FINITE ELEMENT SIMULATION OF A 2-D NON-LINEAR HEAT TRANSFER PROBLEM USING A MINUMUM PRINCIPLE

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Abstract. *This article studies the steady-state conduction-radiation heat transfer in an opaque and convex rigid body at rest. The mechanical model gives rise to a second-order partial differential equation subjected to non-linear boundary conditions, treated as a sequence of linear problems, which are solved by employing the minimization of quadratic functionals, proposed in this work. A finite element method approximates this solution. Some numerical results illustrate the procedure.*

Keywords*: Conduction-radiation, variational principle, finite elements.*

1. INTRODUCTION

Since real bodies are at a temperature above absolute zero, an opaque body immersed in a non-opaque medium emits thermal radiation from its surface. Whatever magnitude it may have, it is present, being caused by the finite temperature of a body. However only in high temperature applications thermal radiation – which is of order of σT^4, in which the Stephan-Boltzmann constant σ is of order 10^{-8} in SI units and T is the absolute temperature – is relevant compared with conduction and convection, which are of the order of ΔT. If the body is rigid and is at rest, the energy transfer process inside this body is heat conduction.

An important application of conduction-radiation problems is the enhancing of thermal insulation. The most largely used insulation materials entrap air as the isolator gas so the thermal conductivity of air, which is about 0.025 W/mK, would set the limit of performance for such materials. Thermal insulation could be enhanced by employing vacuum insulating panels (VIP), based on micro-nano-porous core materials (such as silica powders) under partial air vacuum, since the small size of the pores allows significantly reducing the conductive heat transfer when a primary vacuum is applied. Considering opacified nanoporous medium under partial air vacuum, Rettelbach et al. (1995) verified that equivalent conductivities of only several mW/mK could be reached at ambient temperature for the opacified silica aerogel, Caps and Fricke (2000) and Caps et al (2001) observed that the addition of opacifiant particles gave rise to equivalent conductivities close to 3 mW/mK under partial air vacuum. Enguehard (2005) have modeled the radiative heat transfer in VIP, when thermal radiation is relevant due to the density, observing that in order to reduce the radiative contribution a relatively small amount of particle with suitable sizes (diameter close to 1μm) could be introduced. Rochais, et al. (2005) computed the thermal conductivity and diffusivity of a nanoporous material by employing fractal representation to study transient and steady-state conductive heat transfer.

Different simulation methodologies are employed when thermal radiation is accounted for. The first engineering application of the discrete ordinate method (DOM) was performed by Fiveland (1984) to solve the radiative transport problem in a 2-D square enclosure with cold black boundaries containing isothermal absorbing-emitting medium. Mishr et al. (2011) analyze radiative transport with and without conduction in a finite concentric cylindrical enclosure containing absorbing, emitting, and scattering medium. In all problems, radiative information is computed using the modified discrete ordinate method (MDOM), and, when a conducting-radiating medium is considered, the lattice Boltzmann method (LBM) is used to formulate and solve the energy equation. The authors compare results obtained from finite volume and finite difference methods.

Saldanha da Gama (1992) presented a minimum principle, suitable for a large class of energy transfer problems, to study the energy transfer phenomenon in a rigid and opaque body exchanging energy on its boundary by convection and by diffuse thermal radiation, considering a constant conductivity. Saldanha da Gama (1996) studied the conduction-radiation energy transfer process in a rigid, opaque, nonconvex and black cylindrical body surrounded by an atmosphere-free space that gives rise to a partial differential equation subjected to a nonlinear boundary condition representing the conduction-radiation coupling on a body boundary, considering a constant conductivity. The problem is solved as the limit of a sequence whose elements (approximated by finite elements) are obtained from the minimization of a functional. Saldanha da Gama (2004) analyzed the conduction-radiation energy transfer process (considering a constant conductivity) in an opaque convex body with radiation properties depending on the wavelength, in which a nonlinear boundary condition involves Planck's law. An adequate description is obtained by the proposition of a modified version of Planck's law and the construction of a minimum principle, proving existence and uniqueness of the solution.

