NUMERICAL SIMULATION OF POWER-LAW FLUID FLOWS IN A POROUS CHANNEL

Jesús Alfonso Puente Angulo, jpuente@id.uff.br
Maria Laura Martins-Costa, laura@mec.uff.br
Heraldo da Costa Mattos, heraldo@mec.uff.br
Laboratory of Theoretical and Applied Mechanics (LMTA), Mechanical Engineering Graduate Program, Universidade Federal Fluminense, Rua Passo da Pátria 225, 24210-240, Niterói, RJ, Brazil

Abstract. *A porous channel limited by two impermeable flat plates is modeled by employing a mixture theory that treats fluid saturated porous media by considering the fluid and the porous matrix as superimposed continuous constituents of a binary mixture - each of them occupying its whole volume. The flow of a power-law fluid through a porous channel, for both shear-thinning and shear-thickening responses is simulated using a Runge-Kutta method coupled with a shooting strategy. Despite the strong nonlinearity of the problem, the methodology provides stable and accurate results.*

Keywords*: Mixture theory, power-law fluid, fluid saturated porous medium.*

1. INTRODUCTION

In this work the steady-state flow of an incompressible generalized Newtonian fluid through a plane saturated porous channel (rigid solid matrix) is modeled by a mixture theory approach. The Mixtures Theory generalizes the classical Continuum Mechanics and has been specially developed to describe multiphase phenomena. Fluid saturated porous media are modeled as superimposed continuous constituents (the fluid and the porous matrix) of a chemically non-reacting binary mixture - each of them occupying its whole volume. This approach is distinct from the widespread volume-averaging technique, discussed by Whitaker (1969), that has been successfully employed to describe most transport phenomena in porous media (see Alazmi and Vafai, 2000 and references therein).

The governing equations – namely solve mass and momentum conservation equations for the fluid constituent (to model an isothermal flow of a fluid through a saturated rigid porous matrix) coupled with constitutive assumption for the partial stress tensor (analogous to Cauchy stress tensor) and a momentum source to account for the momentum interaction between both constituents of the mixture – give give rise to a non-linear two-point boundary-value problem in ordinary differential equations. Such a problem can be numerically approximated using a Runge-Kutta method coupled to a shooting technique. This latter consists of an iterative algorithm, which attempts to identify appropriate initial conditions for a related initial value problem that provides the solution to the original boundary value problem.

2. MECHANICAL MODEL AND NUMERICAL SIMULATION

The mechanical model combines mass and momentum balance equations for the fluid constituent, since the solid constituent (porous matrix) is rigid and at rest with constitutive assumptions for the power-law fluid constituent. The balance equations (Atkin and Craine, 1976; Rajagopal and Tao, 1995) are given by

$$\begin{aligned} &\frac{\partial \rho_F}{\partial t} + \nabla\cdot(\rho_F \mathbf{v}_F) = 0 \\ &\rho_F\left[\frac{\partial \mathbf{v}_F}{\partial t} + \left(\nabla \mathbf{v}_F\right)\mathbf{v}_F\right] = \nabla\cdot\mathbf{T}_F + \mathbf{m}_F + \rho_F \mathbf{g} \end{aligned} \tag{1}$$

in which $\rho_F = \varphi\rho$ is the fluid constituent mass density (with φ representing the fluid fraction and ρ the actual fluid density) $\mathbf{v}_F$ its velocity, $\mathbf{T}_F$ is the partial stress tensor associated with the fluid constituent and $\mathbf{m}_F$ is an interaction force per unit volume acting on the fluid constituent due to its interaction with the other constituent of the mixture.

