ANALYSIS OF LAMINAR FLOW OF NON-NEWTONIAN FLUIDS IN ECCENTRIC ANNULAR DUCTS USING THE BIPOLAR COORDINATE SYSTEM

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Abstract. *In this problem a hybrid numerical-analytical solution based on the Generalized Integral Transform Technique (GITT) is obtained for the hydrodynamically fully developed and thermally developing flows in annular ducts for non-Newtonian fluids that follow the power-law rheological model. In this paper, it is employed the bipolar coordinate system to map the eccentric annular duct. Therefore, results for the product of the friction factor by the Reynolds number were compared with those of a previous contribution and shown excellent agreement.*

Keywords: Annular ducts, Non-Newtonian flow, Integral transforms, Bipolar coordinate system.

1. INTRODUCTION

Industrial applications in which processing of materials behaving as non-Newtonian fluids are those commonly encountered in the chemical, cosmetics, food processing, polymer and petrochemical industries. The petrochemical industries are in search of solutions for the velocity and temperature field of the fluid flow with characteristics typically non-Newtonian. In these applications, the power-law model can described adequately the rheology of such fluids.

The developing laminar flow and heat transfer in the annular passages have been investigated by Heaton et al. (1964), Feldman et al. (1982), in this latter, it is solved laminar developing flow in eccentric annular ducts using the bipolar coordinate system. Others problems were also solved numerically using bipolar coordinates, such as those in the work of Heyda (1959). The author determined the Green's function in bipolar coordinates for a potential flow and obtained a solution for the momentum equation. El-Shaarawi et al. (1998) use the bipolar coordinate system for determined developing laminar forced convection in eccentric annuli, the author has based the analysis on the work of El-Saden (1961), where it was studied heat conduction in an eccentrically hollow, infinitely long cylinder.

The objective of the present paper is to obtain a hybrid solution through the GITT approach for the fully developed flow of non-Newtonian fluids in eccentric annular ducts by using a bipolar coordinate system to map the region of such annular duct. Also, it is intend to develop a numerical algorithm to solve the transformed equation. Therefore, the numerical results will be confronted with results from the literature (Monteiro et al., 2010).

2. MATHEMATICAL FORMULATION

We consider fully developed laminar flow in the eccentric doubly connected duct geometry. The transformation equation from the cylinder coordinate system to this bipolar coordinate system is used to map the duct walls. It was considered that the two-dimensional flow is laminar and incompressible and stationary, the fluid follows the rheological power-law model, the properties of the fluid are constant and that the duct walls are impermeable and non-slip (Fig. 1).

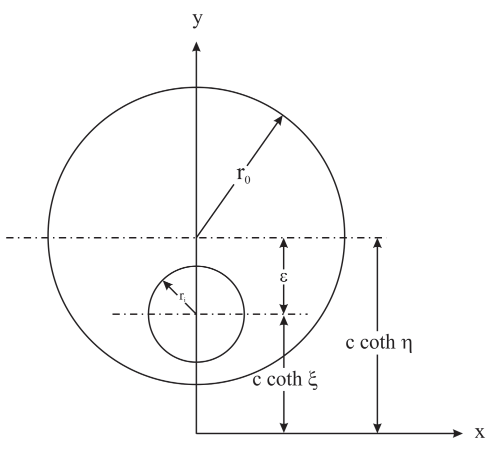


Figure 1. Geometric configuration of the doubly connected duct analyzed with angular symmetry.

The mathematical formulation of the flow problem is given by the momentum conservation equation in the axial direction, in dimensionless form, as follows:

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial \mathrm{V}_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial \mathrm{V}_z}{\partial y}\right)=-c \tag{1}$$

where,

$$\mu=\left[\left(\frac{\partial \mathrm{V}_z}{\partial x}\right)^2+\left(\frac{\partial \mathrm{V}_z}{\partial y}\right)^2\right]^{\frac{n-1}{2}} \tag{2}$$

The boundary condition for the present problem is $\mathrm{V}_z=0$ on the surface. Also, the velocity distribution must be symmetrical about the x-axis.