Although conduction-radiation heat transfer is a very frequent process, there is no systematic procedure for its simulation. Assuming that the body can receive thermal radiation from external sources and can dissipate energy internally (eg due to the passage of an electric current), the steady-state conduction-radiation heat transfer may be represented as (Siegel and Howell, 2002; Slattery, 1999)

$$\nabla\cdot(k\,\nabla T)+\dot{q}=0 \qquad \text{in} \quad \Omega \tag{1}$$

$$-k\,\nabla T\cdot\mathbf{n}=\sigma\left|T\right|^{3}T-\gamma \quad \text{on} \quad \partial\Omega \tag{2}$$

where $k = k\,(T)$ is the thermal conductivity, $\dot{q}$ is the rate of heat generation per unit volume, σ is the Stefan-Boltzmann constant, γ is a known non-negative function (representing the external thermal radiation sources effect), $\mathbf{n}$ is the unit outward nornal and T is the unknown absolute temperature.

Equation (1) is the conservation of energy within the body Ω, while equation (2) represents the coupling between the heat conduction and thermal radiation on the boundary of the body $\partial\Omega$. It is important to note that this is an inherently nonlinear problem.

Also, in equation (2) instead of T^4, $|T|^3T$ is used, ensuring the existence and the uniqueness of the solution and preserving the physical meaning of the mathematical description. This non-linear term is frequently approximated by linearized expressions. Sometimes, iterative procedures are used to estimate its value. In this work the original problem is treated exactly, eliminating physically inconsistent values for the temperature field, such as negative and complex ones.

2. VARIATIONAL FORMULATION

This article studies the heat transfer process inside the body Ω, considering a temperature-depent thermal conductivity k and a constant heat generation $\dot{q}$. Its main contribution is the simulation of a non-linear system as a sequence of linear problems, and the proposal of a class of functionals. The limit of the sequence is achieved by the minimization of the functionals and represents the exact solution. The non-linear system presented in Eqs. (1)-(2) is regarded as the limit of a sequence of linear problems, which are solved by means of the minimization of the following quadratic functional

$$I_n\left[u\right]=\int_\Omega \left(\nabla u\right)\cdot\left(\nabla u\right)dV-2\int_\Omega \dot{q}u\,dV+\int_{\partial\Omega}\left(\alpha u^2-2\beta_n u\right)dS \tag{3}$$

where α is a positive constant sufficiently large to assure that the field ω_n that minimizes the functional I_n is smaller than or equal to the field ω_{n+1} that minimizes I_{n+1} . In other words,

$$\omega_{n+1}\geq\omega_n \quad in\ \bar{\Omega} \tag{4}$$

The field β_n is defined as follows

$$\beta_n=\alpha T_{n-1}-\sigma\left|T_{n-1}\right|^{3}T_{n-1}+\gamma \tag{5}$$

in which the fields T_n are obtained from the inverse of a Kirchhoff transform. Here $T_0\equiv 0$.

The Kirchhoff transform may be defined in terms of the variable ω as follows (Arpaci, 1966)

$$\omega=\int_{T_0}^{T}\hat{k}\left(\xi\right)d\xi=\hat{f}(T) \quad \text{so that} \quad T=\hat{f}^{-1}(\omega) \tag{6}$$

The field ω_n, which minimizes the functional defined in Eq. (3), is the solution of the classical conduction-convection heat transfer problem given by

$$\begin{aligned} &\Delta\omega_n + \dot{q} = 0 \qquad \text{in} \quad \Omega \\ &-\nabla\omega_n \cdot \mathbf{n} = h\left(\omega_n - \omega_n^\infty\right) = \alpha\omega_n - \beta_n \quad \text{on} \quad \partial\Omega \end{aligned} \tag{7}$$

where the constant $h=\alpha$ represents the convection heat exchange such that $\omega_n^\infty = \beta_n / h$, in which the function ω_n^∞ represents a temperature outside the body Ω, an environmental temperature.

The solution of the original problem, the field T, is given by

$$T = \lim_{n\to\infty} T_n = \lim_{n\to\infty} \hat{f}^{-1}(\omega_n) \quad \text{in} \quad \Omega \tag{8}$$

The problem consists of finding the elements of the sequence ω_n that minimize the functional I_n, defined in Eq. (3). The temperature, which is the solution of the original problem, stated in Eqs. (1)-(2), is obtained by Eq. (8).

In other words, the exact solution construction presented in this article induces a scheme for approximating the solution by means of an approximation procedure, which could employ a finite elements methodology, for example.