Considering a power-law fluid, Cauchy tensor may be stated as $\mathbf{T} = -p\mathbf{I} + 2\eta(\mathbf{D}\cdot\mathbf{D})^n\mathbf{D}$ (Bird et al., 1987, Tanner, 2000), in which p is the hydrostatic pressure acting on the fluid, η and n are the power-law rheological parameters that characterize the fluid behavior and $\mathbf{D}$ is the strain rate tensor acting on the fluid. It is important to note that the usual power-law equation, given by $\boldsymbol{\tau} = 2\kappa\left(\dot{\gamma}\right)^{m-1}\mathbf{D}$ (Slattery, 1999), in which κ is a consistency index and m a power-law index, could be recovered, making $\eta = 2^{(m-1)/2}\kappa$ and $n = (m-1)/2$. Considering the above-stated Cauchy tensor, the partial stress tensor and the momentum source, are given by (Martins-Costa et al., 2000; Costa Mattos et al. 1995)

$$\mathbf{T}_F = -p\varphi\mathbf{I} + 2\varphi\,\beta(\mathbf{D}_F\cdot\mathbf{D}_F)^n\mathbf{D}_F \quad \text{with } \beta{=}\varphi\eta$$
$$\mathbf{m}_F = -\varphi\,\alpha\left\|\mathbf{v}_F\right\|^{2n}\mathbf{v}_F, \quad \text{with } \alpha = \hat{\alpha}\left(\varphi,\eta,K,n\right) = \frac{\varphi\,\eta}{3K}\left(\frac{4n+3}{2n+1}\right)^{2n+1}\left(\frac{\varphi}{6K}\right)^{n} \tag{2}$$

Assuming a steady-state flow of an incompressible fluid constituent though a porous channel with height $2H$, equations (1)-(2) and the no-slip boundary condition give rise to

$$\begin{aligned} &\nabla\cdot\mathbf{v}_F = 0 \\ &\nabla\cdot\left[-p\varphi\mathbf{I} + 2\varphi\,\beta(\mathbf{D}_F\cdot\mathbf{D}_F)^n\mathbf{D}_F\right] - \varphi\alpha\left\|\mathbf{v}_F\right\|^{2n}\mathbf{v}_F + \rho_F\mathbf{g} = 0 \\ &\mathbf{v}_F = 0 \qquad\qquad \text{on } y = \pm H \end{aligned} \tag{3}$$

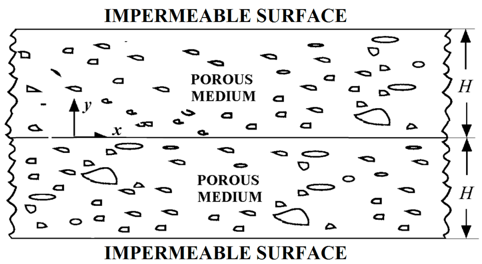


Figure 1. Flow through a plane porous channel.

Neglecting gravitational effects and making $\mathbf{v}_F{=}v_F\,\mathbf{i}$ and $v_F = w$ in Eq. (3), for the geometry depicted in Fig. 1, the fully developed steady-state flow may be expressed as

$$\left.\begin{aligned} &-\frac{dp}{dx} + \frac{2n+1}{2^{2n}}\beta\left|\frac{dw}{dy}\right|^{2n}\frac{d^2w}{dy^2} - \alpha\left|w\right|^{2n}w = 0 \qquad -H \le y < +H \\ &w = 0 \qquad\qquad \text{at} \quad y = \pm H \end{aligned}\right\rangle \Rightarrow \; w_{\max} = \left(\frac{-1}{\alpha}\frac{dp}{dx}\right)^{\frac{1}{2n+1}} \tag{4}$$

where $w_{\max}$ represents the maximum value of w.

Equations (4) form as two-point boundary value problem; which can be approximated by using a fourth-order Runge-Kutta methodology coupled with a shooting technique, described below. The first equation of Eq. (4) may be conveniently rewritten by considering the convenient variables redefinition $z_1 = w$ and $z_2 = dw/dy$ as $\frac{dp}{dx} = \frac{2n+1}{2^{2n}}\beta\left|z_2\right|^{2n}\frac{dz_2}{dy} - \alpha\left|z_1\right|^{2n}z_1$, giving rise to the following system of first order differential equations

$$\frac{dz_2}{dy} = \frac{2^{2n}}{\left(2n+1\right)\beta}\left|z_2\right|^{-2n}\left(\frac{dp}{dx} + \alpha\left|z_1\right|^{2n}z_1\right) \qquad \text{and} \qquad \frac{dz_1}{dy} = z_2 \tag{5}$$

