INTEGRAL TRANSFORM ANALYSIS OF PARALLEL FLOW REGENERATORS IN EULERIAN FORMULATION

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Abstract. *This paper presents a closed-form analytical solution for calculating the effectiveness of parallel-flow balanced and symmetric heat regenerators using the Generalized Integral Transform Technique (GITT). The problem is analyzed in a Eulerian form, in which both the fluid streams and the regenerator matrix move in cross flow through when a fixed position in space is analyzed. While most solutions of periodic flow exchangers are based on models that follow a channel motion in the regenerator (Lagrangian form), the alternate Eulerian form allows a periodic solution in terms of the regenerator angle and axial position (flow direcion) to be obtained. The usage of the GITT with linear algebra, finally allows the regenerator effectiveness to be calculated in by a simple closed-form expression. The results are verified by comparing to a finite-volumes solution and a very good convergence rate is seen.*

Keywords: *heat regenerator, eigenfunction expansion, analytical solution*

1. INTRODUCTION

Regenerative heat exchangers have a substantial number of applications in several processes where indirect heat transfer between two process streams with a compact construction is required. The traditional thermal design of these exchangers are based on solving a system of two energy transfer equations, one for the process streams and the other one for the solid matrix. The heat transfer theory for thermal theory for heat regenerators, or periodic heat exchangers, is well established and can be found on textbooks (Kays and London, 1998; Shah and Sekulic, 2002). The first formulation can be traced by to the early works of Hausen (1930), and a method of analysis based on physically meaningful dimensionless groups for the regenerator problem was presented by Coppage and London (1953). Different than recuperator problems, in which closed-form expressions for the effectiveness of the exchanger can be obtained in terms of the number of transfer units and capacity ratios, for regenerative heat exchangers, there are apparently only numerical and approximate solutions for the effectiveness calculation. The classic text by Shah and Sekulic (2002) clearly states that “no closed form exact solution of the theoretical model is available presently.” Bačlić (1985) obtained a closed-form expression for the effectiveness for balanced and symmetric regenerators using the Galerkin method, and later on for unbalanced and asymmetric exchangers (Romie and Bačlić, 1988; Bačlić and Dragutinovic, 1991). Numerical solutions for the problem were also obtained by a number of other authors such as Lambertson (1958), Theoclitus and Erckrich (1966) and Willmott and Knight (1993).

Different the the previous numerical and approximate solutions, this paper proposes a fully-analytical solution to the parallel-flow problem in balanced and symmetric regenerators using the Generalized Integral Transform Technique (Cotta, 1990, 1994). The regenerator problem is analyzed in an Eulerian form in which instead of following a channel during its rotation through the matrix, a fixed position is chosen and both the fluid flow and the matrix move through it in a cross-flow like arrangement. Effectiveness results are compared with those of a finite volumes solution, showing a very good agreement.

2. PROBLEM FORMULATION AND INTEGRAL TRANSFORM SOLUTION

A balanced, symmetric, and parallel (concurrent) flow regenerator is considered. Using the usual simplifying assumptions for heat regenerators (Shah and Sekulic, 2002), the following normalized equations are obtained, for the fluid streams and the solid matrix, during the hot and cold periods, respectively:

$$\frac{\partial T^*}{\partial t_h^*}\,\tau_{dw}^* + \frac{\partial T^*}{\partial z^*} \,=\, 2\,\mathrm{NTU}_o\,(T_m^* - T^*), \qquad \mathrm{C}_r^*\,\frac{\partial T_m^*}{\partial t_h^*} \,=\, -2\,\mathrm{NTU}_o\,(T_m^* - T^*), \qquad \text{for} \qquad 0 \le t_h^* \le 1, \tag{1}$$

$$\frac{\partial T^*}{\partial t_c^*}\,\tau_{dw}^* + \frac{\partial T^*}{\partial z^*} \,=\, 2\,\mathrm{NTU}_o\,(T_m^* - T^*), \qquad \mathrm{C}_r^*\,\frac{\partial T_m^*}{\partial t_c^*} \,=\, -2\,\mathrm{NTU}_o\,(T_m^* - T^*), \qquad \text{for} \qquad 0 \le t_c^* \le 1. \tag{2}$$

