VORTEX METHOD SIMULATION OF BLASIUS' FLAT PLATE BOUNDARY LAYER

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Abstract. *The Vortex Method has been extensively used to simulate external flows around bluff and streamlined bodies. It relies on the discretization of the vorticity field into a cloud of vortex blobs to simulate the convective-diffusive transport of vorticity. Vortex blobs are generated in the neighborhood of the solid wall in order to satisfy the no-slip and the no-penetration boundary conditions, and they move in a Lagrangian manner to solve the vorticity transport equation. Despite the tremendous development that this powerful mesh-free technique has recently achieved, the numerical implementation of the wall boundary conditions is currently under intense investigation, since it is intimately connected to the vorticity generation process in the vicinity of the body surface. In this paper we describe an efficient two-dimensional vortex method algorithm, with emphasis on a new model for the vortex creation near the surface that increases the accuracy of the simultaneous implementation of the wall boundary conditions. We employ the Adaptive Fast Multipole Method to calculate the induced velocities and the Corrected Core-Spreading Method to simulate the vorticity diffusion in the boundary layer and wake. The method is second-order accurate in space when gaussian vortex blobs are used and second-order accurate in time when the Adams-Bashforth scheme is used to march the integration process in time. The algorithm is tested against the well-known two-dimensional, incompressible boundary-layer flow over a flat plate. The observed agreement between the numerical results and the exact Blasius solution indicates that the algorithm provides an excellent representation of the vorticity field.*

Keywords*: vortex method, corrected core-spreading method, adaptive fast multipole method, Blasius flat-plate boundary layer, wall boundary conditions.*

1. INTRODUCTION

The flow around bluff and streamlined bodies that usually arise in engineering applications is quite difficult to calculate due to the presence of several complex phenomena, such as boundary layer development and separation, vortex shedding and turbulence. For external flows of this type, the use of Eulerian methods may not be the most adequate approach because they require the construction of a numerical grid to discretize an infinite domain, which brings up mesh refinement and domain size issues that are not easily resolved. On the other hand, Lagrangian mesh-free numerical methods, such as the vortex method, present the advantage that no grid is necessary to compute the flow (Leonard, 1980). The vortex method is based on the numerical discretization of the vorticity field by a linear combination of moving basis functions (the vortex blobs). The convective term is replaced by ordinary differential equations for the blobs trajectories, which follow the local velocity.

The first version of the vortex method that considered viscous effects was developed by Chorin (1973), who proposed the random-walk method to model viscous difusion. Since then, vortex methods have been substantially developed, despite facing difficulties coming from the evaluation of the Biot–Savart law, the Lagrangian modeling of viscous effects, and the loss of accuracy due to lagrangian distortion of the computational elements (Barba *et al*., 2005). This latter difficulty requires the overlap of blobs to accurately reconstruct a continuous field variable and guarantee convergence of the method (Barba *et al*., 2005). The first two difficulties have been successfully overcome by the application of the fast multipole method (Carrier *et al*., 1988) and by a variety of viscous schemes available in the literature (Barba *et al*., 2005), respectively. The third problem may be dealt with by the use of either global field interpolation methods, which are based on either radial basis function collocation schemes or techniques that use approximate solutions to the reverse heat equation, or remeshing schemes (Barba and Rossi, 2010).

Despite the tremendous development that the mesh-free vortex method has recently achieved, the numerical implementation of the wall boundary conditions is currently under intense investigation, since it is intimately connected to the vorticity generation process in the vicinity of the body surface. In this paper we describe an efficient two-dimensional vortex method algorithm, with emphasis on a new model for the vortex creation near the wall surface that increases the accuracy of the simultaneous implementation of the no-slip and the no-penetration boundary conditions. We use gaussian vortex blobs, which turn the method second-order accurate in space, and we integrate the Lagrangian motion of the blobs using the Adams-Bashforth second-order time-marching scheme, which turns the method second-order accurate in time. We employ the Adaptive Fast Multipole Method (Carrier *et al.*, 1988) to calculate the induced velocities and the Corrected Core-Spreading Method (Rossi, 1996, 1997) to simulate the vorticity diffusion in the boundary layer and wake. The algorithm is tested against the well-known two-dimensional, incompressible boundary-layer flow over a flat plate. Next, we briefly describe the vortex method and some sample results compared with the Blasius solution for a flat plate.

