LOCAL DIFFERENTIAL QUADRATURE METHOD WITH RADIAL BASIS FUNCTIONS: NUMERICAL ASPECTS AND APPLICATIONS IN FLUID DYNAMICS AND HEAT TRANSFER PROBLEMS

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Abstract. *The Local Differential Quadrature Method with Radial Basis Functions (LDQM-RBF) is a collocation method for numerical solution of partial differential equations. It has been recently proposed for treating complex discretizations including structured and non-structured meshes and meshless schemes. In this work, numerical experiments are carried out using a Poisson-type equation in a unit square domain with uniform equidistant meshes and various stencils. The influence of relevant parameters of LDQM-RBF (multiquadric shape parameter c, mesh size h and stencil structure) is discussed. Then, two applications of the LDQM-RBF in physical problems are made: a problem of incompressible hydrodynamics and a problem of natural convection, both in a square domain. Results are compared with the literature and discussed.*

Keywords: *Local Differential Quadrature Method, Radial Basis Functions, Multiquadric, Fluid Dynamics and Heat Transfer*

1. LOCAL DIFFERENTIAL QUADRATURE METHOD WITH RADIAL BASIS FUNCTIONS

By the Local Differential Quadrature Method (LDQM) a partial derivative of any order of a function in $\mathbb{R}^n$ in a reference point **x** can be approximated by a weighted sum of function values at some neighboring discrete points including **x** itself (support nodes of **x**). For example, the approximation method for m-th derivative with respect to x_1 of a function $f(\mathbf{x})$ at a point $\mathbf{x}_i$ of $\mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$, can be expressed by (Ding *et al.*, 2005):

$$\frac{\partial^m f(\mathbf{x}_i)}{\partial x_1^m} = \sum_{j=1}^{n_s} w_{i,j}^{mx_1} f(\mathbf{x}_j), \quad i = 1, 2, ..., N \tag{1}$$

where $\mathbf{x}_j$, $j = 1, \dots n_s$ are the support nodes of **x** and $w_{i,j}^{mx_1}$ denotes the respective weighting coefficients. The index i refers to the reference node in a global discretization of N nodes while j is a local index for the respective support nodes. This approach can be naturally applied to any dimension. In the case of a structured mesh the local support is called $stencil$.

The key of LDQM is the determination of weighting coefficients, $w_{i,j}$. For this, a set of n_s basis functions is required. When Basis Radial Functions (RBF) are employed the method is referred to LDQM-RBF. Among the various RBF, the multiquadric function (Mq) is chosen due its to accuracy, stability and efficiency (Franke, 1982). The basis functions, φ_k, generated by the Mq are:

$$\varphi_k(\mathbf{x}) = \sqrt{(\|\mathbf{x} - \mathbf{x}_k\|_2)^2 + c^2}, \quad k = 1, 2, ..., n_s \tag{2}$$

where **x** = $(x_1, x_2, ...x_{n_s})$ and c is a shape parameter. Substituting this set of radial basis functions in Equation (1), we obtain the following system of linear algebraic equations for the weighting coefficients:

$$\frac{\partial^m \varphi_k(\mathbf{x}_i)}{\partial x_1^m} = \sum_{j=1}^{n_s} w_{i,j}^{mx_1} \varphi_k(\mathbf{x}_j), \quad k = 1, 2, ..., n_s \tag{3}$$

In matrix form, the vector of weighting coefficients $\{w\}_i$ can be obtained by:

$$\left\{ \frac{\partial^m \varphi_k(\mathbf{x}_i)}{\partial x_1^m} \right\} = [A]\{w\}_i \tag{4}$$

After the solution of the system, the values of the weighting coefficients can be used to approximate the derivatives. For the Mq, Micchelli (1986) showed that the matrix $[A]$ may have cases of singularity. On the other hand, Hon and Schaback (2001) proved that these cases are extremely rare and can be disregarded in practical situations.

When using the formulation LDQM-RBF for an equidistant uniform mesh, the weighting coefficients are calculated only once and stored for any domain discretization.

2. TESTS OF LQDM-RBF FOR A POISSON-TYPE EQUATION INCLUDING FIRST ORDER DERIVATIVES

This section aims to evaluate some relevant numerical information about the LQDM-RBF for PDE solutions. For this, the LQDM-RBF was applied to a equation of Poisson-type including a non-linear term with first order derivatives in a two-dimensional unitary square domain. Several parameters were tested, among them the multiquadric shape parameter c, the mesh size h and the $stencil$ structure (number and distribution of nodes).