An alternate formulation may be employed by bearing in mind the fact that the regenerator operates at a constant angular velocity such that:

$$\theta \,=\, \pi\,t_h^*, \quad \text{for} \quad 0 < \theta \le \pi, \qquad \theta \,=\, \pi \,+\, \pi\,t_c^*, \quad \text{for} \quad \pi < \theta \le 2\pi, \tag{3}$$

which leads to the following eulerian-type formulation for the heat transfer in the periodic exchanger:

$$\pi\,\tau_{dw}^*\,\frac{\partial T^*}{\partial \theta} + \frac{\partial T^*}{\partial z^*} \,=\, 2\,\mathrm{NTU}_o\,(T_m^* - T^*), \qquad \pi\,\frac{\partial T_m^*}{\partial \theta} = -\frac{2\,\mathrm{NTU}_o}{\mathrm{C}_r^*}\,(T_m^* - T^*), \qquad \text{for} \qquad 0 < \theta \le 2\,\pi, \tag{4}$$

where the inlet conditions are given by:

$$T^*(0,\theta) = 1 \quad \text{for} \quad 0 \le \theta \le \pi, \qquad T^*(0,\theta) = 0 \quad \text{for} \quad \pi \le \theta \le 2\pi, \tag{5}$$

The integral transform solution of the problem is based on the following integral transform pairs:

$$T_m^* \,=\, \sum_{i=0}^{\infty} \bar{Y}_i^*(z^*)\,\tilde{\Psi}_i(\theta), \qquad \bar{Y}_i^*(z^*) \,=\, \int_0^{2\,\pi} T_m^*(\theta, z^*)\,\tilde{\Psi}_i(\theta)\,\mathrm{d}\theta, \tag{6}$$

$$T^* \,=\, \sum_{i=0}^{\infty} \bar{T}_i^*(z^*)\,\tilde{\Psi}_i(\theta), \qquad \bar{T}_i^*(z^*) \,=\, \int_0^{2\,\pi} T^*(\theta, z^*)\,\tilde{\Psi}_i(\theta)\,\mathrm{d}\theta, \tag{7}$$

where the eigenvalue problem is a Helmholtz problem with periodic conditions:

$$\tilde{\Psi}'' \,+\, \mu^2\,\tilde{\Psi} \,=\, 0, \qquad \tilde{\Psi}(0) \,=\, \tilde{\Psi}(2\,\pi), \qquad \tilde{\Psi}'(0) \,=\, \tilde{\Psi}'(2\,\pi), \tag{8}$$

which gives the following solutions:

$$\tilde{\Psi}_i \,=\, \cos\big(i/2\,\theta\big)\Big/\sqrt{N_i} \qquad \text{for} \quad i \;\text{ even}, \qquad \tilde{\Psi}_i \,=\, \sin\big((i+1)/2\,\theta\big)\Big/\sqrt{N_i} \qquad \text{for} \quad i \;\text{ odd}, \tag{9}$$

where the norms are given by:

$$N_0 \,=\, 2\,\pi, \qquad N_i \,=\, \pi \quad (\text{for } i > 0). \tag{10}$$

The transformation of the problem is started by multiplying the equations by the eigenfunction and integrating within the θ-domain, which leads, after simplification and substitution of the inversion formula to the non-transformable terms, to:

$$\frac{\mathrm{d}\bar{T}_i^*}{\mathrm{d}z^*} - \pi\,\tau_{dw}^*\,\sum_{j=0}^{\infty} A_{i,j}\,\bar{T}_j^* \,=\, 2\,\mathrm{NTU}_o\,(\bar{Y}_i^* - \bar{T}_i^*), \qquad -\pi\,\sum_{j=0}^{\infty} A_{i,j}\,\bar{Y}_j^* \,=\, -\,\frac{2\,\mathrm{NTU}_o}{\mathrm{C}_r^*}\,(\bar{Y}_i^* - \bar{T}_i^*), \tag{11}$$

where the matrix coefficients are given by:

$$A_{i,j} \,=\, \int_0^{2\,\pi} \tilde{\Psi}_i'\,\tilde{\Psi}_j\,\mathrm{d}\theta. \tag{12}$$

If system (11) is truncated to a finite order it can be written in vector form:

$$\frac{\mathrm{d}\bar{\boldsymbol{T}}^*}{\mathrm{d}z^*} = (\pi\,\tau^*_{dw}\,\boldsymbol{A} - 2\,\mathrm{NTU}_o\,\boldsymbol{I})\,\bar{\boldsymbol{T}}^* + 2\,\mathrm{NTU}_o\,\bar{\boldsymbol{Y}}^*, \qquad \left(-\frac{\pi\,\mathrm{C}^*_r}{2\,\mathrm{NTU}_o}\,\boldsymbol{A} + \boldsymbol{I}\right)\bar{\boldsymbol{Y}}^* = \bar{\boldsymbol{T}}^*, \tag{13}$$

and the two previous equations can then be combined into a single form:

$$\frac{\mathrm{d}\bar{\boldsymbol{T}}^*}{\mathrm{d}z^*} = \left(\pi\,\tau^*_{dw}\,\boldsymbol{A} - 2\,\mathrm{NTU}_o\,\boldsymbol{I} + 2\,\mathrm{NTU}_o\,\boldsymbol{M}^{-1}\right)\bar{\boldsymbol{T}}^*, \qquad \text{where} \qquad \boldsymbol{M} = -\frac{\pi\,\mathrm{C}^*_r}{2\,\mathrm{NTU}_o}\,\boldsymbol{A} + \boldsymbol{I} \tag{14}$$

