho^1 \frac{\ln(t+\sqrt{t^2-\rho^2})}{t^2+c^2}\,dt. \tag{18}$$

A problem of finding stresses σ_{ij} and displacements (u, ω) in the body containing the crack opened due applied temperature on its surfaces is reduced to the solution of the following dual integral equations [5]:

$$\int_0^\infty \eta\psi(\eta)J_0(\rho\eta)d\eta = h(\rho) \quad for\ 0 \le \rho \le 1, \qquad \int_0^\infty \psi(\eta)J_0(\rho\eta)d\eta = 0 \quad for\ \rho \ge 1 \tag{19}$$

where

$$h(\rho) = (1+\nu)\alpha a^2\int_0^\infty Q(\eta)J_0(\rho\eta)d\eta, \qquad Q(\eta) = \int_0^1 \rho q(\rho)J_0(\rho\eta)d\rho, \tag{20}$$

and $q(\rho)$ representing the heat flux across crack surfaces is given by Eq. (12).
The solution of the dual integral equations (19) has the following form [5]

$$\psi(\eta) = \frac{2}{\pi}\int_0^1 \sin(\eta s)\left(\int_0^s \frac{xh(x)}{\sqrt{s^2-x^2}}\,dx\right)ds \tag{21}$$

and the corresponding crack opening ω is read as

$$\omega = \int_0^\infty \psi(\eta)J_0(\rho\eta)d\eta \quad for\ 0 \le \rho \le 1. \tag{22}$$

From Eqs. (20-22) the crack opening $\omega(\rho)$ can be given as follows

$$\omega(\rho) = 2A_0\left[B(c)\ln(1+\sqrt{1-\rho^2}) + C(c)\ln\rho + \ln\frac{c+\sqrt{1+c^2}}{c+\sqrt{\rho^2+c^2}} + \frac{c}{\sqrt{1+c^2}}L(\rho,c)\right] \tag{23}$$

where

$$B(c)=1-\frac{c}{\sqrt{1+c^2}}-\frac{1}{\pi}\arctan\frac{1}{c},\quad C(c)=\frac{1}{\pi}\left[\arctan\frac{1}{c}-\frac{c}{\sqrt{1+c^2}}\ln\frac{\sqrt{1+c^2}+1}{\sqrt{1+c^2}-1}\right],$$
$$L(\rho,c)=\frac{1}{\pi}\int_{\rho}^{1}\frac{1}{t}\ln\frac{\sqrt{1+c^2}-\sqrt{1-t^2}}{\sqrt{1+c^2}+\sqrt{1-t^2}}dt+\frac{1}{\pi}\int_{\rho}^{1}\frac{1}{t}\arctan\sqrt{\frac{1-t^2}{c^2+t^2}}dt \tag{24}$$

Finally the total crack's opening $\omega^*(\rho)$ is given by

$$\omega^*(\rho)=\omega^{(1)}(\rho)+\omega(\rho) \tag{25}$$

where $\omega^{(1)}(\rho)$ and $\omega(\rho)$ are given by Eq. (17) and Eq. (23), respectively.

4. CLOSURE OF CRACKS WITH VARIOUS INITIAL OPENINGS

The developed model and obtained closed-form solutions will now be used for the analysis of crack closure in materials exploited to the heating from two point heat sources placed symmetrically to the crack surfaces at different distances from them. The crack are penny-shaped. The closer the heat sources to the crack, the higher the maximum of the heat flux (formula (12)) through the crack surfaces (Figure 1), and the greater the crack closure (formula (23)) due to thermal compression (Figure 1).