Where, in Eqs. (1) and (2) above the following dimensionless groups were employed:

$$x=\frac{x^*}{r_0};y=\frac{y^*}{r_0};\gamma=\frac{r_i}{r_0};\mathrm{V}_z=\mathrm{V}_z^*\left[\left(-\frac{dp}{dz}\right)\frac{\mathrm{D}_h^{n+1}}{\mathrm{K}}\right]^{-1/n};\mu=\mu^* r_0^{-1+n}\left[\left(-\frac{dp}{dz}\right)\frac{D_h^{n+1}}{K}\right]^{-n+1/n};c=\left(\frac{r_0}{D_h}\right)^{n+1}=\frac{1}{\left[2\left(1-\gamma\right)\right]^{n+1}} \tag{3-6}$$

The related transformation equations from the Cartesian coordinate system to this bipolar coordinate system are given below:

$$x=a\sinh(\eta)/\cosh(\eta)-\cos(\xi);\quad y=a\sin(\xi)/\cosh(\eta)-\cos(\xi);\quad a=\gamma\sinh(\eta_i)=\sinh(\eta_0) \tag{7-9}$$

Making the transformation of coordinate systems by using Eqs. (7) to (9) above, we obtain the following equations:

$$\frac{\partial}{\partial \xi}\left(\mu\frac{\partial V_z}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\mu\frac{\partial V_z}{\partial \eta}\right)=-c\,a^{n+1}/\left(\cosh(\eta)-\cos(\xi)\right)^2 \tag{10}$$

$$V_z(\xi,\eta_i)=0;\quad V_z(\xi,\eta_0)=0;\quad \frac{\partial V_z(0,\eta)}{\partial \xi}=0;\quad \frac{\partial V_z(\pi,\eta)}{\partial \xi}=0 \tag{11-14}$$

$$\mu=\left\{\left[\cosh(\eta)-\cos(\xi)\right]^2\left[\left(\frac{\partial V_z}{\partial \xi}\right)^2+\left(\frac{\partial V_z}{\partial \eta}\right)^2\right]\right\}^{n-1/2} \tag{15}$$

In order to obtain the solution of Eq. (10), we rewritten such equation as:

$$\begin{aligned}&\left\{\mu+\left(n-1\right)F^{n-3/2}A^2(\xi,\eta)B^2(\xi,\eta)\right\}\frac{\partial^2 V_z}{\partial \eta^2}=-\frac{c\,a^{n+1}}{A^2(\xi,\eta)}-\\&\left\{\mu+(n-1)F^{n-3/2}(\xi,\eta)A^2(\xi,\eta)C^2(\xi,\eta)\right\}E(\xi,\eta)-\\&-(n-1)F^{n-3/2}(\xi,\eta)\left\{A(\xi,\eta)\left[B^2(\xi,\eta)+C^2(\xi,\eta)\right]\left[\sinh(\eta)B(\xi,\eta)+\sin(\xi)C(\xi,\eta)\right]+\right.\\&\left.2A^2(\xi,\eta)B(\xi,\eta)C(\xi,\eta)D(\xi,\eta)\right\}\end{aligned} \tag{16}$$

Where the coefficients are defined by:

$$A(\xi,\eta)=\cosh(\eta)-\cos(\xi);\quad B(\xi,\eta)=\frac{\partial V_z}{\partial \eta};\quad C(\xi,\eta)=\frac{\partial V_z}{\partial \xi};\quad D(\xi,\eta)=\frac{\partial^2 V_z}{\partial \xi\partial \eta};\quad E(\xi,\eta)=\frac{\partial^2 V_z}{\partial \xi^2}; \tag{17-21}$$

$$\mathrm{F}(\xi,\eta)=\left\{\left[\cosh(\eta)-\cos(\xi)\right]^2\left[\left(\frac{\partial V_z}{\partial \xi}\right)^2+\left(\frac{\partial V_z}{\partial \eta}\right)^2\right]\right\}=\left\{A^2(\xi,\eta)\left[B^2(\xi,\eta)+C^2(\xi,\eta)\right]\right\} \tag{22}$$