This system can be solved analytically, yielding the following closed-form solution in terms of a matrix exponential:

$$\bar{\boldsymbol{T}}^* = \boldsymbol{C}\,\bar{\boldsymbol{f}}, \qquad \text{where} \qquad \boldsymbol{C} = \exp\left(-\boldsymbol{F}\,z\right), \qquad \boldsymbol{F} = -\pi\,\tau^*_{dw}\,\boldsymbol{A} + 2\,\mathrm{NTU}_o\,\boldsymbol{I} - 2\,\mathrm{NTU}_o\,\boldsymbol{M}^{-1} \tag{15}$$

and $\bar{\boldsymbol{f}}$ is the transformed inlet condition vector, given in terms of the following coefficients:

$$\bar{f}_i = \int_0^{\pi} \tilde{\Psi}_i(\theta)\,\mathrm{d}\theta. \tag{16}$$

For balanced and symmetric exchangers, the effectiveness is calculated through:

$$\epsilon = 1 - T^*_{h,out} = T^*_{c,out} \tag{17}$$

where the average outlet temperatures, for the hot and cold periods are given by:

$$T^*_{h,out} = \frac{1}{\pi}\int_0^{\pi} T^*(\theta,1)\,\mathrm{d}\theta, \qquad T^*_{c,out} = \frac{1}{\pi}\int_{\pi}^{2\pi} T^*(\theta,1)\,\mathrm{d}\theta. \tag{18}$$

With the inversion formulas, this corresponds to calculating:

$$T^*_{h,out} = \frac{1}{\pi}\sum_{i=0}^{\infty}\bar{T}^*_i(1)\int_0^{\pi}\tilde{\Psi}_i(\theta)\,\mathrm{d}\theta, \qquad T^*_{c,out} = \frac{1}{\pi}\sum_{i=0}^{\infty}\bar{T}^*_i(1)\int_{\pi}^{2\pi}\tilde{\Psi}_i(\theta)\,\mathrm{d}\theta, \tag{19}$$

such that an analytical expression for the effectiveness is readily obtained:

$$\epsilon = \frac{1}{\pi}\sum_{i=0}^{\infty}\bar{T}^*_i(1)\int_{\pi}^{2\pi}\tilde{\Psi}_i(\theta)\,\mathrm{d}\theta = \frac{1}{\pi}\left(\exp(-\boldsymbol{F})\,\bar{\boldsymbol{f}}\right)\cdot\boldsymbol{b}, \tag{20}$$

where the vector $\boldsymbol{b}$ is given by the coefficients:

$$b_i = \int_{\pi}^{2\pi}\tilde{\Psi}_i(\theta)\,\mathrm{d}\theta. \tag{21}$$

3. RESULTS AND DISCUSSION

Table 1 presents effectiveness values calculated with the current formulation for different truncation orders $i_{\max}$ and different combinations of NTU_o and C^*_r values for a case with $\tau^*_{dw} = 10^{-5}$. The results are compared with a four-digit converged Finite Volume Method solution of the same problem. As can be seen from these results, the GITT solution has a remarkable convergence behavior, producing six-digit converged effectiveness values with 10 terms in the series for almost all cases. The exception is the lower NTU_o and lower C^*_r values; however, even for this case, 10 terms yield 5 converged digits in the effectiveness values.

Table 1. Calculated effectiveness values for different NTU_o and C_r^* values with $\tau_{dw}^* = 10^{-5}$.

$i_{\max}$	$\mathrm{NTU}_o = 5$, $\mathrm{C}_r^* = 1$	$\mathrm{NTU}_o = 10$, $\mathrm{C}_r^* = 1$	$\mathrm{NTU}_o = 5$, $\mathrm{C}_r^* = 2$	$\mathrm{NTU}_o = 10$, $\mathrm{C}_r^* = 2$
1	0.094715	0.094715	0.094715	0.094715
5	0.613497	0.704636	0.459895	0.398172
10	0.658419	0.749507	0.504939	0.443207
20	0.658421	0.749507	0.504939	0.443207
40	0.658422	0.749507	0.504939	0.443207
80	0.658422	0.749507	0.504939	0.443207
FVM	0.6584	0.7495	0.5049	0.4432

4. CONCLUSIONS

This paper presented a closed-form analytical solution for calculating the effectiveness of balanced and symmetric parallel-flow regenerators in terms of the number of transfer units, the matrix-to-fluid heat capacity ratio and the dimensionless dwell time. The methodology was based on the Generalized Integral Transform Technique.

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