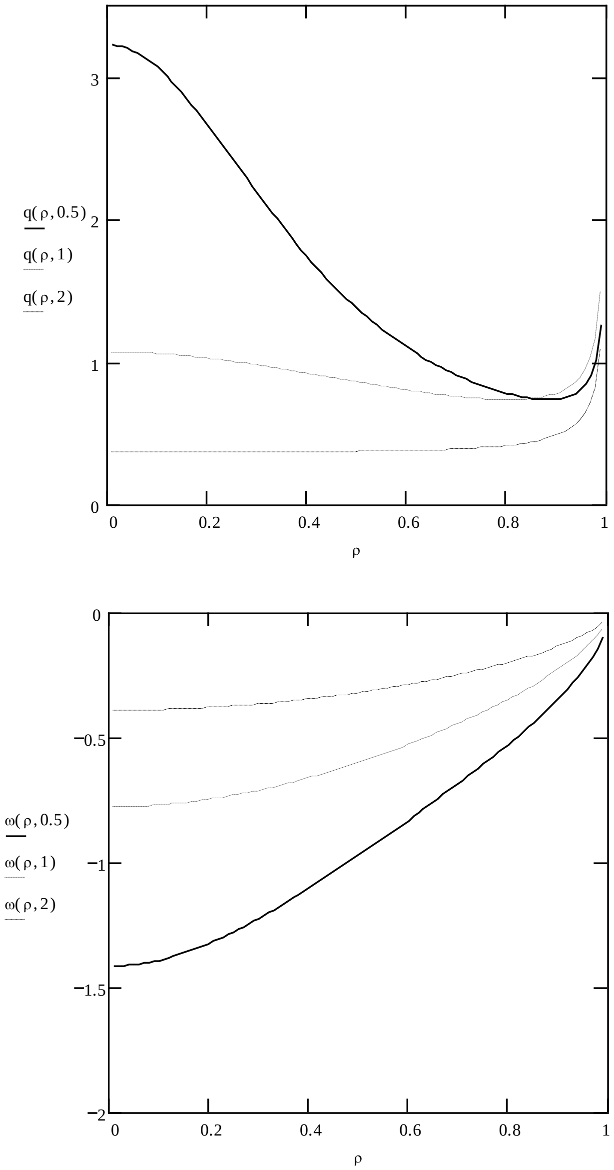

Figure 1. Dependence of heat flux $q(\rho, c)$ through the crack surfaces and the thermal crack closure $w(\rho, c)$ for different distances from the heat sources to the crack plane, c = 0.5; 1.; 2 (ρ is a dimensionless crack radius).

Let the initial crack opening be $w_0 = b\,(1-\rho^2)^{0.5\,\beta}$ with $b = 1.2$; $\beta = 0.5$; 1; 2; ρ is a dimensionless crack radius. The parameter β is responsible for the crack shape: cracks with $\beta = 1$ have an elliptical initial aperture, whereas $\beta = 0.5$ gives a more vertical opening of the crack on its contour and $\beta = 2$ corresponds to a smooth touching of the crack surfaces along its contour. As the heat sources approach the crack surfaces, thermal stresses and corresponding thermal crack closures (Figure 1) increase and they superpose into the initial crack openings (Figure 2), which finally leads to the overlapping and the surface contact.

$$\mathrm{w0}(\rho,\beta) := 1.2\left(1-\rho^2\right)^{0.5\cdot\beta}$$

$$\rho := 0.01\;0.02\,.\,0.99$$

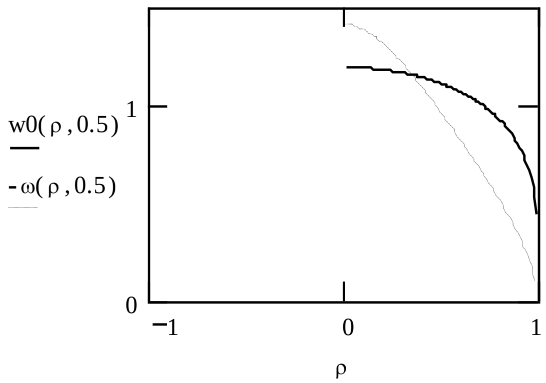

Figure 2. The initial crack opening w_0 and negative thermal crack opening w (analytical solution from formula (23),(24)), superposition of which gives the total crack opening, $R(\rho, c)$, where c is the distance from the heat sources to the crack *(* (because of the symmetry the picture is given for $0 \le \rho \le 1$).

Graphs presented on Figures 3-5 show the process of gradual approaching the crack surfaces to each other upon approaching the heat sources to the crack, which eventually, at some moment, leads to the surface contact (because of the symmetry the pictures are given for $0 \le \rho \le 1$). The obtained results reveal some intriguing features. On Figures 3 and 4, where the initial crack shape was $w_0 = b\,(1-\rho^2)^{0.5\,\beta}$ correspondingly with $\beta = 0.5$ and $\beta = 1$, the crack contact occurs in the center. However, on Figure 5 the contact zone is located along the crack contour (with $\beta = 2$). This fact is easy to explain: a parameter $\beta = 2$ gives the initial crack opening with smoothly overlapping of the crack surface along its contour, and it is easier for the heat sources to close the crack there. In contrast, $\beta = 0.5$ and $\beta = 1$ contribute to the shape of the crack aperture making its surfaces “steeper” on its contour, which eventually facilitates the initial contact at the crack center.

