HEAT CONDUCTION ANALYTICAL SOLUTIONS TO BE APPLIED IN BOUNDARY CONDITIONS OBTAINED FROM DISCRETE DATA

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Abstract. *Analytical solutions have varied uses. One is to provide solutions that can be used in verification of numerical methods. Another is to provide relatively simple forms of exact solutions that can be used in estimating parameters, thus, it is possible to reduce computation time in comparison with numerical methods. In this paper, an alternative procedure is presented. Here is used a hybrid solution based on Green's function and real characteristics (discrete data) of the boundary conditions.*
Keywords: *Heat conduction, analytical solutions, discrete data*

1. NOMENCLATURE

$X22$	cartesian problem x-coordinate with boundary conditions of the second kind
T_0	initial temperature, ^oC
x	cartesian coordinate, m
L	plate dimension, m
t	time, s
$T(t)$	temperature, ^oC
$q(t)$	heat flux, W/m^2
k	thermal conductivity, $W/m.K$
c_i	constant, $i = 1, 2$
Greek Symbols	
α	thermal diffusivity, m^2/s
Δ	on the discrete-time
Subscripts	
m	eigenvalue index, $m = 1, 2, 3, ..$
n	flux components index, $n = 1, 2, 3, ..., N$

2. INTRODUCTION

Analytical solutions have varied uses. One is to provide solutions that can be used in verification, that is, to provide solutions which the accuracy of approximate methods, such as finite difference and finite element solutions, can be investigated (Beck *et al.*, 2004); (Beck *et al.*, 2006); (McMasters *et al.*, 2002(1); (McMasters *et al.*, 2002(2); (Roache, 1998). Another is to provide relatively simple forms of exact solutions that can be used in estimating parameters (Gustafsson *et al.*, 1984); (Cole, 2005).

Realistic applications in heat conduction usually have transient variations at the boundaries. Many examples can be cited such as manufacturing like welding (Gonçalves *et al.*, 2010), cutting (Carvalho *et al.*, 2006) or drilling (Huang *et al.*, 2007) process heating of brake drums of cars and cooking of foods (Beck *et al.*, 2008). However, few exact solutions for transient boundary conditions exist.

The main feature of these problems is that heat flux is not always described by one mathematical expression, but it can be obtained through discrete measurements, estimating techniques or optimization procedures and even by curve fitting.

Therefore, purely analytical solutions are almost impossible to obtain due to discrete nature of the boundary conditions. In this paper, an alternative proposal is presented. It is about of using hybrid solution where is employed all analytical formulation of thermal problem. However the real characteristics (discrete data) of the boundary conditions are applied only in time integrals of the general solution obtained based on Green's function.

In this way, the solution is provided assuming that heat flux data can be represented such as a vector by its components that are constants at each acquisition time (experimental or estimated). The obtained approximation is very important in investigation of the inverse heat conduction problems (IHCPs) because it gives a convenient expression for the temperature in terms of the heat flux components.

This research will be concerned specifically to the analytical solution obtained by Green's functions for a one-dimensional heat conduction problem. This problem is referred by $X22$ (Beck *et al.*, 1992). In this case, (Fig. 1), this means that boundary conditions are of the second kind in $x = 0$ (imposed heat flux) and in $x = L$ (insulated thermal).

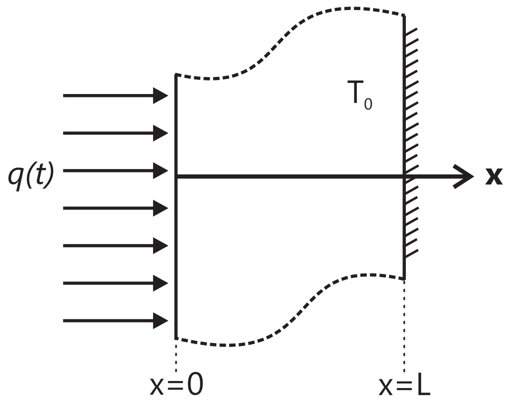


Figure 1. One-dimensional problem ($X22$)

