ANALYSIS OF STRONG STABILITY PRESERVING TIME-MARCHING SCHEMES FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. *Several unsteady problems in transport phenomena require highly accurate solutions. It is, therefore, strongly desirable to increase the order of accuracy in time, while still maintaining the same stability properties. Strong stability preserving (SSP) methods are designed to achieve this goal. This paper was concerned with the main concepts in the field of the so-called non-linear monotone, high-order methods. In order to numerically investigate some features about this class of methods, some numerical tests were made on the shock tube problem.*

Keywords: *Strong Stability Preserving, Runge-Kutta methods, Total Variation Diminishing.*

1. INTRODUCTION

Several problems in transport phenomena require highly precise solutions. The employment of highly accurate numerical methods is strongly recommended whenever computing thermal and hydrodynamical instability, aeroacoustic noise generation and turbulent flows. In fact, the efficiency and accuracy of simulations strongly depend on the numerical properties of the discretization scheme and iterative method employed for solving the system of equations describing these phenomena. According to Alves (2009) reaching high temporal resolution can be a rugged task mainly because of stiffness. Consequently, physical-time steps dictated by numerical stability are much smaller than required by accuracy considerations, drastically increasing computer time for conditionally stable transient schemes.

Gottliebs' research shows that when the time step is limited by a linear stability requirement, or even by a nonlinear stability requirement involving an inner-product norm, there exist some well-known classes of implicit methods that allow the use of arbitrarily large time steps. However, for higher order schemes, stability conditions still remain a problem (Gottlieb *et al.*, 2009).

In this fashion, conditional stability in higher order multi-step schemes led several researchers towards their multi-stage counterparts to minimize the physical-time step size (H. Bijl, 2002; e Ascher *et al.*, 1997). This choice was motivated by the success achieved by these schemes when simulating unsteady compressible flows at high Mach numbers (Yoh and Zhong, 2004, Pareschi and Russo, 2005 e Ferracina and Spijker, 2008). They are known as implicit Runge-Kutta (IRK) methods (Butcher, 2008). High accuracy-order with strong numerical stability is achieved through an increase in the number of intermediate stages used between physical-time steps.

Recently, Alves (2010) developed methodology that allows to efficiently use implicit Runge-Kutta schemes in physical-time, where the preconditioning matrix is introduced into the intermediate stages in an appropriate manner. Hence, the immense field of existing diagonally-and fully-implicit Runge-Kutta methods, with their superior numerical stability properties, can now be used to achieve arbitrary accuracy orders in physical-time.

Nevertheless, it is important to realize that Preconditioned Implicit Runge-Kutta (PIRK) methods were developed based on the original Butcher formulation. In other words, the PIRK method on its original form fulfills just linear requirements. Hence, there is a need for an extension towards nonlinear stability properties in order to increase robustness, especially when dealing with discontinuous solutions.

Initially termed Total Variation Diminishing (TVD), strong stability preserving (SSP) high order time discretizations were developed to guarantee nonlinear stability properties necessary in the numerical solution of hyperbolic partial differential equations and those with discontinuous solutions (Gottlieb *et al.*, 2009). In this manner, the starting point for ensuring nonlinear stability to PIRK method is to investigate this nonlinear numerical features and also, how the existent literature overcome this kind of problem.

This paper discusses the main aspects of Strong Stability-Preserving Runge-Kutta (SSPRK) methods. Moves then, in section 3, to some numerical tests on Euler equations which verifies the necessity of preserving the strong stability properties. Finally, concluding remarks are given in section 4.

2. NUMERICAL METHODS

2.1 Strong Stability Preserving Methods

Strong Stability Preserving methods are high-order time discretization schemes that preserve nonlinear stability properties of first-order Euler time stepping. This methodology has been widely employed for solving hyperbolic PDEs. Shu *et al.* (2011) concluded that this strong stability property guarantee is obtained whenever the time discretization can be decomposed into convex combinations of forward Euler steps. Therefore any convex functional property satisfied by forward Euler will be preserved by the higher-order time discretizations. Shu and Osher (1988) stated the concept of strong stability preserving methods starting from a method of lines approximation of hyperbolic conservation law:

$$u_t = -f(u)_x \tag{1}$$

where the spatial derivative, $f(x)_x$, is discretized by a TVD finite difference and denoted by $-L(u)$. In this manner the spatial discretization has the property that when it is combined with the first-order forward Euler time discretization,

$$u^{n+1} = u^n + \Delta t L(u^n) \tag{2}$$

and for a sufficiently small time step dictated by the Courant-Friedrichs-Levy (CFL) condition, the total variation (TV) of the one-dimension discrete solution does not increase in time, the so-called TVD property holds:

$$TV(u^{n+1}) \leq TV(u^n), \tag{3}$$

$$TV(u^n) = \sum_j \mid u^n_{j+1} - u^n_j \mid . \tag{4}$$

In this manner, the main goal of the SSP Runge−Kutta methods is to maintain the strong stability property while achieving higher order accuracy in time, perhaps with a modified CFL restriction (Shu *et al.*, 2001).