The equation used is shown below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = f(x, y) \tag{5}$$

The boundary conditions are given by the Dirichlet condition, i.e., the u values are given at the domain boundary. Similarly, when using local $stencils$ that make use of nodes outside the domain, the u values for these points must also be known.

Given an analytical solution $u(x, y)$ is possible to obtain the $f(x, y)$ according to Eq. (5). For numerical tests, we adopted the analytical solution u_p, taken from Shu *et al.* (2003), given by:

$$u_p(x, y) = \frac{\frac{5}{4} + \cos(5.4y)}{6 + 6(3x - 1)^2} \tag{6}$$

After discretization of Eq. (5) by LQDM-FBR, the numerical solution is obtained by the method of Successive Over Relaxation (SOR) with the first derivatives terms treated explicitly. The relative error used to measure the accuracy of the method is defined as

$$\|\varepsilon\| = \frac{\sqrt{\sum_{i=1}^{N_{int}} (u_{num} - u_{exact})_i^2}}{\sqrt{\sum_{i=1}^{N_{int}} (u_{exact})_i^2}} \tag{7}$$

First, the mesh was fixed at 41×41 and the shape parameter c was varied in the interval $0.06 \leq c \leq 2.00$ for the $stencils$ shown in “Fig. 1”. The results for the variation of relative error with the shape parameter is shown in “Fig. 2a)”. It may be noted the dependence of optimal values of c in relation to the structure of the $stencils$, as well as numerical instabilities for some $stencils$ and for shape parameter above a certain value. for some from a certain value (c). Best results are produced as the extent of the $stencil$ increases.

For verifying the influence of the shape parameter c in the numerical error in conjunction with the mesh refinement, the $stencil$ 1 was fixed and the shape parameter varied in the same interval as before with different meshes. It may be noted

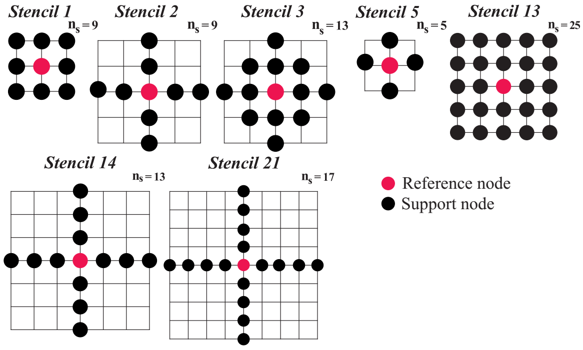

Figure 1. Node number and distribution for the tested stencils.

in “Fig 2b)” a change in the slope of the curves for values for the shape parameter between 0.4 and 0.5. The tendency of variation of the relative error and therefore there optimal values of c are independent of the mesh refinement h. These results support theoretical studies of Bayona *et al.* (2010).

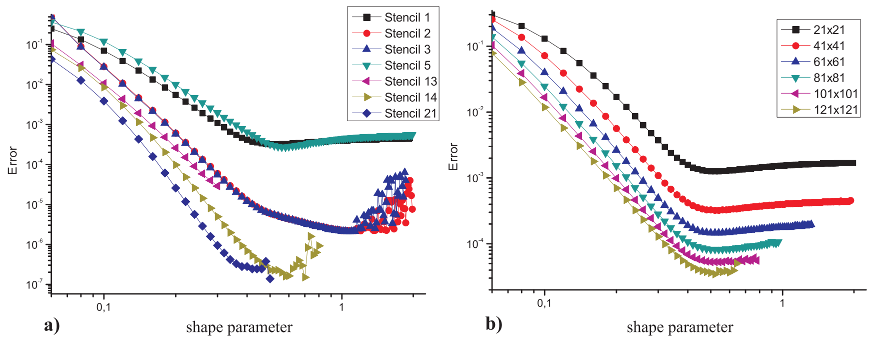

Figure 2. Relative error variation with the shape factor (a) for the 41x41 mesh and different stencils and (b) for the *stencil* 1 and different meshes.

3. LDQM-RBF APPLICATIONS IN FLUID DYNAMICS AND HEAT TRANSFER PROBLEMS

The LDQM-RBF was used for obtaining numerical solutions of two classical benchmark problems of incompressible 2-D flow: (a) the driven cavity flow problem with a constant velocity imposed on the upper wall (“Figure 3(a)”); (b) the natural convection in a square enclosure with adiabatic horizontal walls and isothermal vertical walls subjected to a temperature difference (“Figure 3(b)”). The stream function-vorticity formulation in non-dimensional form was employed in both cases. For problem (b) a superconductor model was used for treating exterior nodes close to the isothermal walls.