The scheme for constructing the solution may be illustrated as follows:

Step 1 → find ω_1 from the minimization of $I_1\left[u\right] = \int_\Omega \left(\nabla u\right)\cdot\left(\nabla u\right) dV - 2\int_\Omega \dot{q}u\, dV + \int_{\partial\Omega}\left(\alpha u^2\right) dS$.

Step 2 → evaluate T_1 from $\omega_1 = \int_{T_0}^{T_1} \hat{k}\left(\xi\right) d\xi$, considering a previously chosen T_0.

Step 3 → evaluate β_2 from $\beta_2 = \alpha T_1 - \sigma\left|T_1\right|^3 T_1 + \gamma$

Step 4 → find ω_2 from the minimization of $I_2\left[u\right] = \int_\Omega \left(\nabla u\right)\cdot\left(\nabla u\right) dV - 2\int_\Omega \dot{q}u\, dV + \int_{\partial\Omega}\left(\alpha u^2 - 2\beta_2 u\right) dS$

Step 5 → evaluate T_2 from $\omega_2 = \int_{T_0}^{T_2} \hat{k}\left(\xi\right) d\xi$, considering a previously chosen T_0.

Step 6 → evaluate β_3 from $\beta_3 = \alpha T_2 - \sigma\left|T_2\right|^3 T_2 + \gamma$

....

Step N → find ω_n from the minimization of $I_n\left[u\right] = \int_\Omega \left(\nabla u\right)\cdot\left(\nabla u\right) dV - 2\int_\Omega \dot{q}u\, dV + \int_{\partial\Omega}\left(\alpha u^2 - 2\beta_n u\right) dS$

Step N+1 → evaluate T_n from $\omega_n = \int_{T_0}^{T_n} \hat{k}\left(\xi\right) d\xi$, considering a previously chosen T_0.

Step N+2 → evaluate β_{n+1} from $\beta_{n+1} = \alpha T_n - \sigma\left|T_n\right|^3 T_n + \gamma$.

3. SIMULATION PROCEDURE

In order to illustrate the previously described procedure, the following problem is considered: solve the system (1)-(2) and find the temperature distribution $T = \hat{T}\left(x, y\right)$ in the two-dimensional domain, defined as

$$\Omega \equiv \left\{\left(x, y\right) \text{ such that } 0 < x < L \text{ and } 0 < y < H\right\} \tag{9}$$

The numerical methodology consists of employing classical triangular element meshes to compute the functional by a finite element methodology (Bathe, 1986). Since the functional I is convex and coercive (Jost and Li-Jost, 1998), there exists a unique solution of $\sum_{i=1}^{N} \partial I / \partial u_i = 0$ (Rudin, 1991) that minimizes the functional, where u_i is the approximation for the Kirchhoff transform at each node. The employed mesh, for a unitary square geometry, is composed by $2\text{x}100^2$ triangular elements.

The associated linear sparse system, arising from $\partial I / \partial u_i = 0$, with $i = (1, 100^2)$, may be expressed as

$$\left(\frac{k}{4} A + \frac{\alpha}{12} B\right)\cdot u = F \tag{10}$$

in which A and B are $n^2 \times n^2$ dimension matrixes in $n \times n$ blocks and F is a $n^2 \times 1$ vector. The system of linear equations gives rise to a positive-definite symmetric matrix. So, it may be solved by employing Cholesky decomposition (van der Vorst, 2003).

4. RESULTS

The temperature distribution for a 100×100 finite element mesh is shown in figures 1 to 3, each figure representing a distinct iteration for the field T_n, which approximates the functions for each n, for the domain Ω. The numerical simulations, performed by employing a *Matlab* numerical code (Matlab, 2008) were validated by the comparison of the total energy generation with the conduction energy flux through the boundary of the body. Problem 1 considers a constant heat source while problem 2 considers a heat source depending on position. In problem 1 the absolute difference between these two cases is smaller than 10^{-12}, indicating that the result of the sequence remains unchanged after its third element. In problem 2 the result of the sequence remains unchanged after its fourth element, since the absolute difference between these two cases is smaller than 10^{-10}.

