APPLICATION OF THE DOUBLE INTEGRAL METHOD TO TRANSIENT NONLINEAR HEAT TRANSFER PROBLEMS

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***Abstract**. The objective of this work is to use the Double Integral Method to solve problems of transient unidimensional conduction heat transfer in semi-infinite body with a non homogeneous Dirichlet, non homogeneous Neumann and Robin boundary conditions. The double integral method is a mathematical technique that can be used to obtain approximate solutions to transient heat transfer problems. This method transforms the non-linear boundary value problem into an initial-value problem, whose solution can often be expressed in a closed analytical form. In the double integral method the partial differential equations are integrated twice, the first integration being performed within the domain and the second along the phenomenological distance. This double integration allows the gradient vector at the surface to be approximated using the Simple Integral Method. Thus improvements can be attained by changing the derivative at the boundary by an integral relation, since the process of differentiation amplifies any difference between the assumed temperature profile and the exact solution. This double integration allows the calculation of the gradient vector at the surface to be approximated by an integral relationship, thus replacing the calculation of the derivative in the boundary which amplifies any difference between the exact solution and the assumed profile, by an integral relationship. In this work, the results obtained were compared with analytical exact solutions found in the literature and approximate analytical solutions provided by the method of Goodman to show that the temperature profiles obtained with the double integral method are better than those presented by Goodman, the improved temperature distribution being closely related to improvements made by the double integral method in the approximation of heat flow along the body*

***Keywords**: Double Integral Method, Approximate Analytical Solution, Refinement of the Karman-Pohlhausen Method.*

1 INTRODUCTION

The proliferation of numerical and computational techniques and the availability of software packages have neglected analytical methods for solving heat transfer problems. There is no doubt that computer programs represent a breakthrough especially in problems of irregular geometries. However, it is important to study and develop exact and approximate analytical methods so that computer programs are optimized demanding less processing time. In this sense this work makes use of the Double Integral Method proposed by Volkov (1965) to solve boundary layer equations. In his work the author conducted a thorough study of the Karman-Pohlhausen (1921) Simple Integral Method and proposed a refinement to this method by means of the double integration of the boundary layer equations.

The results presented by Volkov (1965) are better than those obtained by Karman-Pohlhausen method and this fact encouraged investigators to develop other studies using this method. Tse-Fou (1976) addressed the boundary layer problem considering large variations of the Prantdl number. El-Genk &Cronenberg, (1979) presented two articles, the first to test the double integral method to verify the accuracy of the solution in phase change problems and the second to obtain an approximate solution for the growth or shrinkage of an ice thickness on a cold plate in contact with a forced flow. In the present work the double integral method was applied to transient heat transfer problems in semi-infinite body with boundary conditions of the first, second and third kinds. In each case the profiles used are polynomials of degrees two and three. The objective is to show the influence of increasing the degree of the polynomial in the description of the temperature profile and heat flow along the body.

2 APPLICATIONS OF THE DOUBLE INTEGRAL METHOD

In this work, all the applications of the double integral method considered a semi-infinite solid with the geometry illustrated in Fig.1. A semi-infinite solid has as its main characteristic to extend infinitely in all directions except one in a way that this solid has a unique identifiable surface. If a sudden change in the boundary condition is imposed on this surface, unidimensional transient conduction occurs. This type of geometry provides a useful idealization in many practical problems, for example, it can be employed in the determination of transient heat transfer near the earth's surface or can be used to approximate the transient response of a finite solid as a thick plate. This last application is reasonable for the initial portion of the transient regime, during which the temperature inside the plate at points distant from the surface are essentially not influenced by the change in the boundary conditions.