3. FUNDAMENTALS

A mathematical description of this problem (Fig. 1) is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1a}$$

subject to boundary conditions

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q(t); \qquad \frac{\partial T}{\partial x}\bigg|_{x=L} = 0 \tag{1b}$$

and initial condition

$$T(x,0) = T_0 \tag{1c}$$

Equation solution of Eqs. (1a)- (1c) using Green's function can given by (Fernandes, 2009)

$$T(x,t) = T_0 + \frac{\alpha}{k}\left[\frac{1}{L}\int_0^t q(\tau)d\tau + \frac{2}{L}\sum_m^{\infty} e^{-\left(\frac{m\pi}{L}\right)^2\alpha t}\cos\left(\frac{m\pi x}{L}\right)\int_0^t q(\tau)e^{\left(\frac{m\pi}{L}\right)^2\alpha\tau}d\tau\right] \tag{2}$$

If the heat flux data can be represented by its components (Fig. 2), $q(t) = [q_1, q_2, q_3, ..., q_n]$ with q_n constant at each $\Delta t = t_{n+1} - t_n$, with $n = 1, 2, ..., N-1$.

Generally Eq. (2) can be written by

$$\int_0^t q(\tau)d\tau = \int_{t_1=0}^{t_2} q_1 d\tau + \int_{t_2}^{t_3} q_2 d\tau + ... + \int_{t_n}^{t_{n+1}} q_n d\tau = \sum_{n=1}^{N-1} q_n(t_{n+1} - t_n) \tag{3}$$

and the second integral of Eq. (2) can be written by

$$\begin{aligned}\int_0^t e^{\left(\frac{m\pi}{L}\right)^2\alpha\tau}q(\tau)d\tau &= \int_{t_1=0}^{t_2} e^{\left(\frac{m\pi}{L}\right)^2\alpha\tau}q_1 d\tau + \int_{t_2}^{t_3} e^{\left(\frac{m\pi}{L}\right)^2\alpha\tau}q_2 d\tau + ... + \int_{t_n}^{t_{n+1}} e^{\left(\frac{m\pi}{L}\right)^2\alpha\tau}q_n d\tau \\ &= \frac{1}{\left(\frac{m\pi}{L}\right)^2\alpha}\sum_{n=1}^{N-1} q_n\left(e^{\left(\frac{m\pi}{L}\right)^2\alpha t_{n+1}} - e^{\left(\frac{m\pi}{L}\right)^2\alpha t_n}\right)\end{aligned} \tag{4}$$

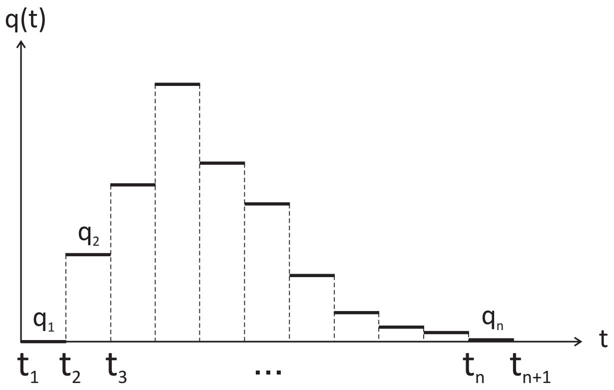


Figure 2. Experimental heat flux

Eq. (2) can then be re-written by

$$T(x,t) = T_0 + \frac{\alpha}{k}\frac{1}{L}\sum_{n=1}^{N-1} q_n(t_{n+1} - t_n) + \frac{\alpha}{k}\frac{2}{L}\sum_{m=1}^{M}\frac{\cos\left(\frac{m\pi x}{L}\right)}{\left(\frac{m\pi}{L}\right)^2\alpha}\sum_{n=1}^{N-1} q_n\left(e^{-\left(\frac{m\pi}{L}\right)^2\alpha(t-t_{n+1})} - e^{-\left(\frac{m\pi}{L}\right)^2\alpha(t-t_n)}\right) \quad (5)$$

Analytical (and exact) solution considering heat flux function, $q(t) = c_1 e^{-c_2 t}$ to be applied in Eq. (2) solved the integrals of time has

$$T(x,t) = T_0 + \frac{\alpha}{k}\frac{1}{L}\frac{c_1}{c_2}\left(1 - e^{-c_2 t}\right) + \frac{\alpha}{k}\frac{2}{L}\sum_{m}^{\infty}\cos\left(\frac{m\pi x}{L}\right)\frac{c_1}{\left[\left(\frac{m\pi}{L}\right)^2\alpha - c_2\right]}\left(e^{-c_2 t} - e^{-\left(\frac{m\pi}{L}\right)^2\alpha t}\right) \quad (6)$$

Thus, in the next section will be showed the comparison between of results obtained through Eqs. (5) and (6). The proposal of this work is verify that this hybrid solution (Eq. (5)) is a convenient expression for describe the temperature in terms of the heat flux components.