2.1. Solution methodology

The Generalized Integral Transform Technique (GITT) is then employed in the hybrid numerical-analytical solution of the problem (Cotta, 1993). For this purpose, the following auxiliary eigenvalue problem is chosen:

$$\frac{d^2\psi_i}{d\xi^2} + \mu_i^2\psi_i = 0 \tag{23}$$

$$\frac{d\psi_i(0)}{d\xi} = 0; \quad \frac{d\psi_i(\pi)}{d\xi} = 0 \tag{24,25}$$

Equation above can be analytically solved, to yield the eigenfunctions and eigenvalues, respectively as:

$$\psi_i(\xi) = \cos(\mu_i\xi); \quad \mu_i = i-1, \quad i = 1,2,3,... \tag{26,27}$$

It can be shown that the eigenfunctions, $\psi_i(\xi)$, obey the following orthogonality property, where N_i is the normalization integral:

$$\int_0^\pi \psi_i\,\psi_j\,d\xi = \begin{cases} 0, & i \neq j \\ N_i, & i = j \end{cases}; \quad N_i = \int_0^\pi \psi_i^2\,d\xi = \begin{cases} \pi, & i = 1 \\ \dfrac{\pi}{2}, & i > 1 \end{cases} \tag{28,29}$$

Equations (23) to (25) together with the respective orthogonality properties allow the definition of the integral transform pair for the velocity field as:

$$\bar{V}_{z,i}(\eta) = \int_0^\pi \psi_i(\xi)\,V_z(\xi,\eta)\,d\xi, \quad \text{transform} \tag{30}$$

$$V_z(\xi,\eta) = \sum_{i=1}^{\infty} \frac{\psi_i(\xi)}{N_i}\bar{V}_{z,i}(\eta), \quad \text{inverse} \tag{31}$$

To obtain the resulting system of differential equations for the transformed potentials, $\bar{V}_{z,i}$, the partial differential Eq. (16) is multiplied by $\psi_i(\xi)$, integrated over the domain [0, π] in the ξ-direction, and the inverse formula is employed in place of the velocity distribution $V_z(\xi,\eta)$, resulting in the following transformed ordinary differential system:

$$\sum_{j=1}^{\infty} G_{ij}(\eta)\frac{d^2\bar{V}_{z,j}}{d\eta^2} = H_i(\eta) \tag{32}$$

$$\bar{V}_{z,i}(\eta_0) = 0; \quad \bar{V}_{z,i}(\eta_i) = 0 \tag{33,34}$$

where, $G_{ij}(\eta)$ and $H_i(\eta)$ are given by:

$$\begin{aligned}
&G_{ij}(\eta) = \int_0^\pi \frac{\psi_i(\xi)\,\psi_j(\xi)\,I(\xi,\eta)}{N_j}d\xi; \quad H_i(\eta) = \int_0^\pi \psi_i(\xi)\,J(\xi,\eta)\,d\xi; \\
&I(\xi,\eta) = \left[\mu + (n-1)F^{n-3/2}(\xi,\eta)\;A^2(\xi,\eta)\,B^2(\xi,\eta)\right]; \\
&J(\xi,\eta) = -\frac{c\,a^{n+1}}{A^2(\xi,\eta)} - \left\{\mu + (n-1)\,F^{n-3/2}(\xi,\eta)A^2(\xi,\eta)C^2(\xi,\eta)\right\}E(\xi,\eta) - \\
&(n-1)\,F^{n-3/2}(\xi,\eta)\left\{A(\xi,\eta)\left[B^2(\xi,\eta)+C^2(\xi,\eta)\right]\left[\sinh(\eta)B(\xi,\eta)+\sin(\xi)C(\xi,\eta)\right]+\right. \\
&\left.2A^2(\xi,\eta)B(\xi,\eta)\,C(\xi,\eta)\,D(\xi,\eta)\right\}
\end{aligned} \tag{35-38}$$

The coefficients $G_{ij}(\eta)$ and $H_i(\eta)$ depend on the transformed potentials and vary along η, Eqs. (32) to (34) form an infinite nonlinear boundary value problem, which has to be truncated in a sufficiently high order NT, followed by computation of the transformed potentials of the velocity field, $\bar{V}_{z,i}(\eta)$, to within a user prescribed precision goal. For the solution of such a system, due to the expected stiff characteristics, specialized subroutines have to be employed such as the DVPFD from the IMSL Library (1991).