2.2 Runge-Kutta Methods (The Shu-Osher formulation)

Each stage of an explicit Runge-Kutta method presented in Shu and Osher (1988) form is given by

$$u^{(0)} = u^n, \tag{5}$$

$$u^{(i)} = \sum_{j=0}^{i-1} (\alpha_{i,j} u^{(j)} + \Delta t \beta_{i,j} F(u^{(j)})) \tag{6}$$

$$u^{n+1} = u^{(s)} \tag{7}$$

for Runge-Kutta methods, consistency requires that $\sum_{k=0}^{i-1} \alpha_{i,k} = 1$. If all the coefficients are non-negative, each stage of the Runge-Kutta method can be rearranged into convex combinations of forward Euler steps (Eq.8), with Δt replaced by $\frac{\beta_{i,k}}{\alpha_{i,k}}\Delta t$,

$$\|u^{(i)}\| = \left\| \sum_{j=0}^{i-1} \left(\alpha_{i,j} u^{(j)} + \Delta t \beta_{i,j} F(u^{(j)}) \right) \right\| \leq \sum_{j=0}^{i-1} \alpha_{i,j} \left\| u^{(j)} + \Delta t \frac{\beta_{i,j}}{\alpha_{i,j}} F(u^{(j)}) \right\| \tag{8}$$

This observation motivates the following theorem presented by Shu and Osher (1988) :

Theorem 1 : *If the forward Euler method applied to eq.(1) is strongly stable under time step restriction $\Delta t \leq \Delta t_{FE}$, i.e. eq.(2) holds, if $\alpha_{i,j}, \beta_{i,j} \geq 0$, then the solution obtained by Runge-Kutta method satisfies the strong stability bound under the time step restriction*

$$\Delta t \leq \mathbf{C}(\alpha, \beta)\Delta t_{FE} \qquad where \qquad \mathbf{C}(\alpha, \beta) = min_{ij} \frac{\alpha_{ij}}{\beta_{ij}} \tag{9}$$

According to Shu *et al.* (2011) this approach can easily be generalized to all Runge-Kutta methods. Thus, it provides sufficient conditions for strong stability preservation of high-order explicit and implicit Runge-Kutta methods whenever the SSP time step restriction is not exceeded. At this point, it is important to understand that the main goal of strong stability preserving methods is to begin with a method-of-lines semi-discretization that is strongly stable in a certain norm, semi-norm, or convex functional under forward Euler time stepping, when the time step Δt is suitably restricted, and then try to find a higher order time discretization that maintains strong stability for the same norm, perhaps under a different time step restriction (Shu *et al.*, 2011).

This paper aims to prospect the main concepts about SSP theory, especially strongly stable time step restrictions. So that, it will be possible to propose conditions to ensure strong stability preservation for the PIRK method. In this context, numerical investigations on strong stability preservation where done for two well-known schemes of Runge-Kutta methods, presented as follows.

2.2.1 Explicit Scheme

The optimal SSP coefficient for all possible representation of explicit two-stage second order Runge-Kutta methods in the Shu-Osher form is $\mathbf{C} = 1$(Shu *et al.*, 2001) . So that, is common to refer to this method as **SSPRK(2,2)**:

$$u^{(1)} = u^n + \Delta t F(u^n), \tag{10}$$

$$u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t F(u^{(1)}). \tag{11}$$

2.2.2 *Implicit Scheme*

A two-stage second order implicit Runge-Kutta method based on midpoint rule is given as follows. For this method the SSP coefficient is $\mathbf{C} = 2$.

$$u^{(1)} = u^n + \frac{1}{2}\Delta t F(u^n) \tag{12}$$

$$u^{n+1} = u^n + \Delta t F(u^{(1)}). \tag{13}$$

2.2.3 *Spatial discretization*

For both explicit and implicit method the spatial derivative was handled in the next section by using a second-order upwind scheme:

$$u_x = \frac{u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x} + O(\Delta x^2). \tag{14}$$

3. NUMERICAL EXPERIMENTS

The application of the SSP schemes to the 1-D Euler equations is demonstrated in this section. The test case is a Riemann problem in a constant area tube. The left and right states are represented by the subscripts L and R. The domain is [0, 1], the number of points in the domain is represented by N. Sods problem is given by $\{p_L, \rho_L, u_L\}=\{1.0.10^5, 1.0, 0.0\}$ and $\{p_R, \rho_R, u_R\}=\{0.1, 0.125, 0.0\}$. For both explicit and implicit simulations, $N = 1001$.