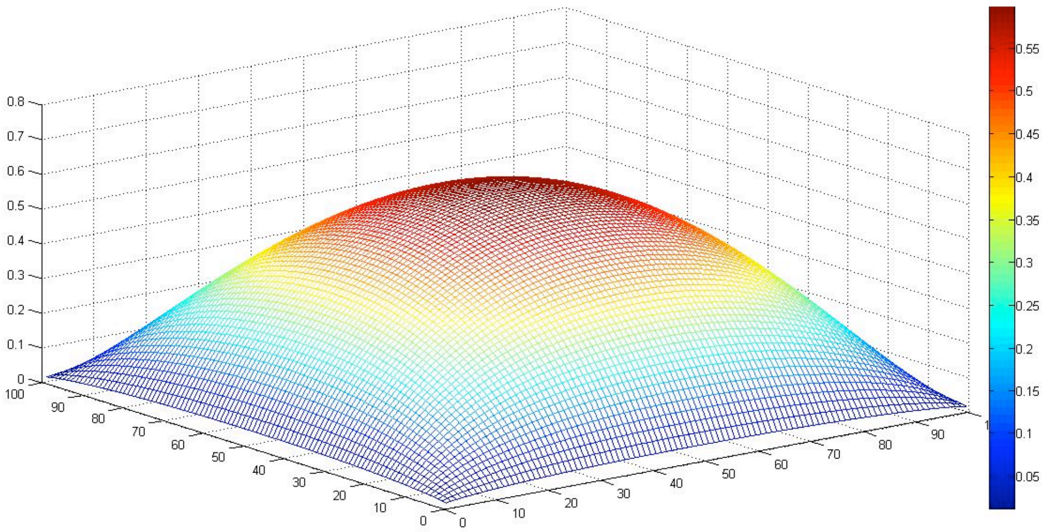

Figure 1. Temperature distribution: first element of the sequence of problem 1.

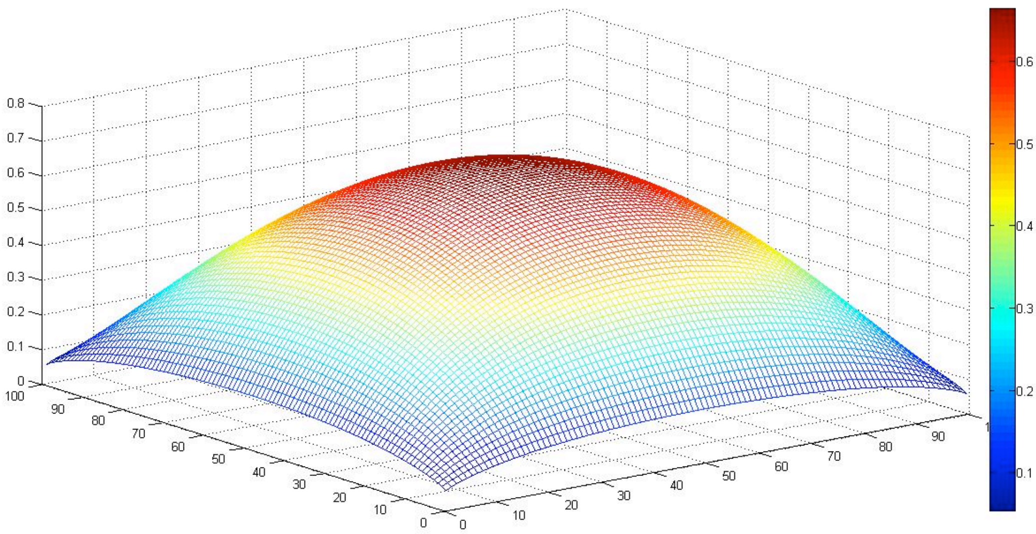

Figure 2. Temperature distribution: second element of the sequence of problem 1.