4. RESULTS

Solutions have been implemented in Matlab©. The following physical and geometrical data were used, sample time $dt = 7\ [s]$, thermal conductivity, $k = 0.159\ [W/m.K]$, thermal diffusivity, $\alpha = 1.57E-07\ [m^2/s]$; initial temperature, $T_0 = 25\ [^oC]$ and thickness $L = 50E-03\ [m]$.

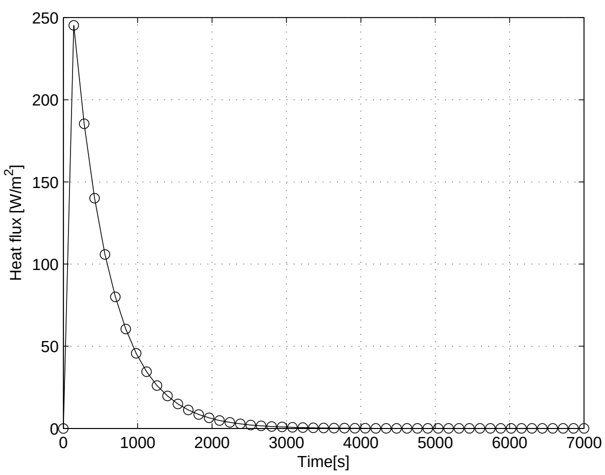
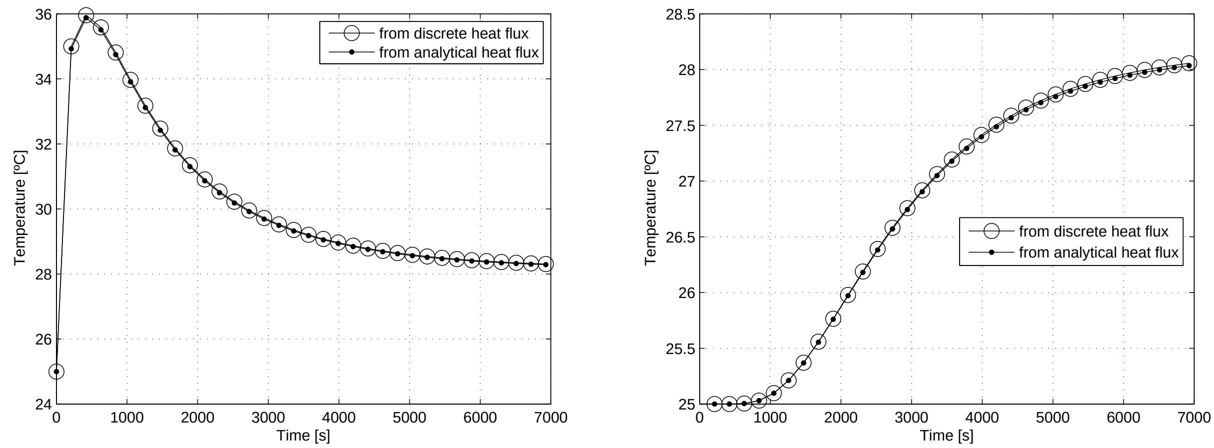


Figure 3. Discrete heat flux for Eq. (5) where $q_n = [0 \quad c_1 \exp(-c_2 t)]$ with $c_1 = 320$ and $c_2 = .002$