The one dimensional Euler equations of gas dynamics are given by:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{15}$$

where **U**, the vector of conserved variables and F, the flux vector are defined by,

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix} \tag{16}$$

$$\mathbf{F} = \begin{bmatrix} \rho u \\ p + rhou^2 \\ (e+p) + u \end{bmatrix} \tag{17}$$

ρ, u, p are density, velocity, and pressure respectively.

Figure (1) shows the density evolution. As can be seen, the second order SSPRK method was capable of maintaining the numerical stability of the forward Euler method and produce a second order accurate solution.

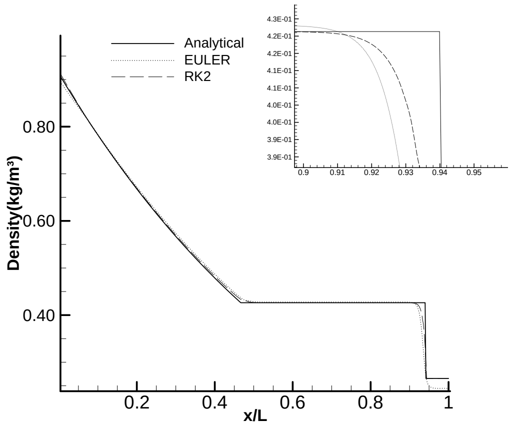

Figure 1. Density evolution on shock tube problem by Euler(Δt,Δx^2), and RKSSP(Δt^2,Δx^2) methods.

3.0.4 Explicit SSPRK (2,2)

Figures (2,3,4) present some results on SSP explicit scheme for the shock tube problem.

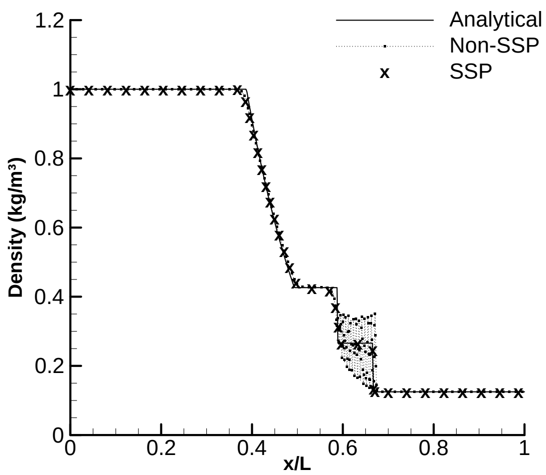

Figure 2. Density evolution on shock tube problem at $t = 0.006s$ and $CFL = 0.2$.

From these results, it can be seen that the SSPKK(2,2) method of time integration is beneficial in the sense that it generate non-oscillatory numerical solutions. In terms of computational cost, most SSP methods have the same cost as traditional solvers. It is also true that the time step Δt might need to be smaller to prove the SSP property. However in many situations Δt can be taken larger without causing instability (Shu *et al.*, 2011). It is important to realize that spurious oscillations or an overshoot may appear whenever the theoretical SSP time step restriction is exceeded. Such oscillations may render difficulties when physical problems are solved, such as the appearance of negative density and pressure (Laney, 1998).

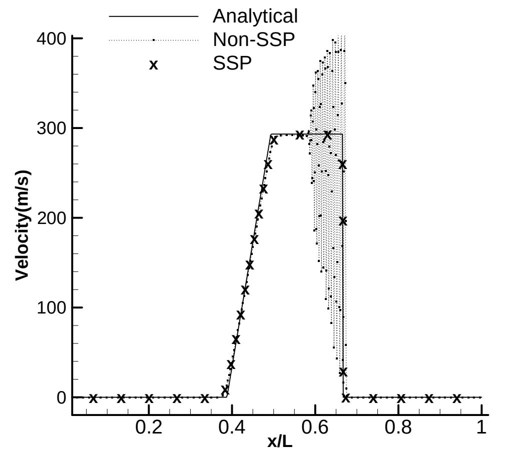

Figure 3. Velocity evolution on shock tube problem at $t = 0.006s$ and $CFL = 0.2$.