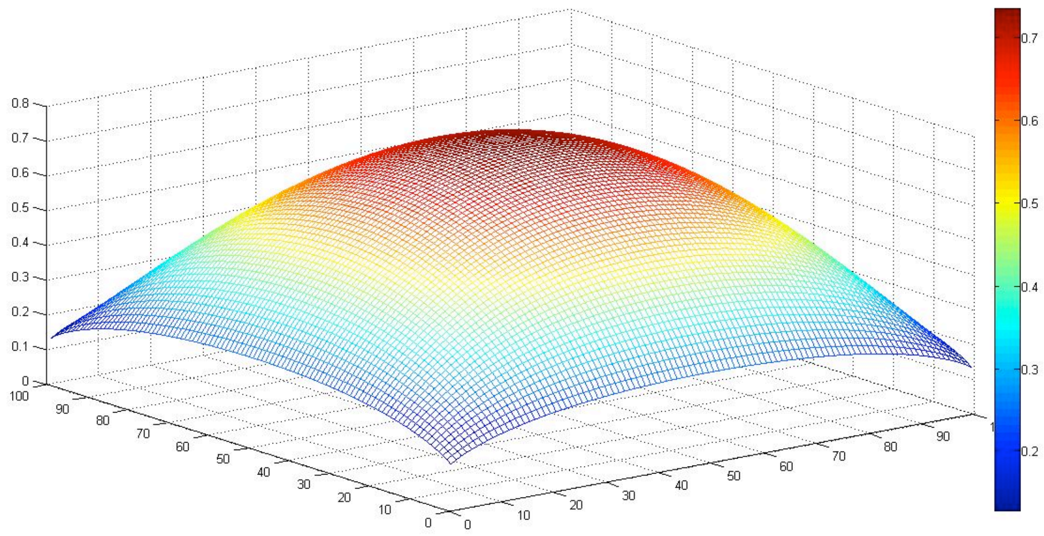

Figure 3. Temperature distribution: third element of the sequence of problem 1.

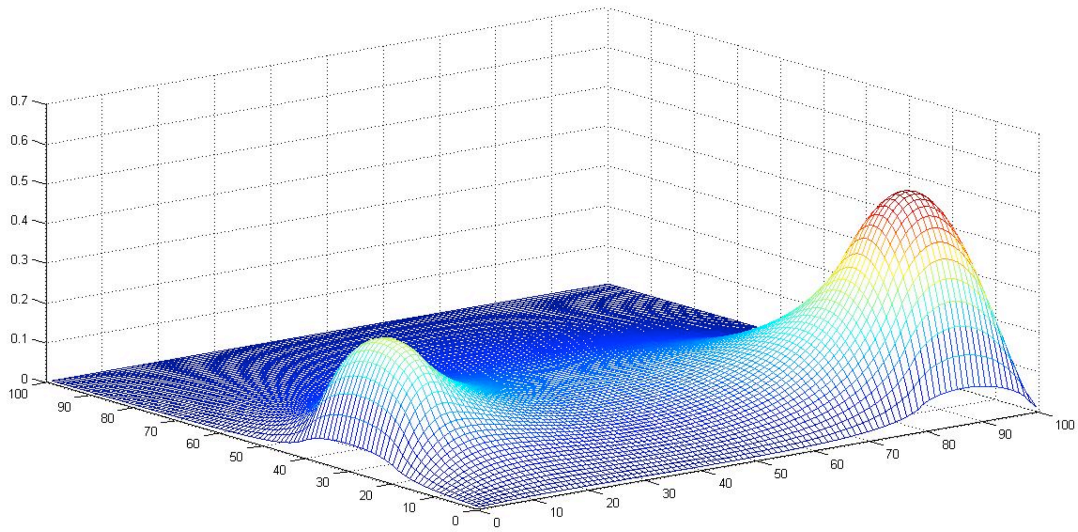

Figure 4. Temperature distribution: first element of the sequence of problem 2.