5. THE CONCLUSIONS

The problems of penny-shaped cracks in materials have attracted a lot of attention, and it is not possible to mention all the papers on this topic. However much fewer deal with penny-shaped cracks in elastic bodies under heating conditions (e.g. [6], [7], [8]). Presence of a crack in material causes distribution of the heat and therefore of the surrounding crack thermal stresses, which makes the problem much more complicated than in absence of the thermal loads, and that is why only few mixed analytical-numerical solutions are available (e.g. [7], [8]). In present work a problem on closure of a penny-shaped crack due to heat sources is considered. Analytical result is obtained by solution step by step of heat conductivity problem with application of principle of superposition and dual integral equations method and of thermoelastic problem with use of principle of superposition and Hankel transform. The formulas of a heat flux into crack from two sources and thermal closure of the crack are derived. Graphs of the crack surface displacements depending on the distance from the thermal sources up to the crack surface and pictures of closing of the cracks due to the thermal sources are obtained. For three cracks with different initial opening the graphs of their closing are given and analysis is conducted how the contact zone depends on various distances between the crack surfaces and the thermal sources. The graphs demonstrate that the closing occurs different ways depending on the crack initial opening: in case of smooth surfaces overlapping the closing occurs on the crack contour, and in other two cases the crack starts closing in the center.