The following boundary-value problem approximates the velocity profile at the porous channel: *Find* $z_1 : [-H,+H] \to \mathbb{R}$ *and* $z_2 : [-H,+H] \to \mathbb{R}$*, such that*

$$\begin{aligned} &\frac{dz_2}{dy} = \frac{2^{2n}}{\left(2n+1\right)\beta}\left|z_2\right|^{-2n}\left(\frac{dp}{dx} + \alpha\left|z_1\right|^{2n}z_1\right) \\ &\frac{dz_1}{dy} = z_2 \end{aligned} \qquad \text{with} \quad \begin{cases} z_1 = 0 & \text{at} \quad y = -H \\ z_1 = 0 & \text{at} \quad y = +H \end{cases} \tag{6}$$

The problem stated in Eq. (6) is equivalent to finding the root of a scalar function represented as $\Phi : \mathbb{R} \to \mathbb{R}$; $t \to \Phi(t) = z_1(y = +H; t)$, where for a given $t \in \mathbb{R}$, representing an initial estimate, the value $\Phi(t)$ is the value of the variable z_1 at point $y{=}{+}H$, obtained by solving the following initial boundary value problem

$$\frac{dz_2}{dy} = \frac{2^{2n}}{(2n+1)\beta}\left|z_2\right|^{-2n}\left(\frac{dp}{dx} + \alpha\left|z_1\right|^{2n} z_1\right) \qquad \text{for } -H \le y < +H \text{ ; such that } \begin{cases} z_1 = 0 & \text{at} \quad y = -H \\ z_2 = t & \text{at} \quad y = -H \end{cases} \tag{7}$$
$$\frac{dz_1}{dy} = z_2$$

Essentially, this procedure is a shooting technique in which t represents the initial estimate of the derivative (dw/dy) at the point $y = -H$. The initial boundary value problem is approximated by a Runge-Kutta technique (Dahlquist and Bjorc, 1969) and the root of the function $\Phi(t)$ is obtained by an unconditionally convergent procedure, the Bisection method (Dahlquist and Bjorc, 1969). It is important to remark that the above-proposed change of variables is only adequate when $z_2 \neq 0$.