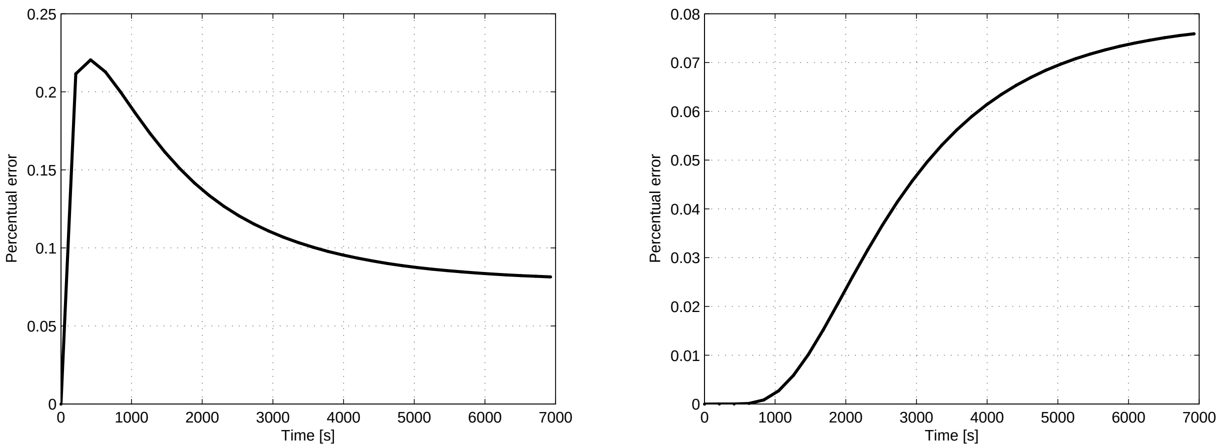
The discrete heat flux shown in Fig. 3 is applied in Eq. (5), i.e., $q(t)$ is represented by the components $q = [0 \;\; c_1 \exp(-c_2 t)]$. Figures 4(a) and 4(b) show a comparison between temperature obtained from Eqs. (5) and (6) at $x = 0$ and $x = L$. Residues are shown in Figs. 5(a) e 5(b).

Considering the sample time, $dt = 7\ [s]$, the greater difference is lower than $2,5\%$ for $x = 0$ and lower than $0,08\%$ for $x = L$. However, if sample time is lower a great agreement can be reached, as shown in Table 1. This table presents a comparison between both solution for $dt = 1\ [s]$; $0,5\ [s]$; $0,1\ [s]$ e $0,01\ [s]$ at $x = 0$, which shows the accuracy between the solutions when making dt increasingly smaller.



(a) $x = 0$ (b) $x = L$

Figure 4. Comparison between Eq. (5) e (6)



(a) $x = 0$ (b) $x = L$

Figure 5. Percentual error between solutions when $dt = 7\ s$

In other test, Fig. 6 shows estimated heat flux with sample time equal $7.03\ [s]$ and the same previous physical and geometrical data that is used. Figures 7(a) and 7(b) show a comparison of experimental and theoretical temperature calculated from positions $x = 0$ and $x = L$.

Residues are shown in Figs. 8(a) and 8(b). Deviate in this case can be attributed to experimental data. It must be informed that the discrete heat flux is not exact but have been obtained from inverse problem. Comparison with temperature measurements should consider the agreement between the thermal model (1D) and the experimental apparatus

Table 1. Comparison between solutions using Eqs. (5) e (6)