2. VORTEX METHOD

The vortex method relies on the numerical discretization of the vorticity field by a linear combination of vortex blobs (or simply vortices) that move with the local flow velocity (Chorin, 1973; Leonard, 1980). The motion of these blobs is governed by the vorticity transport equation, which can be written, for two-dimensional incompressible flow, as

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} \equiv \frac{\partial \omega}{\partial t} + \mathbf{u}\cdot\nabla\omega = \nu\nabla^2\omega, \tag{1}$$

where $\omega(\mathbf{x},t)$ is the vorticity field at a point $\mathbf{x}$ in space and time t, and ν is the fluid's kinematic viscosity. This vorticity field is approximated by a superposition of N_v vortex blobs with a Gaussian basis function given by

$$\omega(\mathbf{x},t) = \sum_{i=1}^{N_v} \frac{\Gamma_i}{4\pi\sigma_i^2} \exp\left(-\frac{\left|\mathbf{x}-\mathbf{x}_i(t)\right|^2}{4\sigma_i^2} \right), \tag{2}$$

where Γ_i is the strength, σ_i is the core radius and $\mathbf{x}_i$ is the position vector of vortex i, with $1 \le i \le N_v$. For the external flow around a body of characteristic length c, immersed in a flow with a freestream velocity $\mathbf{U}_\infty = U\mathbf{i}$ at infinity, where $\mathbf{i}$ is the unit vector in the x direction and U is freestream speed, vortex blobs are generated in the neighborhood of the body surface S_b. These blobs must satisfy the no-slip and the no-penetration boundary conditions, and they move in a Lagrangian manner to approximate the solution of Eq. (1). The velocity field $\mathbf{u}_{v,i}$ associated to Eq. (2) at which vortex i moves may be written as

$$\frac{d\mathbf{x}_{v,i}}{dt} = \mathbf{u}_{v,i} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \sum_{\substack{j=1 \\ j\neq i}}^{N_v} \frac{\Gamma_j}{2\pi} \frac{\mathbf{x}_i - \mathbf{x}_j}{\left|\mathbf{x}_i - \mathbf{x}_j\right|^2} \left[1 - \exp\left(-\frac{\left|\mathbf{x}_i - \mathbf{x}_j\right|^2}{4\sigma_i^2} \right) \right], \tag{3a}$$

whereas vortex i simultaneously undergoes viscous diffusion according to

$$\frac{d\sigma_i^2}{dt} = \nu. \tag{3b}$$

Equations (3) form a dynamical system of ordinary differential equations that approximates the solution to Eq. (1).

The body is discretized using the panel method (Katz and Plotkin, 2001). Hence, for the potential flow that represents the body contribution, the body velocity field $\mathbf{u}_b$ can be written as the gradient of the velocity potential $\phi(\mathbf{x},t)$ according to $\mathbf{u}_b \equiv \nabla\phi$. For incompressible flow, the field $\phi(\mathbf{x},t)$ is the solution to the following boundary-value problem

$$\nabla^2\phi(\mathbf{x}) = 0, \qquad \text{in } V, \tag{4a}$$

$$\nabla\phi(\mathbf{x})\cdot\mathbf{n} = \frac{\partial\phi}{\partial\mathbf{n}} = 0, \quad \text{on } S_b, \tag{4b}$$

$$\lim_{|\mathbf{x}|\to\infty} \nabla\phi(\mathbf{x},t) = \mathbf{U}_\infty, \quad \text{as } |\mathbf{x}| \to \infty. \tag{4c}$$

In Eqs. (4), V is the fluid domain and $\mathbf{n}$ is a unit vector normal to S_b. Using Green's identity (Katz and Plotkin, 2001), it can be shown that the solution to the boundary-value problem defined by Eqs. (4) is

$$\phi(\mathbf{x}) = \int_{S_b} -\frac{\gamma}{2\pi}\frac{\partial}{\partial\mathbf{n}}\left(\ln\mathbf{x}\right) ds + \int_{S_b} \frac{\lambda}{2\pi}\left(\ln\mathbf{x}\right) ds + \phi_\infty . \tag{5}$$

where $\gamma(s)$ is a vorticity distribution and $\lambda(s)$ is a source distribution on the body surface, s is a coordinate along S_b, and $\phi_\infty = Ux$ is the velocity potential associated to the uniform freestream flow.

The solution expressed by Eq. (5) is obtained using the panel method with $\lambda(s) = 0$ and $\gamma(s)$ given by a piecewise-continuous linear-vortex panel method, where the body geometry is divided up into N flat panels, each one with length Δs_i. The middle point of the panel is the control point, where the boundary conditions are enforced, and the endpoints are the panel nodes. We assume that the vortex singularity distribution is linear over the panel, and the values γ_i of this function at the nodes are the unknowns. The γ_i's are calculated in order to satisfy the wall boundary conditions.