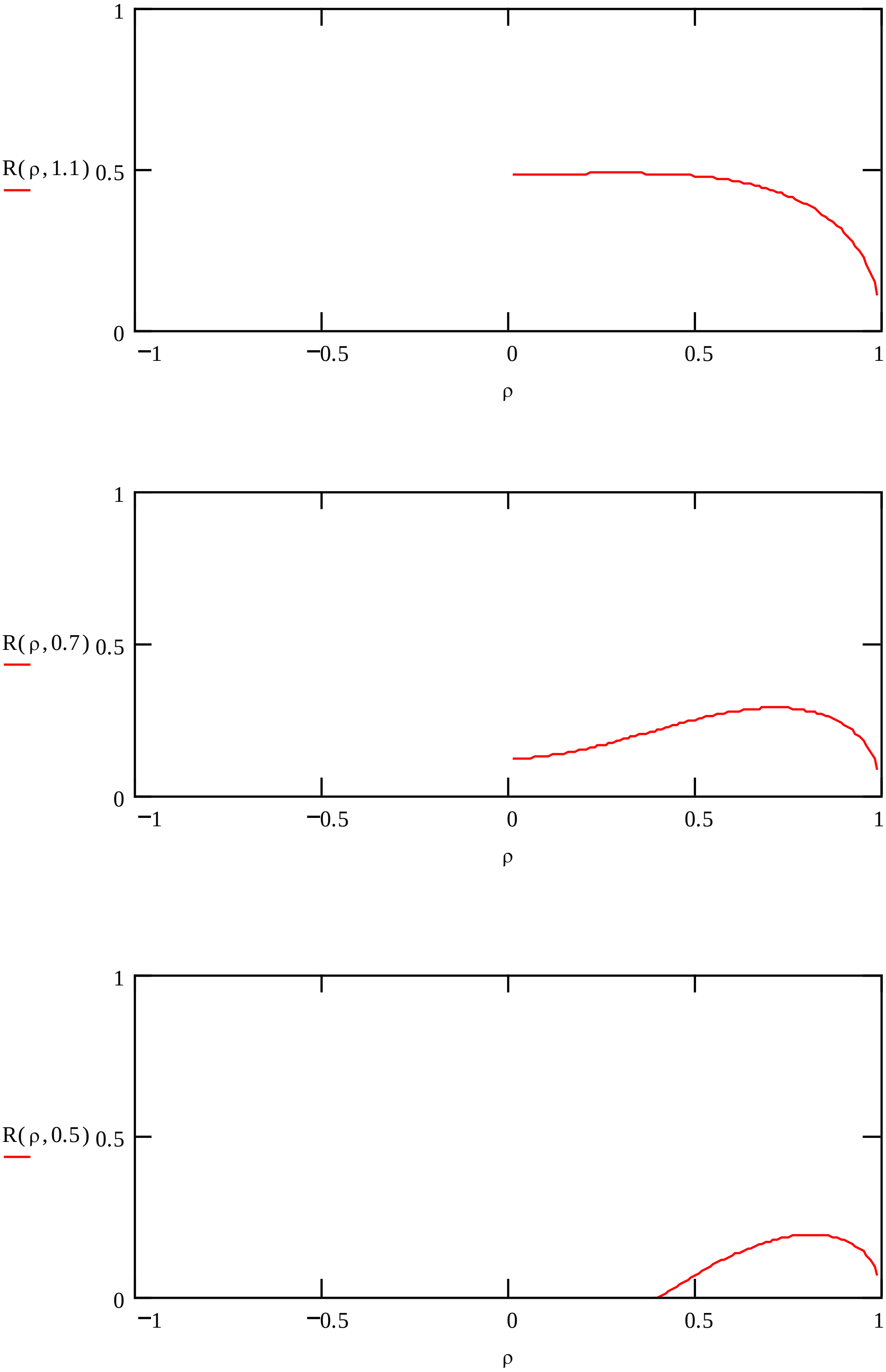

Figure 3. The total crack apertures $R(\rho, c) = \omega_0(\rho, c) - \omega(\rho, c)$ for the initial crack shape with β = 0.5 as result of superposition of the initial aperture $\omega_0(\rho,c)$ and negative thermal aperture $\omega(\rho,c)$ for the distances from the heat sources to the crack surfaces c = 1.1; 0.7; 0.5.