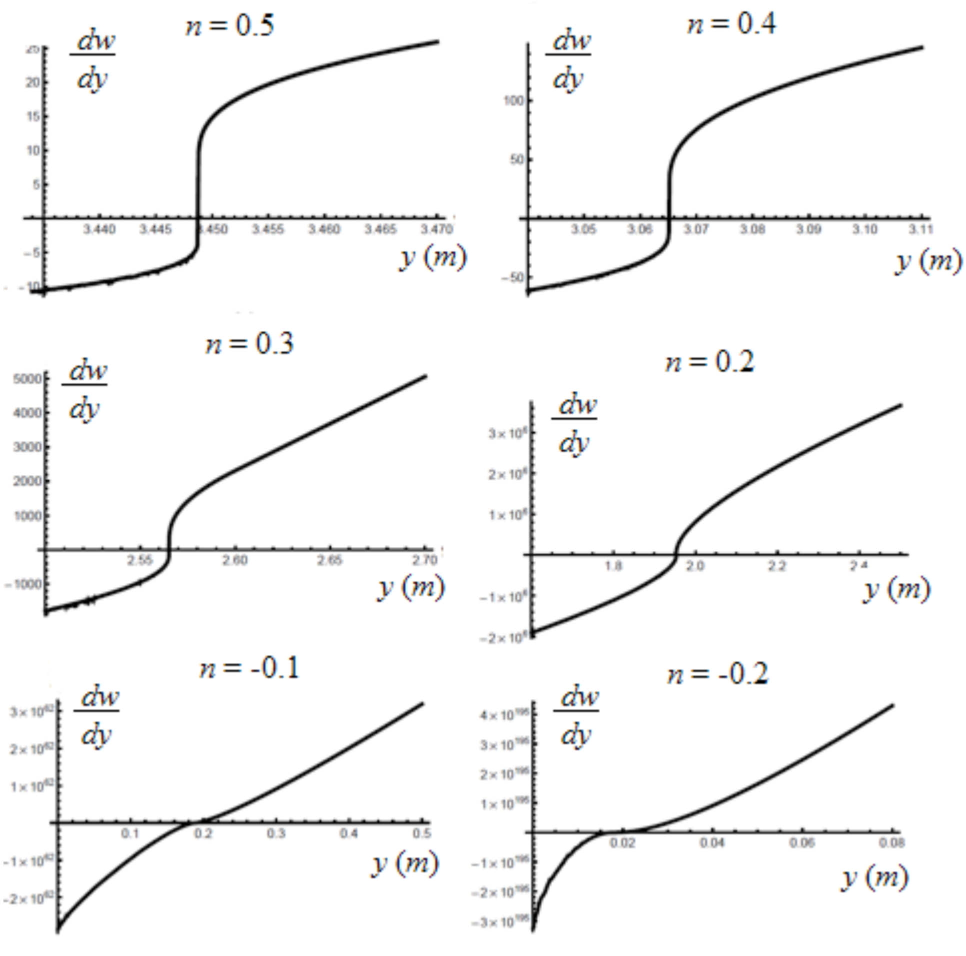


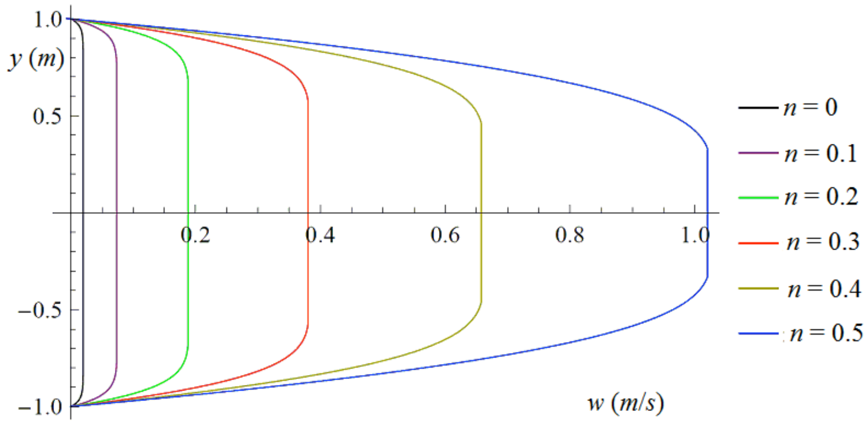
Figure 2. Behavior of the function $\Phi(t)$ for distinct values of power-law index.

It is important to note that $w_{\max}$ occurs exactly when $z_2 \neq 0$, giving rise to numerical instabilities in a neighborhood of y=0, in the process of searching for the root of the function $\Phi(t)$. Taking advantage of the symmetry, this problem may be circumvented, but in order to find the roots an unconditionally convergent methodology is required, due to the behavior of the function $\Phi(t)$, depicted in Figure 2. It is important to observe that distinct scales have been employed in this figure. The nonlinear nature of the function $\Phi(t)$ implies that a small variation of one parameter (y, for n>0 or dw/dy, for n<0) in the neighborhood of the root causes a huge variation of the other one.