In order to compute the product of the friction factor by the Reynolds number, first it is necessary to calculate the average velocity, and then from the introduction of the inverse formula, Eq. (33), into its usual definition, one obtains:

$$V_{z,m} = \frac{2a^2}{\pi(1-\gamma^2)} \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} K_{i,m} L_{i,m} \tag{39}$$

The coefficients in Eq. (39) above are defined as follows:

$$K_{i,m} = \int_0^{\pi} \frac{\psi_i(\xi)\cos(\lambda_m \xi)}{N_i M_m} d\xi; \quad L_{i,m} = \int_{\eta_0}^{\eta_i} \frac{e^{-\lambda_m \eta}(\lambda_m + \coth(\eta))\bar{V}_{z,i}(\eta)}{\sinh^2(\eta)} d\eta \tag{40,41}$$

From the definition of the friction factor and Reynolds number, it is concluded that the product fRe is given by:

$$f\,\mathrm{Re} = \frac{1}{2V_{z,m}^n} \tag{42}$$

3. RESULTS AND DISCUSSIONS

Numerical results for the product of the Fanning friction factor-Reynolds number and for the velocity field were obtained from a code developed in the FORTRAN 90 programming language.

In Table (1), it is shown the convergence analysis of the results of the product fRe for Newtonian fluids in eccentric annular ducts, the values were calculated for different values of aspect ratios (γ = 0.2; 0.5 and 0.8) and different values of eccentricity (ε = 0.1; 0.5 and 0.9) depending on the number of terms NT. There is a good convergence of results even for low number of terms. It is observed that with the gradual increase of the eccentricity, the convergence is reached with higher truncation orders. It can be seen clearly that for γ = 0.2 and ε = 0.1, the convergence occurs with NT=3, for ε = 0.5, the values converge with NT=9, while for ε = 0.9, it is observed a fully convergence with number of terms, NT=25. The most critical cases are those with ε = 0.9, in which the convergence is obtained with NT between 15 and 25 terms. Also, in Table (1), it was performed a verification of the present results with those given in the work of Monteiro et al. (2010), showing that there is a good agreement between the two set of results.

Table 1. Convergence analysis of the product fRe in eccentric annular ducts for Newtonian fluids.

NT	fRe								
	γ = 0.2			γ = 0.5			γ = 0.8		
	ε = 0.1	ε = 0.5	ε = 0.9	ε = 0.1	ε = 0.5	ε = 0.9	ε = 0.1	ε = 0.5	ε = 0.9
1	23.271	28.714	68.701	23.949	27.980	57.701	24.103	27.743	55.335
3	22.829	18.350	18.635	23.481	17.788	15.614	23.628	17.598	14.952
9	22.829	18.196	13.269	23.481	17.671	11.471	23.628	17.480	10.945
11	22.829	18.196	13.163	23.481	17.671	11.433	23.628	17.480	10.912
13	22.829	18.196	13.126	23.481	17.671	11.425	23.628	17.480	10.905
15	22.829	18.196	13.112	23.481	17.671	11.423	23.628	17.480	10.903
17	22.829	18.196	13.108	23.481	17.671	11.422	23.628	17.480	10.903
19	22.829	18.196	13.106	23.481	17.671	11.422	23.628	17.480	10.903
25	22.829	18.196	13.105	23.481	17.671	11.422	23.628	17.480	10.903
27	22.829	18.196	13.105	23.481	17.671	11.422	23.628	17.480	10.903
b	22.830[b]	18.197[b]	13.105[b]	23.481[b]	17.671[b]	11.422[b]	23.628[b]	17.480[b]	10.903[b]

b - Monteiro et al. (2010)