The total velocity vector $\mathbf{u}_i$ induced at vortex i has three components: the uniform flow at infinity, $\mathbf{U}_\infty$; the velocity $\mathbf{u}_{v,i}$ induced by the entire cloud of M vortices present in the flow at time t, given by Eq. (3a); and the velocity $\mathbf{u}_{b,i}$ induced by the N panels that discretize the body. Hence, we may write that each vortex blob i moves according to

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{u}_i \equiv \mathbf{U}_{\infty,i} + \mathbf{u}_{v,i} + \mathbf{u}_{b,i} . \tag{6}$$

Integration of Eq. (6) over a time step Δt using the 2nd-order Adams-Bashforth scheme yields

$$\mathbf{x}_i(t+\Delta t) = \mathbf{x}_i(t) + [1.5\mathbf{u}_i(t) - 0.5\mathbf{u}_i(t-\Delta t)]\Delta t . \tag{7}$$

The evaluation of the velocity component $\mathbf{u}_{v,i}$ in Eq. (7) for a cloud of M vortices using direct calculation based on Eq. (3a) requires M^2 operations. Since this computational cost is too high, we use the Adaptive Fast Multipole Method (Carrier *et al*., 1988) to calculate the induced velocities. This is a box-box multipole expansion method, which groups vortices into boxes of different sizes, according to the number of particles in the box. Direct vortex-vortex calculations are performed within the same box, whereas box-box calculations are carried out according to a multipole expansion in terms of the distance between box centers. This algorithm reduces the operation count to order M.

To simulate the diffusive transport of vorticity we employ the Corrected Core-Spreading Method (CCSM) proposed by Rossi (1996) and implemented using the merging algorithm also proposed by Rossi (1997). This method is based on the integration of Eq. (3b), which predicts that the vortex core radius grows in time according to

$$\Delta\sigma^2 = \nu\Delta t . \tag{8}$$

The CCSM limits the core radius to fall within the range $\alpha l \le \sigma \le l$, where l is the largest core radius in the simulation and $0 < \alpha \le 1$. These two parameters determine the frequency of the spatial refinement. Hence, when the core radius of a vortex with strength Γ_i reaches a value greater than or equal to l, it is replaced four new vortices with strength $\Gamma_i/4$ and radius αl. These four new vortices are positioned at 90º from each other and at a distance r from the center of the original vortex, where r is calculated from $r = 2\sigma(1-\alpha^2)^{1/2}$. These equations guarantee that the first- and second-order moments are conserved and also allow overlapping among the four cores to be imposed. This method is deterministic, easily implemented and completely mesh-free.

3. NEW MODEL FOR THE BOUNDARY CONDITIONS AT THE WALL

The no-slip and the no-penetration boundary conditions on the body surface are simultaneously imposed through a model for the calculation of the instantaneous vorticity flux on the wall. This model is based upon the superposition of two vortex sheets (Santiago, 2008): one distributed over the body surface, represented by the linear vortex panels with strength $\bar{\gamma}_k \Delta s_k$, where the mean vorticity of panel k is $\bar{\gamma}_k = (\gamma_k + \gamma_{k+1})/2$; and the other, represented by a layer of nascent vortices with strength Γ_k positioned a distance ε off the wall and along the panels. The vorticity flux that diffuses from the wall into the fluid is determined by imposing explicitly the no-slip and the no-penetration boundary conditions at the panel control points, in addition to an equation for the conservation of circulation (Kelvin's theorem). These equations provide a linear system of algebraic equations for the unknown $\bar{\gamma}_k$ and Γ_k. The linear system of algebraic equations is comprised of $(2N + 1)$ equations and $(2N + 1)$ unknowns, that is,

$$\sum_{k=1}^{N+1} A_{jk}\gamma_k(t) + \sum_{k=1}^{N} B_{jk}\Gamma_k(t) = b_j(t), \quad 1 \le j \le 2N+1. \tag{6}$$

The elements of the matrices A_{jk} e B_{jk} depend only on the position of the control points and the location where the nascent vortices are generated and, therefore, are calculated only once. The vector $b_j(t)$ is updated every time step. As soon as these quantities are determined, $N \times (L+1)$ new vortices are generated into the flow such that they are now free to convect and diffuse, as described in section 2, where N vortices have strength $\bar{\gamma}_k \Delta l_k$ and $(N \times L)$ vortices have strength Γ_k/L.