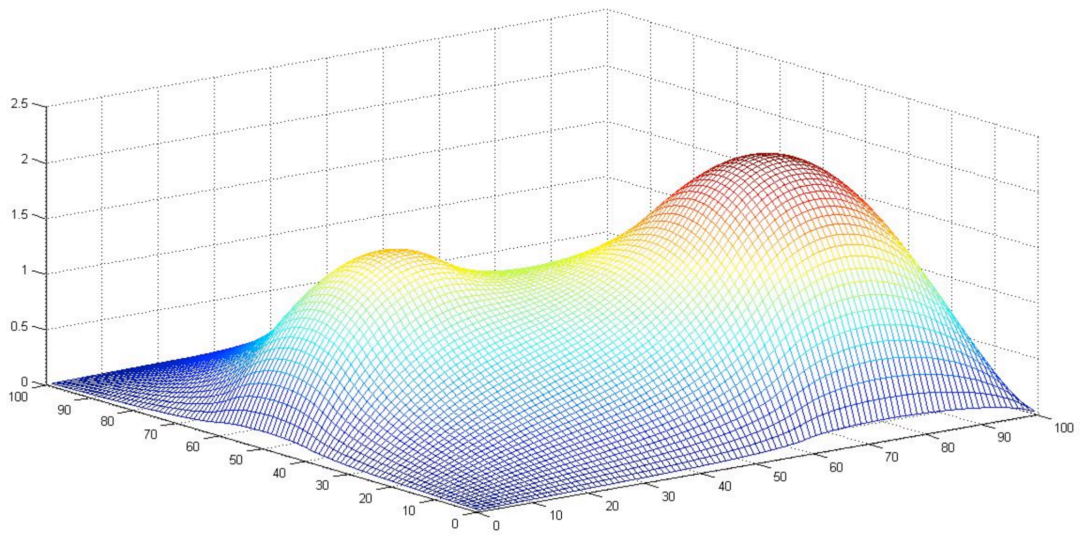

Figure 5. Temperature distribution: second element of the sequence of problem 2.

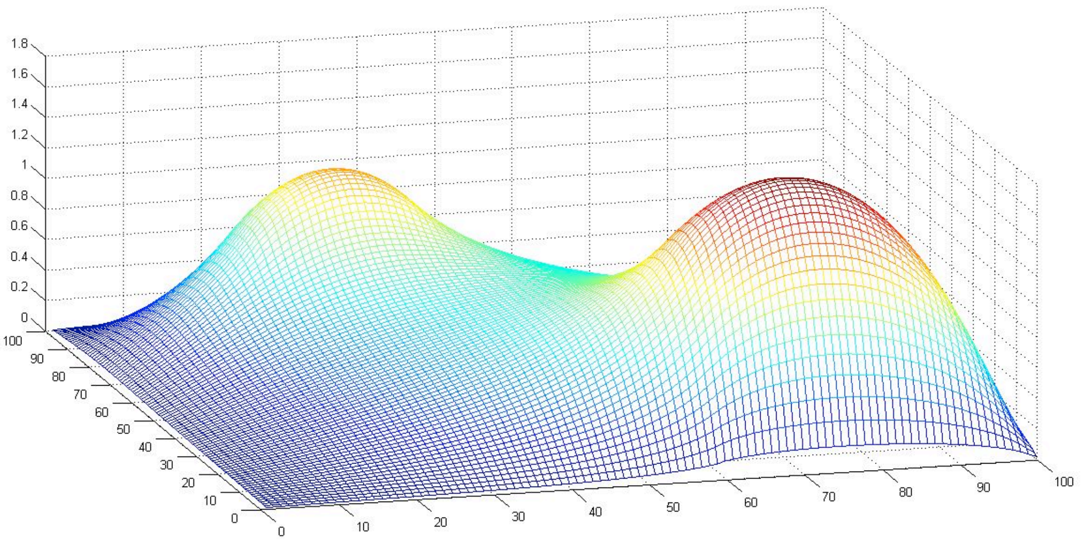

Figure 6. Temperature distribution: third element of the sequence of problem 2.