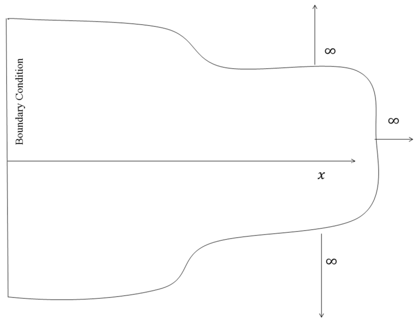

Figure 1. Semi-infinite solid

2.1Application of the double integral method with boundary condition of first kind

The double integral method will be applied to solve a unidimensional thermal conduction problem involving a semi-infinite body extending along the axis $x > 0$, having a non-homogeneous Dirichlet type boundary condition at $x = 0m$, initially at temperature of zero degrees Celsius. Thus, the physical system is well established and the mathematical model is described by the equation of heat conduction and the corresponding boundary conditions given below.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

$$T(0,t) = T_S \tag{2}$$

$$T(\delta,t) = 0 \tag{3}$$

$$\frac{\partial T}{\partial x}(\delta,t) = 0 \tag{4}$$

$$\frac{\partial^2 T}{\partial x^2}(\delta,t) = 0 \tag{5}$$

By applying the double integral method in Eq. (1) results:

$$\int_0^{\delta(t)} \int_0^{x} \alpha \frac{\partial^2 T}{\partial x^2} = \int_0^{\delta(t)} \int_0^{x} \frac{\partial T}{\partial t} dx dx \tag{6}$$

$$\int_0^{\delta(t)} \alpha \frac{\partial T}{\partial x}(x,t)dx - \int_0^{\delta(t)} \alpha \frac{\partial T}{\partial x}(0,t)dx = \int_0^{\delta(t)} \frac{\partial}{\partial t}\left(\int_0^{x} T(x,t)dx \right)dx \tag{7}$$

and approximating the gradient vector at the boundary using the simple integral method,

$$\alpha \frac{\partial T}{\partial x}(0,t) = \int_0^{\delta(t)} \frac{\partial T}{\partial t} dx \tag{8}$$

Substituting Eq. (8) into Eq. (7):

$$\int_0^{\delta(t)} \alpha \frac{\partial T}{\partial x}(x,t)dx + \int_0^{\delta(t)} \int_0^{\delta(t)} \frac{\partial T}{\partial t} = \int_0^{\delta(t)} \int_0^{x} \frac{\partial T}{\partial t} \tag{9}$$

Assuming cubic and quadratic polynomial profiles for the temperature distribution given by Eq. (10) and Eq. (11)

$$T(x,t) = T_S \left(1 - \frac{x}{\delta(t)} \right)^2 \tag{10}$$

$$T(x,t) = T_S \left(1 - \frac{x}{\delta(t)} \right)^3 \tag{11}$$

Substituting Eq. (10) and Eq. (11) in Eq. (9) results in the following ordinary differential equations:

$$6\alpha = \delta(t)\frac{d\delta}{dt} \tag{12}$$

$$10\alpha = \delta(t)\frac{d\delta}{dt} \tag{13}$$

Solving both ordinary differential equations with the initial condition given by Eq. (14) and subsequently replacing the quadratic profile given by Eq. (10) and the cubic profile given by Eq. (11) the temperature distributions along the semi-infinite body is completely determined.

$$\delta(0) = 0 \tag{14}$$

Table 1 show the temperature profiles and calculate the heat flux at the surface for similarity method, integral and double integral single.

Table 1. Comparison of temperature distribution profiles.

Method	Temperature Profile $T(x,t)$	Heat Flux at $x=0$	Error
Similarity	$T_s\left(1-erfc(\frac{x}{2\sqrt{\alpha t}})\right)$	$\frac{\sqrt{\frac{1}{\pi}}T_s k}{\sqrt{\alpha t}}$	0%
Double Integral (Quadratic Profile)	$T_s\left(1-\frac{x}{\sqrt{12\alpha t}}\right)^2$	$\frac{\frac{2}{\sqrt{12}}T_s k}{\sqrt{\alpha t}}$	2.3%
Double Integral (Cubic Profile)	$T_s\left(1-\frac{x}{\sqrt{20\alpha t}}\right)^3$	$\frac{\frac{3}{\sqrt{20}}T_s k}{\sqrt{\alpha t}}$	18.8%
Simple Integral (Cubic Profile)	$T_s\left(1-\frac{x}{\sqrt{24\alpha t}}\right)^3$	$\frac{T_s k\sqrt{\frac{3}{8}}}{\sqrt{\alpha t}}$	8.5%