4. RESULTS AND DISCUSSION

We now present sample results for our vortex method applied to the steady incompressible boundary layer flow over a flat plate that is parallel to the freestream flow. The plate has thickness of 2% with respect to its length *c*. In Santos (2010) systematic studies were carried in order to evaluate the performance of the vortex method with respect to the numerical parameters that are intrinsic to the algorithm and to determine their efficient range of values. Based on these studies and in order to produce relatively coarse simulations to highlight the potential of the method, we have chosen the following set of values for the numerical parameters used in the simulations presented here: $\alpha = 0.70$, the parameter that controls the vortex splitting in the CCSM; $h/\sigma = 0.70$, the parameter that prescribes the initial ratio of the spacing between nascent vortices to the core radius, which guarantees core overlapping; $N = 50$, the number of panels for the body discretization; $\Delta\tau = 0.02$, the dimensionless time step. The Reynolds number is $Re \equiv Uc/\nu = 10^3$.

Results for the u/U-velocity profiles versus η at two positions along the plate, $x/c = 0.25$ and $x/c = 0.75$, are presented in Figs. 1 and 2 at $\tau = 4.5$ and compared to the exact Blasius solution. The similarity variable η is defined as $\eta = (y/c)(Re/(x/c))^{1/2}$. These chordwise stations are far enough from the leading and trailing edges of the plate, where the Blasius solution is no longer valid. The results show good agreement between the numerical results and the Blasius solution at $x/c = 0.25$. However, there is some discrepancy in the middle region of the profile at $x/c = 0.75$, which may be attributed to some numerical diffusion coming from the CCSM and to some lack of refinement of N and $\Delta\tau$. However, the main result that needs to be emphasized is the fact that the boundary conditions are exactly satisfied on the plate, as shown in Figs. 1 and 2, which indicates that the flow near the solid wall is calculated very accurately. This is the most important contribution presented here. As a consequence, the shear stress at the plate is also well calculated, as one can see from the derivative of the velocity profile at $\eta = 0$ for these two chordwise stations.

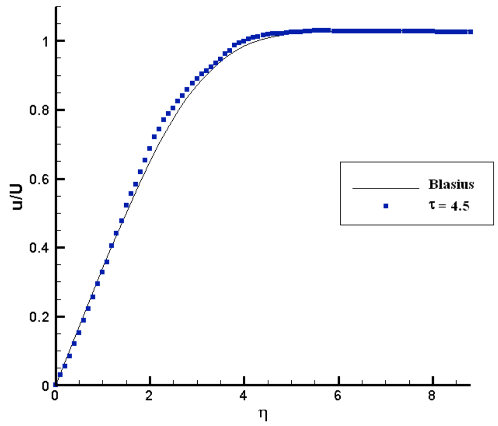

Figure 1. Velocity profile at $x/c = 0.25$.

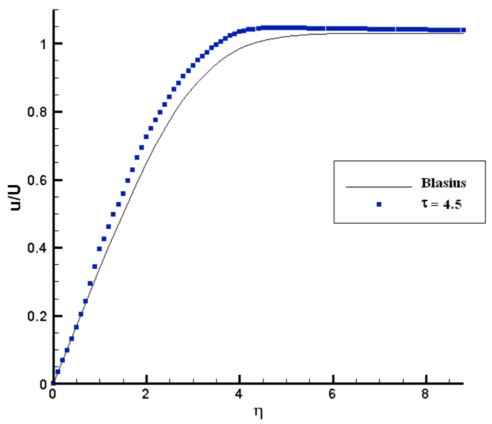

Figure 2. Velocity profile at $x/c = 0.75$.

5. CONCLUSIONS

A new formulation for the wall boundary conditions within the lagrangian vortex method is described and some sample results are presented in this paper. Numerical simulations are carried out for the flow around a flat plate and velocity profiles in the freestream direction at two chordwise stations are compared to Blasius solution. The results show good agreement mainly in the wall region, even for the relatively coarse simulations shown here. The velocity profiles show that the wall boundary conditions are exactly satisfied at the plate and the derivative of the velocity profile, which allows the shear stress at the plate to be calculated, is well calculated.

The results described in this paper indicate that the algorithm has enormous potential to simulate external two-dimensional flows around bluff bodies. The wall model proposed may be extended to three-dimensional flows.

6. REFERENCES

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