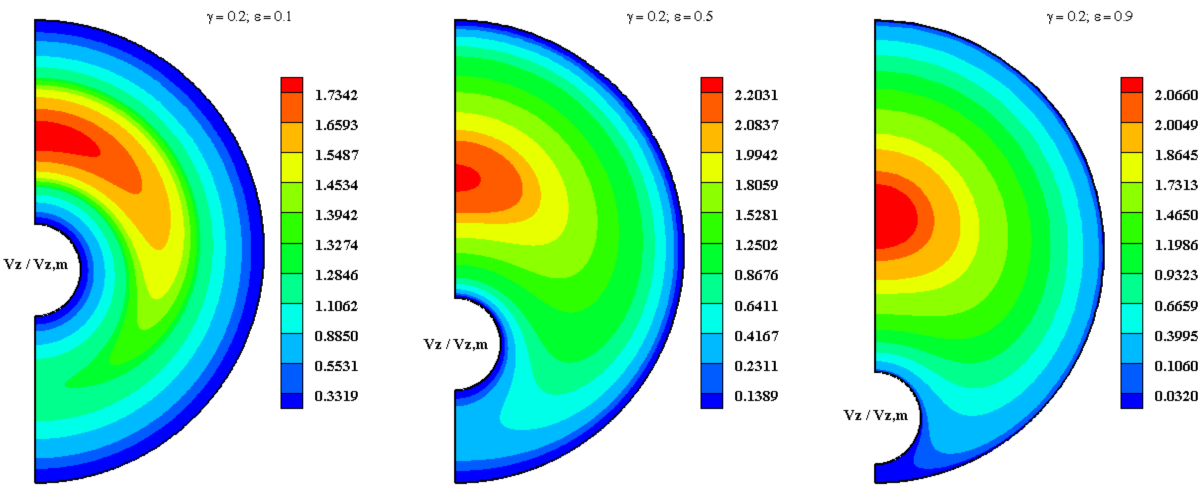
In Table (2) are shown results for the product fRe for different aspect ratios (γ) and dimensionless eccentricities (ε). It is observed that an increase in the aspect ratio, the product fRe decreases when the values of eccentricity is greater than 0.5, and an increase in the product fRe for eccentricity smaller than 0.5. Also, in this table are performed comparisons of the present results with those of Monteiro et al. (2010), where it is observed an agreement in at least four significant digits.

Table 2. Comparison of the product fRe in eccentric annular ducts for Newtonian fluids.

γ	fRe				
	ε = 0.05	ε = 0.2	ε = 0.5	ε = 0.7	ε = 0.9
0.005	19.505[a] 19.505[b]	19.210[a] 19.210[b]	17.857[a] 17.857[b]	16.802[a] 16.802[b]	16.014[a] 16.012[b]
0.03	20.992[a] 20.993[b]	20.509[a] 20.510[b]	18.360[a] 18.360[b]	16.738[a] 16.738[b]	15.486[a] 15.486[b]
0.06	21.726[a] 21.727[b]	21.116[a] 21.117[b]	18.454[a] 18.454[b]	16.479[a] 16.479[b]	14.919[a] 14.919[b]
0.2	23.023[a] 23.023[b]	22.093[a] 22.094[b]	18.196[a] 18.197[b]	15.406[a] 15.407[b]	13.105[a] 13.105[b]
0.5	23.729[a] 23.729[b]	22.541[a] 22.542[b]	17.671[a] 17.672[b]	14.256[a] 14.256[b]	11.422[a] 11.422[b]
0.8	23.891[a] 23.891[b]	22.631[a] 22.631[b]	17.480[a] 17.480[b]	13.882[a] 13.882[b]	10.903[a] 10.903[b]

b - Monteiro et al. (2005)

Now, it is performed an analysis of the effect of eccentricity and aspect ratio on isolines for the velocity ratio $V_z/V_{z,m}$. In Fig. 2, it is observed that the gradual increase of the eccentricity (ε) causes the redirection of velocity peak for the region θ = π, since increasing the eccentricity causes an increase of annular passage in the fluid flow. In Fig. 3, it is observed that the gradual increase in the aspect ratio leads to a decrease in the width of the velocity isolines. Also, one can see that an increase of the aspect ratio provides a tendency of symmetry in the velocity field. This observation is explained by the fact that this geometry approaching that of a parallel-plates channel when the aspect ratio tends to 1.0.