Figure 2 shows the temperature distribution profiles obtained by similarity, simple and double integral methods. It may be observed that the double integral method with quadratic profile exhibits better accuracy to describe the temperature distribution when compared with Goodman's method with cubic profile. This improved accuracy is related to the way each of the methods describe the heat flow along the body, as shown in Figure 3.

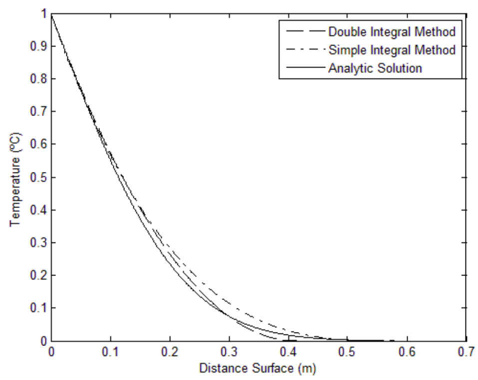

Figure 2. DIM quadratic profile

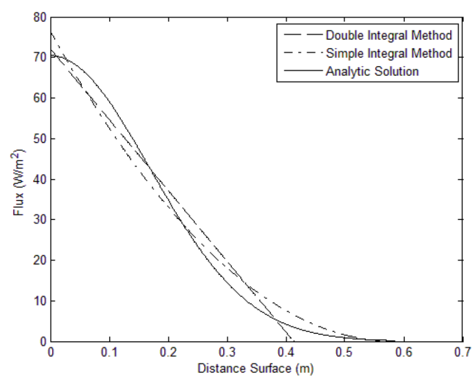

Figure 3. DIM flux quadratic profile

Figure 5 shows the temperature distribution obtained by the methods discussed above, except that in this case the double integral method is used with cubic profile. As it is shown in Fig.3, the curve described by the double integral best fits the exact analytical solution, when compared to the approximation made by the simple integral method. Figure 4 exhibits the distribution of the thermal flow across the body. Notice that the approximation made by the double integral method for heat flux at the boundary is a bit coarse. However throughout the body this approach is best suited to the analytic solution when compared to the method of Goodman. Thus it is evident that the best accuracy in the description of the temperature profile is closely related to better precision in the description of the flow along the body.

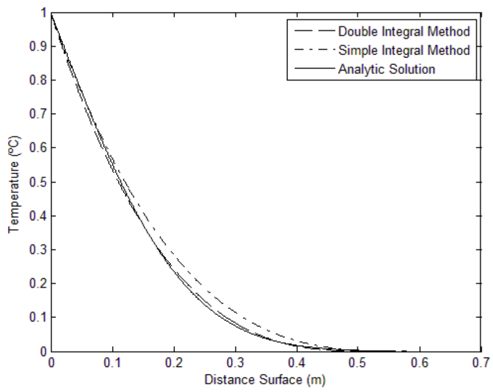

Figure 5. DIM cubic profile

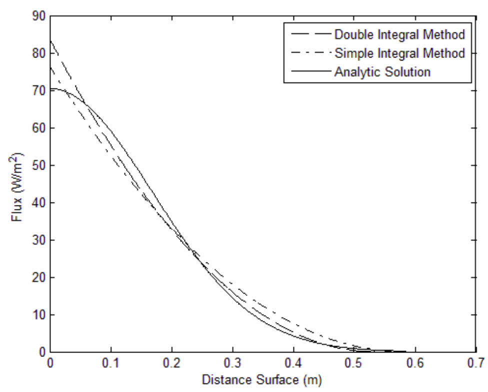

Figure 4. DIM flux cubic profile

2.1 Application of the double integral method with boundary condition of second kind.

Consider a semi-infinite plate extending along the axis $x > 0$ initially at temperature of zero degrees Celsius, assuming its boundary at a specified heat flux and equal to $f(t) = 1000\dfrac{W}{m^2}$. Thus the physical system is completely determined and the mathematical model is described by the conduction equation and the boundary conditions given below.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{15}$$