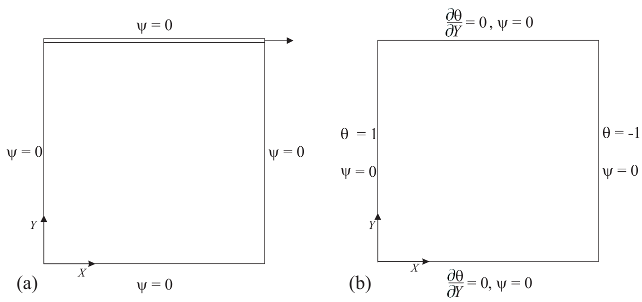

Figure 3. Domain and boundary conditions for the flow problems studied in this work.

“Figure. 3” illustrates the domains and the boundary conditions for each case. The solutions of the equations of stream function were obtained by the SOR method and the equations of vorticity and energy (in the case of natural convection) were obtained by the explicit Euler method. The *stencils* 1, 2 and 3 were used. The shape parameter is fixed at 0.5 and the mesh was 101×101.

The velocities at the vertical and horizontal center lines in problem (a) are deployed in "Fig. 4" for number Reynolds $Re = 400$ and compared with results obtained by Ghia *et al.* (1982). The present results are in good agreement with those of the literature.

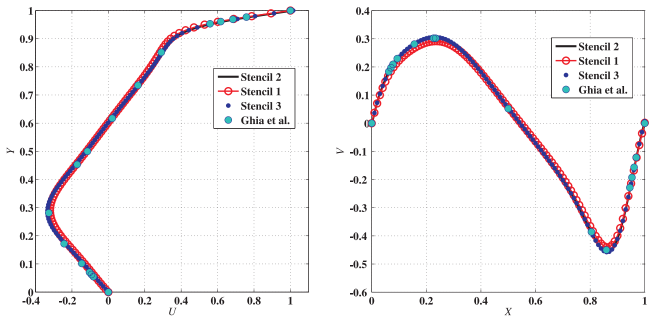

Figure 4. Results for the velocities in the vertical an horizontal centerlines in problem (a) (Re = 400).

For problem (b), the average Nusselt number for $Ra = 10^5$ was calculated in the hot wall ("Tab. 1") and compared with results obtained by Shu *et al.* (2003). The average Nusselt number obtained by the authors was 4.573 (5338 nodes). One can note a good agreement of results for $stencil$ 1 only. $Stencils$ 2 and 3, although potentially more accurate, require the super-conductor model which is not quite compatible with the flow conditions in the physical domain. Better schemes for address this limitation should be considered in the near future.

Table 1. Average Nusselt number for $Ra = 10^5$.

	$stencil$ 1	$stencil$ 2	$stencil$ 3
101×101	4.590	4.178	4.179

4. ACKNOWLEDGEMENTS

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5. REFERENCES

Bayona, V., Moscoso, M., Carretero, M. and Kindelan, M., 2010. "RBF-FD formulas and convergence properties". *Journal of Computational Physics*, Vol. 229, pp. 8281–8295.

Ding, H., Shu, C. and Tang, D., 2005. "Error estimates of local multiquadric-based differential quadrature (LMQDQ) method through numerical experiments". *International Journal for Numerical Methods in Engineering*, Vol. 63, pp. 1513–1529.

Franke, R., 1982. "Scatered data interpolation: tests of some methods". *Mathematics of Computation*, Vol. 38, pp. 181–199.

Ghia, U., Ghia, K. and Shin, T., 1982. "High-re solutions for incompressible flow using the navier-stokes equations and a multi-grid method". *Journal of Computational Physics*, Vol. 48, pp. 387–411.

Hon, Y. and Schaback, R., 2001. "On unsymmetric collocation by radial basis functions". *Applied Mathematics and Computation*, Vol. 119, pp. 177–186.

Micchelli, C., 1986. "Interpolation of scattered data: distance matrices and conditionally positive definite functions". *Constructive Aproximation*, Vol. 2, pp. 11–22.

Shu, C., Ding, H. and Yeo, S., 2003. "Local radial basis function-based differential quadrature method and its application to solve two-dimensional incompressible navier-stokes equations". *Computer Methods in Applied Mechanics and Engineering*, Vol. 192, pp. 941–954.