(a) (b) (c)

Figure 2. Effect of eccentricity on isolines for the velocity ratio $V_z/V_{z,m}$ for Newtonian fluids in eccentric annular ducts: (a) ε = 0.1 and γ = 0.2; (b) ε = 0.5 and γ = 0.2 e (c) ε = 0.9 and γ = 0.2.

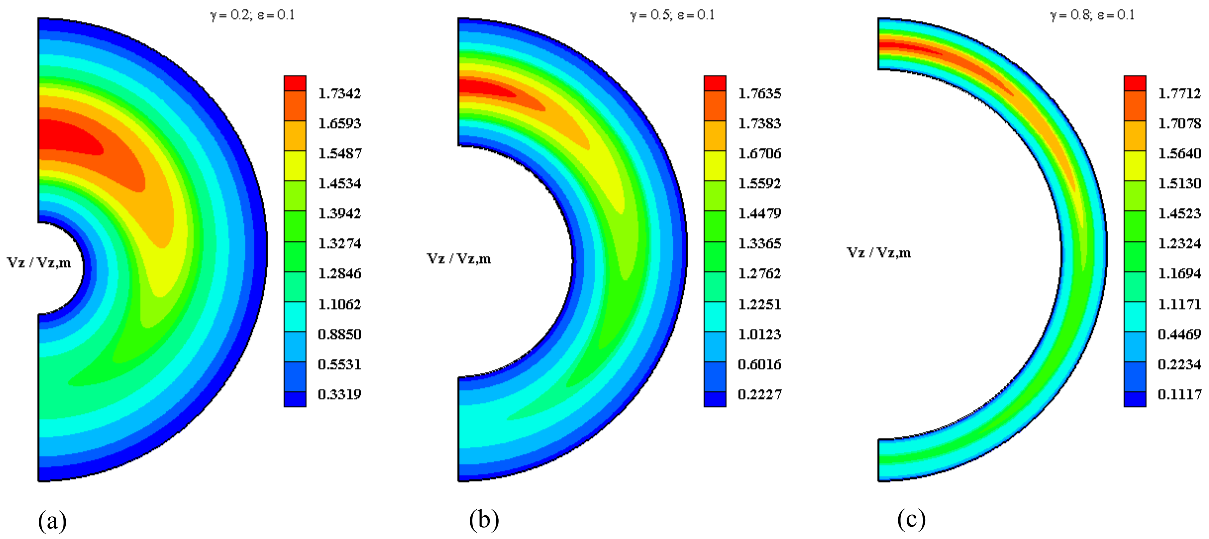


Figure 3. Effect of aspect ratio on isolines for the velocity ratio $V_z/V_{z,m}$ for Newtonian fluids in eccentric annular ducts: (a) $\gamma = 0.2$ and $\varepsilon = 0.1$; (b) $\gamma = 0.5$ and $\varepsilon = 0.1$ and (c) $\gamma = 0.5$ and $\varepsilon = 0.1$.

4. CONCLUSIONS

A solution based on the Generalized Integral Transform Technique (GITT) was developed to predict fully developed laminar flow of non-Newtonian power-law fluids in eccentric annular ducts. The proposed integral transform approach provided reliable and cost effective simulations for the considered cases by employing a bipolar coordinate representation of the solution domain. It was possible to observe the direct influence of eccentricity and aspect ratio on the velocity isolines. Benchmark results for the product of the Fanning friction factor-Reynolds number were systematically tabulated for different values of the governing geometric parameters, demonstrating the usefulness and robustness of the GITT alternative solution procedure. Also,

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6. RESPONSABILITY NOTICE

The authors are the only responsible for the printed material included in this paper.