$x = 0$		$\Delta t = 1\,s$		$\Delta t = 0,5\,s$		$\Delta t = 0,1\,s$		$\Delta t = 0,01\,s$	
	Eq. (5)	Eq. (6)	diff	Eq. (6)	diff	Eq. (6)	diff	Eq. (6)	diff
t	$T(q(t))$	$T(q=[q_i])$		$T(q=[q_i])$		$T(q=[q_i])$		$T(q=[q_i])$	
0,0	25,0000000	25,0000000	0,0000000	25,0000000	0,0000000	25,0000000	0,0000000	25,0000000	0,0000000
1,0	25,8782847	25,8794429	0,0011581	25,8788273	0,0005425	25,8783816	0,0000969	25,8782937	0,0000090
2,0	26,2488556	26,2504052	0,0015496	26,2495887	0,0007331	26,2489898	0,0001341	26,2488683	0,0000127
3,0	26,5320627	26,5339046	0,0018419	26,5329396	0,0008769	26,5322252	0,0001625	26,5320782	0,0000155
4,0	26,7698621	26,7719470	0,0020849	26,7708592	0,0009971	26,7700485	0,0001864	26,7698800	0,0000179
5,0	26,9785294	26,9808264	0,0022970	26,9796316	0,0011022	26,9787367	0,0002072	26,9785494	0,0000200
6,0	27,1664236	27,1689108	0,0024872	27,1676202	0,0011966	27,1666496	0,0002260	27,1664454	0,0000219
7,0	27,3385172	27,3411782	0,0026610	27,3398003	0,0012830	27,3387605	0,0002433	27,3385408	0,0000236
8,0	27,4980549	27,5008767	0,0028218	27,4994180	0,0013631	27,4983142	0,0002592	27,4980801	0,0000252
9,0	27,6472935	27,6502654	0,0029719	27,6487313	0,0014378	27,6475676	0,0002741	27,6473201	0,0000267
10,0	27,7878784	27,7909916	0,0031132	27,7893866	0,0015082	27,7881666	0,0002882	27,7879064	0,0000281

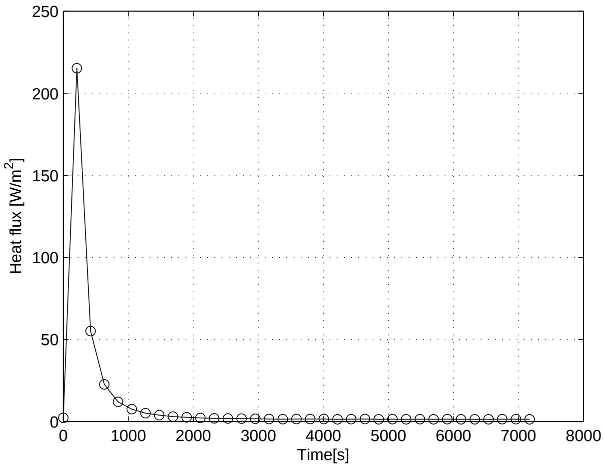
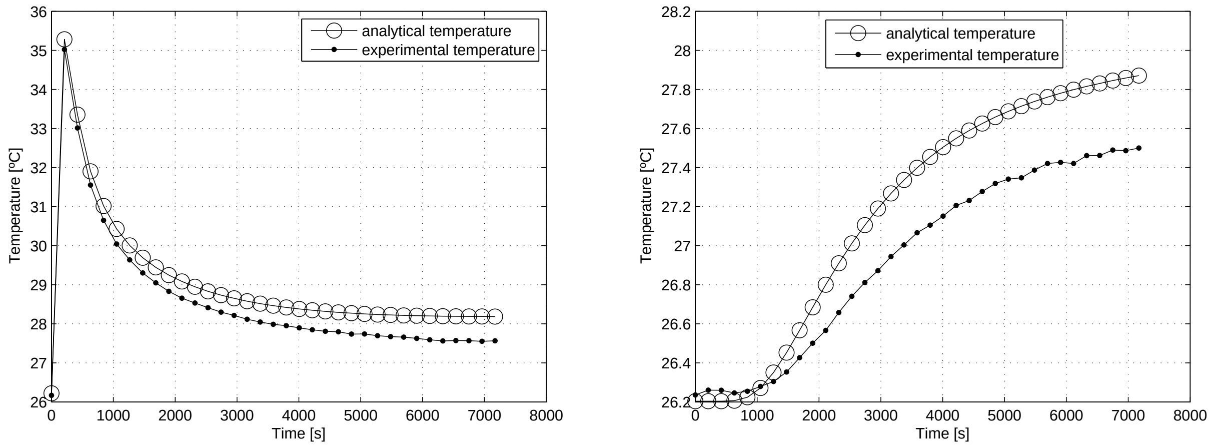