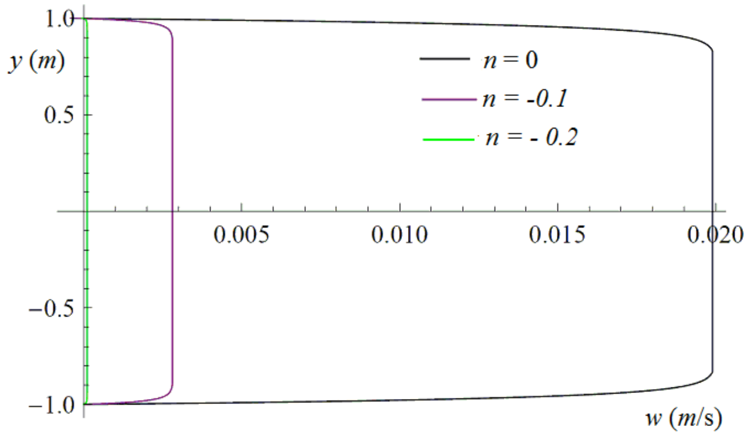
4. RESULTS

Figure 3 shows numerical results for the fluid constituent velocity profile, obtained varying the power-law index *n* (from *n*=-0.2 to *n*=0.5) and considering the flow depicted in Fig. 1 with the following parameters: $dp/dx = 10^{-2}$ Pa/m, $\eta = 10^{-3}$ Pa.s^n, $\varphi = 0.5$, $\lambda = 1$, $\beta = \varphi\eta = 0.5\times10^{-3}$ Pa.s^n and $K = 10^{-3}$ m^{-2}. It may be noted that the velocity profile becomes flatter as *n* decreases, in which there is shear-thinning behavior for *n*<0, shear-thickening for *n*>0 and Newtonian for *n*=0. When strongly shear-thickening behavior is verified, as depicted in figure 3a for *n*=0.5, the velocity profile tends to a parabolic profile for $y > 0.4$ or $-0.4 < y < 1.0$, but even in this case, a flat profile is verified for $-0.4 \le y \le 0.4$.

Actually, although distinct scales in the horizontal axis have been employed in figures 3a and 3b, it may be noted that the velocity profile is almost zero for $n = -0.2$, as depicted in figure 3b, in which it may be observed that $w_{\max} \to 0$ for $n < -0.1$, as confirmed by the graph depicted in figure 4.



(a)



(b)

Figure 3. Velocity profiles for: (a) shear-thickening and (b) shear-thinning behavior.

The problem described by Eq. (4) could be scaled, giving rise to:

$$\frac{dp^*}{dx^*} = \frac{1}{2^{2n}}\frac{d}{dy^*}\left(\left|\frac{dv^*}{dy^*}\right|^{2n}\frac{dv^*}{dy^*}\right) - \frac{\alpha H^{2n+2}}{\beta}\left|w^*\right|^{2n} w^* \tag{8}$$

where: $y^* = \dfrac{y}{H}$, $x^* = \dfrac{x}{H}$, $w^* = \dfrac{w}{w_{\max}}$, $p^* = \dfrac{p}{\beta\left(w_{\max}/H\right)^{2n+1}}$

Figure 4 depicts the behavior of the maximum velocity (in *m/s*) for distinct values of *n*, obtained with the same material parameters employed in figure 3, showing that it approaches from zero for *n*<-0.1, as confirmed in figure 3. This behavior is caused by the shear-thinning feature of the fluid combined with the interaction between the porous matrix and the fluid constituent.

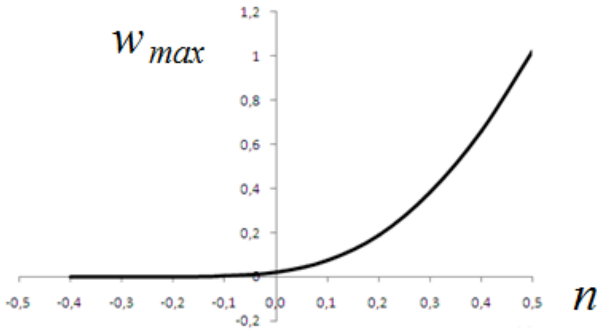
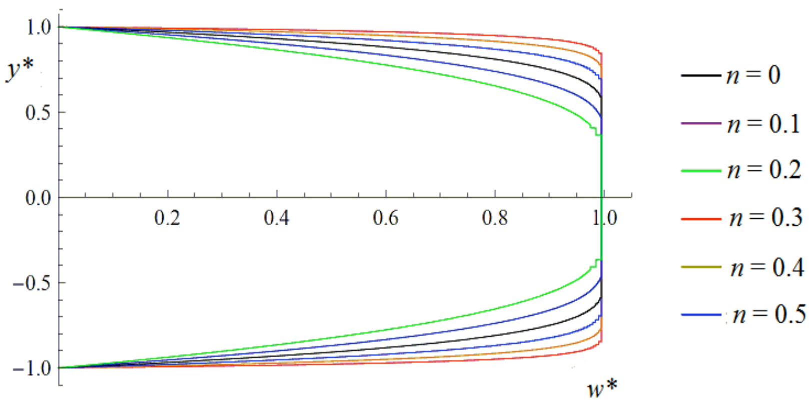
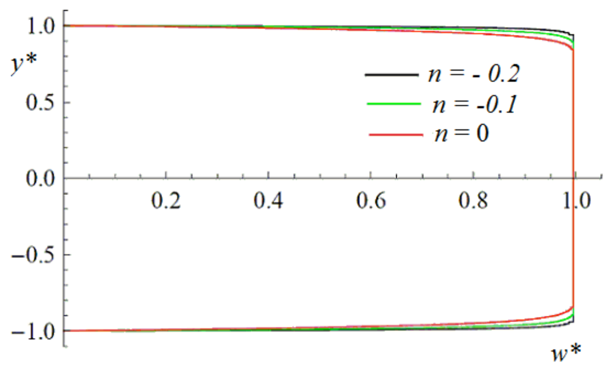