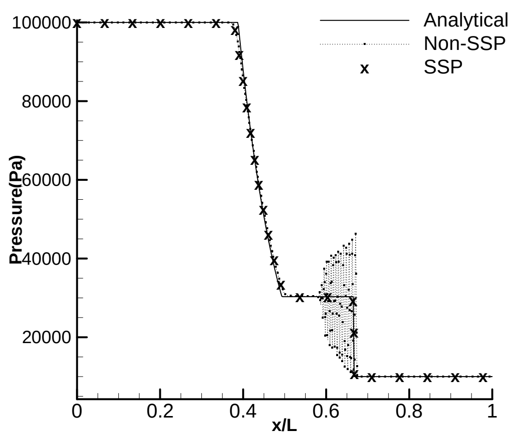

Figure 4. Pressure evolution on shock tube problem at $t = 0.006s$ and $CFL = 0.2$.

3.0.5 Implicit SSPRK (2,2)

In the case of classical stability properties, implicit methods exist that are stable under arbitrarily large time steps. Bellen and Torelli (1997) proved that no Runge-Kutta method of order greater than one can be unconditionally SSP. For the implicit scheme, the same problem was solved under a higher CFL condition. Figures (5,6,7) present density, velocity and pressure profiles obtained by the implicit scheme.

When simulating the shock tube, the implicit scheme was capable of maintaining strong stability more than twice the explicit CFL restriction. Furthermore, as neatly observed by Kraaijevanger (1991), the SSP condition also serves to guarantee that the errors introduced in the solution of the stage equations due to numerical roundoff are not unduly amplified. This is the reason why that major time steps in the implicit scheme produced minor oscillations in results when comparing to the explicit scheme whenever the theoretical barrier is exceeded.

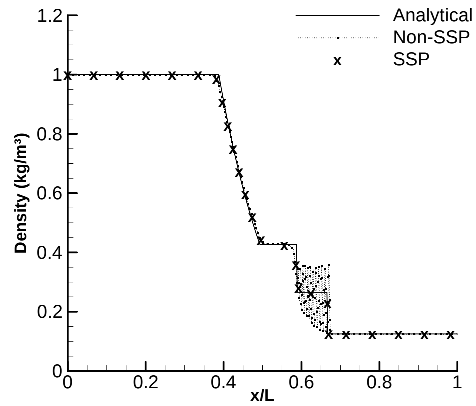

Figure 5. Density evolution on shock tube problem at $t = 0.01s$ and $CFL = 0.7$.

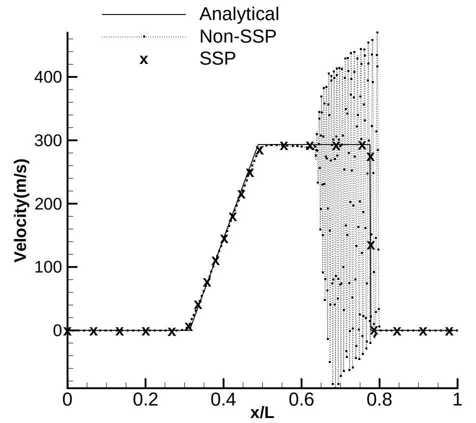

Figure 6. Velocity evolution on shock tube problem at $t = 0.01s$ and $CFL = 0.7$.

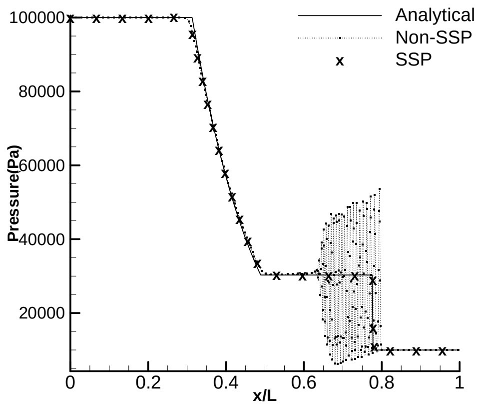

Figure 7. Pressure evolution on shock tube problem at $t = 0.01s$ and $CFL = 0.7$.

4. CONCLUDING REMARKS

In this paper, were presented some basic feature about SSPRK time discretization methods, which preserve stability, in any norm, of Euler first-order time discretizations. Some simulations with explicit and implicit SSPRK schemes were implemented on the 1-D Euler equations. This study represents the first step towards extending PIRK methodology to nonlinear stability properties.

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