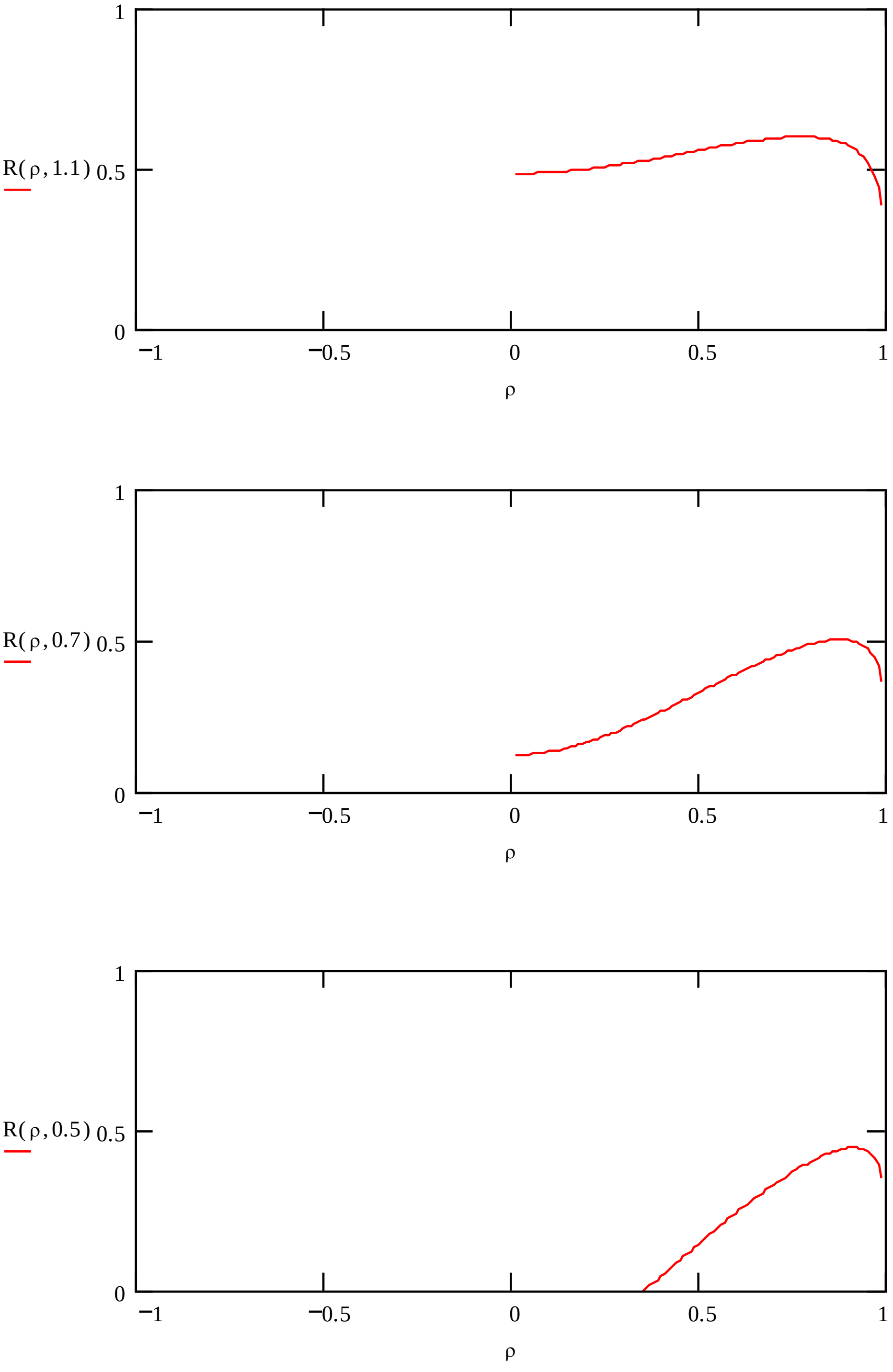

Figure 4. The total crack apertures $R(\rho, c) = \omega_0(\rho, c) - \omega(\rho, c)$ for the initial crack shape with β = 1 for the distances from the heat sources to the crack surfaces c = 1.1; 0.7; 0.5.

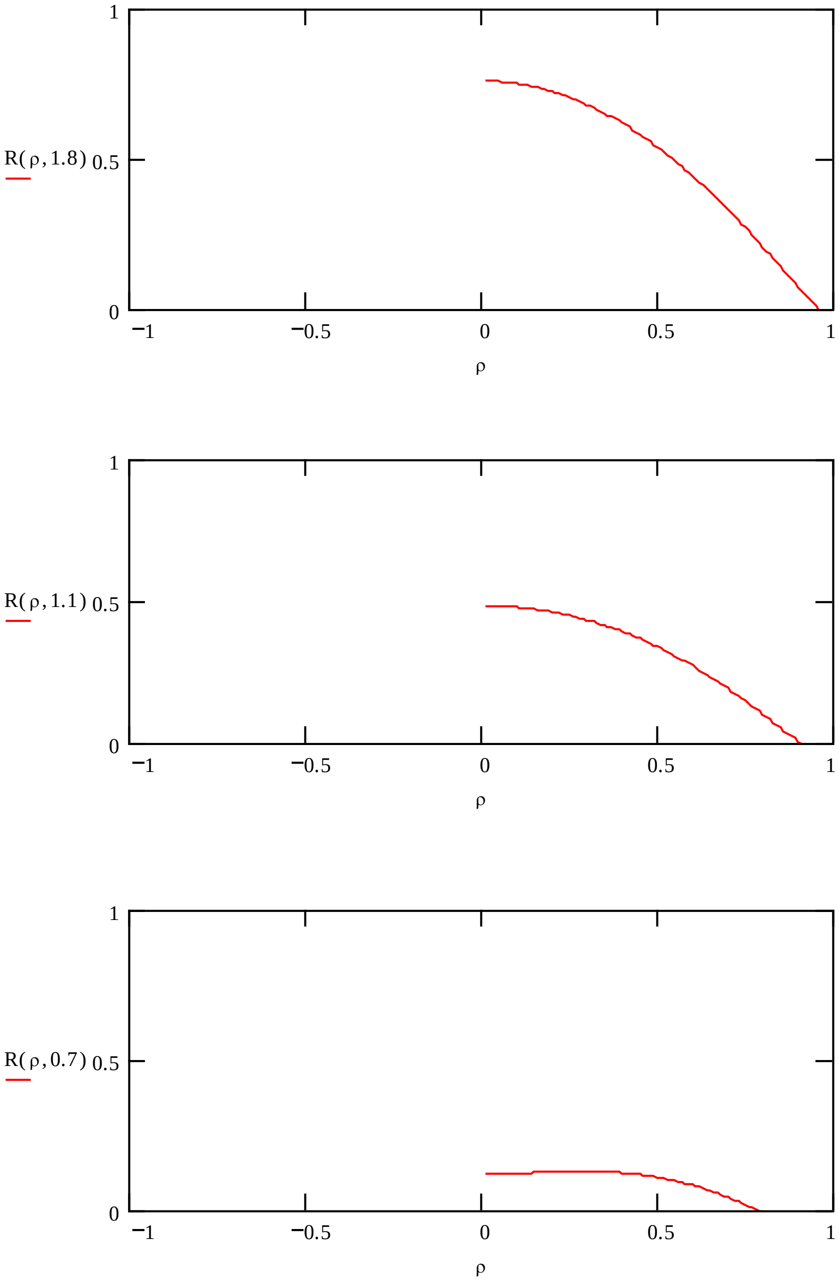

Figure 5. The total crack apertures $R(\rho, c) = \omega_0(\rho, c) - \omega(\rho, c)$ for the initial crack shape with $\beta = 2$. (smooth overlapping of the crack surfaces along its contour) for the distances from the heat sources to the crack surfaces c = 1.1; 0.7; 0.5.

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