Figure 4. Behavior of maximum velocity

The dimensionless velocity profiles, defined by Eq. (8) are presented in figure 5. It may be clearly observed that the velocity profiles become flatter as n decreases, as expected for this model. Actually flat velocity profiles are verified for all considered values of the power-law index and for $n \le 0$ (corresponding to Newtonian and shear-thinning behavior) the velocity is almost constant except for a very thin boundary layer. In these cases the velocity profiles are almost coincident.



(a)



(b)

Figure 5. Dimensionless Velocity Profiles

4. FINAL REMARKSS

This work studies the flow of a power-law fluid through a porous channel limited by two impermeable flat plates employing a mixture theory approach. The numerical simulations were performed combining a fourth-order Runge-Kutta method with a shooting strategy, which proved to be adequate to simulate this kind of problem.

Despite the nonlinear behavior of the function $\Phi(t)$ depicted in figure 2, the numerical methodology is able to produce accurate results for the problem – see Martins-Costa et al. (2011) in which this numerical strategy is compared with some analytical solutions.

5. ACKNOWLEDGEMENTS

The author J.A. Puente A. acknowledges Brazilian agency CAPES for scholarship and the authors M.L. Martins-Costa and H. da Costa Mattos acknowledge Brazilian agency CNPq for financial support.

6. REFERENCES

Alazmi, B. and Vafai, K., 2000, "Analysis of variants within the porous media transport models", *J. Heat Transfer*, Vol. 122, pp. 303-326.

Atkin, R.J. and Craine, R.E., 1976, "Continuum theories of mixtures. Basic theory and historical development", *Quart. J. Mech. Appl. Math*., Vol. 29, pp. 209-244.

Bird, R. B. Armstrong, R. C. Hassager, O., 1987, *Dynamics of Polymeric Liquids*. v.1, John Wiley & Sons, U.S.A.

Costa Mattos, H., Martins-Costa, M.L. and Saldanha da Gama, R.M., 1995, "On the modelling of momentum and energy transfer in incompressible mixtures", *Int. J. Non-Linear Mech*., Vol. 30, No. 4, pp. 419-431.

Dahlquist, G. and Bjorck, A., 1969, *Numerical Methods*, Prentice-Hall, Englewood Cliffs.

Martins-Costa, M.L., Saldanha da Gama, R.M. and Frey, S., 2000, "Modelling of a generalized Newtonian flow through channels with permeable wall", *Mech. Research Comm.,* Vol. 27, No. 6, pp. 707-712.

Martins-Costa, M.L., Puente Ângulo, J.A. and da Costa Mattos, H., 2011, "Generalized Newtonian fluid flows in ducts with permeable walls", *Proceedings of 21st Brazilian Congress of Mechanical Engineering*, Natal, RN, Brazil.

Slattery, J.C., 1999, *Advanced Transport Phenomena*, Cambridge University Press, USA.

Rajagopal, K. R. and Tao, L, 1995, Mechanics of Mixtures, Series on Advances in Mathematics for Applied Sciences, Vol. 35, World Scientific, Singapore.

Whitaker, S., 1969, *Advances in theory of fluid motion in porous media*, Ind. Engng. Chem., Vol. 61, pp. 14-28.

Tanner, R. I., 2000, *Engineering Rheology*, 2nd ed., Oxford Press, Oxford, UK.

5. RESPONSIBILITY NOTICE