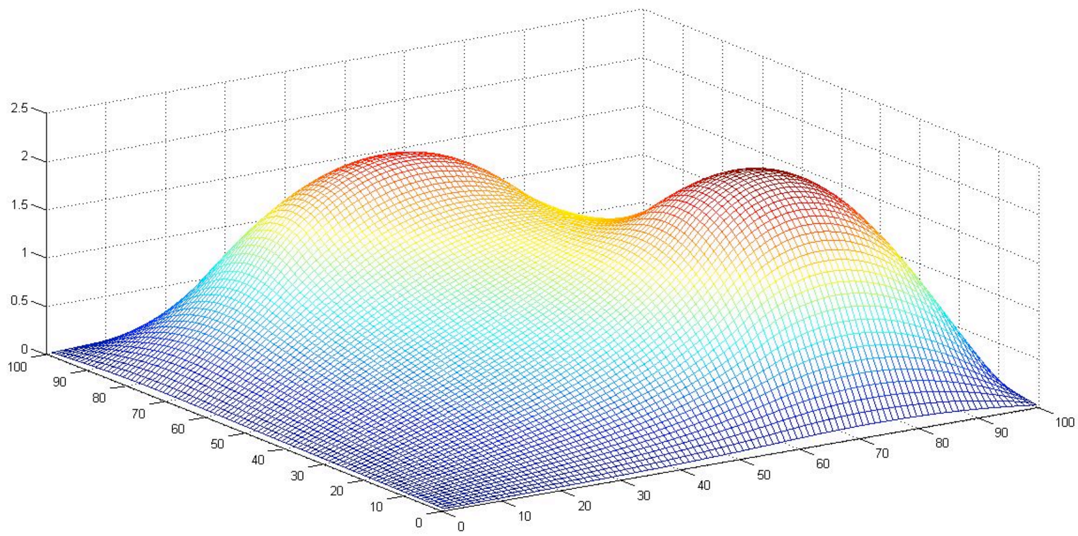

Figure 7. Temperature distribution: fourth element of the sequence of problem 2.

5. FINAL REMARKS

This work presents a systematic procedure for the numerical simulation of a non-linear problem. The procedure employs an infinite-dimension result in a finite-dimension context (since the finite element methodology was employed). Basically the solution of a non-linear problem is obtained by the solution of a sequence of linear problems, which is achieved through the minimization of quadratic functionals. This simple and effective tool can be extended to other problems.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

Arpaci, V. S., 1966, *Conduction Heat Transfer*, Prentice-Hall, Addison-Wesley.

Bathe, K., 1996, *Finite Element Procedures*, Prentice Hall.

Caps, R. and Fricke, J. 2000, "Thermal Conductivity of Opacified Powder Filler Materials for Vacuum Insulations", *International Journal of Thermophysics*, vol. 21/2, pp. 445–452.

Caps, R., Heinemann, U. Ehrmanntraut, M. and Fricke, J., 2001, "Evacuated insulation panels filled with pyrogenic silica powders: properties and applications", *High Temperatures-High Pressures*, vol. 33, pp. 151-156.

Enguehard F., 2005, "Multi-scale modeling of radiation heat transfer through nanoporous superinsulating materials", *Proceedings of the European Conference on Thermophysical Pr*operties, Bratislava, Slovak Republic.

Fiveland, W. A., 1984, "Discrete-ordinates solution of the radiative transport equation for rectangular enclosures" *J. Heat Transfer*, vol. 106, pp. 699–706

Jost, J. and Li-Jost, X.,1998, *Calculus of Variations*, Cambridge University Press.

Mishr, S. C., Krishna, Ch. H. and Kim, M. Y., 2011, "Analysis of conduction and radiation heat transfer in a 2-D cylindrical medium using the modified discrete ordinate method and the lattice Boltzmann method", *Numerical Heat Transfer, Part A*, vol. 60, pp. 254–287.

MATLAB version 7.6.0.324, 2008. Natick, Massachusetts: The MathWorks Inc.

Rettelbach, T., Sauberlich, J., Korder, S. and Fricke, J., 1995, "Thermal conductivity of IR-opacified silica aerogel powders between 10 K and 275 K", *J. Phys. D: Appl. Phys.,* vol. 28/ 3, pp 581-587.

Rochais, D., Domingues, G. and Enguehard, F., 2005, "Numerical simulation of thermal conduction and diffusion through nanoporous superinsulating materials", *Proceedings of the European Conference on Thermophysical Pr*operties, Bratislava, Slovak Republic.

Rudin, W., 1991, *Functional Analysis*, McGraw-Hill.

Saldanha da Gama, R. M., 1992, "Simulation of the steady-state energy transfer in rigid bodies, with convective-radiative boundary conditions, employing a minimum principle", *J. Comput. Physics*, vol. 99, pp. 310-320.
Saldanha da Gama, R. M., 1996, "Numerical simulation of the (nonlinear) conduction-radiation heat transfer process in a nonconvex and black cylindrical body", *J. Comput. Physics*, vol. 128, pp. 341-350.
Saldanha da Gama, R. M., 2004, "On the conduction/radiation heat transfer problem in a body with wavelength-dependent properties", Appl. Math. Model., vol. 28, pp. 795–816.
Siegel, R. and Howell, J., 2002, *Thermal Radiation Heat Transfer*, Taylor and Francis.
Slattery, J.C. 1999, *Advanced Transport Phenomena*, Cambridge University Press.
van der Vorst, H. A., 2003, *Iterative Krylov Methods for Large Linear Systems*, Cambridge University Press.

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