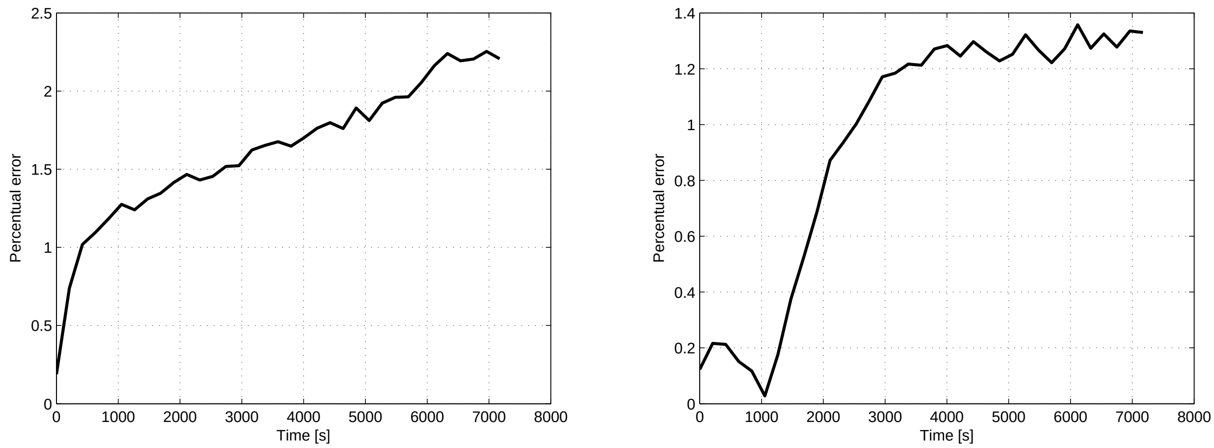
Figure 6. Estimated heat flux using inverse techniques (Borges, 2008)



(a) $x = 0$ (b) $x = L$

Figure 7. Comparison between experimental temperature and calculated temperature from Eq. (5)

used. Despite of this consideration, maximum perceptual errors were $2,5\%$ and $1,4\%$ at $x = 0$ and $x = L$ respectively. Table 2 also presents the comparison between experimental temperatures and the solution by Eq.(5) for several sampling of the time.



(a) $x = 0$ (b) $x = L$

Figure 8. Perceptual error between experimental and calculated temperature

Table 2. Comparison between experimental temperatures and the solution by Eq.(5)

$[dt = 7.03\ s]$		$x = 0$			$x = L$	
t	Texp*	Tana**	diff(%)	Texp*	Tana**	diff(%)
0,00	26,1719	26,2213	0,1888	26,2363	26,2041	0,1227
7,03	26,1846	26,2274	0,1635	26,2285	26,2041	0,0930
14,06	26,2236	26,2987	0,2864	26,2500	26,2041	0,1749
21,09	26,3770	26,4882	0,4216	26,2451	26,2041	0,1562
105,45	30,8887	31,2080	1,0337	26,2441	26,2041	0,1524
203,87	35,0215	35,2823	0,7447	26,2607	26,2041	0,2155
302,29	34,2402	34,5405	0,8770	26,2471	26,2041	0,1638
400,71	33,2344	33,5435	0,9301	26,2432	26,2041	0,1490
506,16	32,3457	32,6805	1,0351	26,2539	26,2044	0,1885
1005,29	30,1924	30,5486	1,1798	26,2900	26,2578	0,1225
2003,55	28,7422	29,1619	1,4602	26,5244	26,7429	0,8238
3001,81	28,1875	28,6347	1,5865	26,8838	27,2091	1,2100
4000,07	27,9014	28,3824	1,7239	27,1689	27,5027	1,2286
5005,36	27,7197	28,2589	1,9452	27,3330	27,6799	1,2692
6003,62	27,6172	28,2032	2,1219	27,4180	27,7888	1,3524
7001,88	27,5635	28,1868	2,2613	27,5020	27,8608	1,3046

*Texp: Experimental temperature; **Tana: Analytical temperature

5. CONCLUSION

The use of analytic solution is a powerful tool to validate numerical methods and to minimize computational cost present in numerical solutions. However, a great difficult can appear when real and discrete boundary conditions are present. This work shows an procedure that assures fast and accuracy results.

Besides, the hybrid solution can also be used to any kind of heat flux (step, triangular, sinusoidal) mainly from discrete measurements without obtaining a new analytical expression. It means the integral term in Eq. (2) doesn't need to be solved analytically for each heat flux shape.

6. ACKNOWLEDGEMENTS

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