$$-k\frac{\partial T}{\partial x} = f(t) \tag{16}$$

$$\frac{\partial T}{\partial x}(\delta, t) = 0 \tag{17}$$

$$T(\delta, t) = 0 \tag{18}$$

$$\frac{\partial^2 T}{\partial x^2} = 0 \tag{19}$$

Considering the quadratic and cubic polynomial profiles for the temperature distribution given respectively by Eq. (20) and Eq. (21),

$$T(x,t) = \frac{f}{2k\delta}(\delta - x)^2 \tag{20}$$

$$T(x,t) = \frac{f}{3\delta^2 k}(\delta - x)^3 \tag{21}$$

Applying the double integral method in the heat conduction Eq. (15)

$$\int_0^{\delta(t)} \alpha \frac{\partial T}{\partial x}(x,t)dx - \int_0^{\delta(t)} \frac{f(t)}{k}dx = \int_0^{\delta(t)} \int_0^{x} \frac{\partial T}{\partial t} dxdx \tag{22}$$

Substituting the temperature profiles given by Eq. (20) and Eq. (21) into Eq. (22) yields the corresponding ordinary differential equations

$$\left(\frac{5}{12}\right)\frac{\delta d\delta}{dt}=\alpha \tag{23}$$

$$\left(\frac{7}{40}\right)\frac{\delta d\delta}{dt}=\alpha \tag{24}$$

The ordinary differential equations Eq. (23) and Eq. (24) are solved with the initial condition given by

$$\delta(0)=0 \tag{25}$$

Table 2 shows a comparison of the solutions for the profiles obtained with the similarity method, simple integral method with cubic profile and double integral method with quadratic and cubic profiles. These solutions were used to calculate the temperature at the surface.

Table 2. Evaluation of surface temperature

Method	Temperature Profile $T(x,t)$	$T(0,t)$	Error
Similarity	$\frac{2f}{k}\left\{\sqrt{\left(\frac{\alpha t}{\pi}\right)}\exp\left(-\frac{x^2}{4\alpha t}\right)-\frac{x}{2}erf\left(\frac{x}{2\sqrt{\alpha t}}\right)\right\}$	$\frac{f\sqrt{\frac{4\alpha t}{\pi}}}{k}$	0%
Double Integral (Quadratic Profile)	$\frac{f}{2k\sqrt{4.8\alpha t}}\left(\sqrt{4.8\alpha t}-x\right)^2$	$\sqrt{1.2}\,\frac{f\sqrt{\alpha t}}{k}$	3.53%
Double Integral (Cubic Profile)	$\frac{f}{3\sqrt{11.428\alpha t}^{2}k}\left(\sqrt{11.428\alpha t}-x\right)^3$	$\sqrt{1.26}\,\frac{f\sqrt{\alpha t}}{k}$	0.66%
Simple Integral (Cubic Profile)	$\frac{f}{k\sqrt{24\alpha t}}\left(\sqrt{24\alpha t}-x\right)^3$	$\sqrt{1.5}\,\frac{f\sqrt{\alpha t}}{k}$	9.0%

Figure 6 shows the approximation of the temperature distribution performed by both integral methods with quadratic profile compared to the exact analytical solution. It may be observed that the double integral method exhibits higher accuracy for distances shorter than $0.1m$. As shown in Fig. 7, better approximation in the description of the temperature profile for distances less than $0.1m$, is linked to the higher accuracy of the double integral method in describing the heat flow in this same distance.

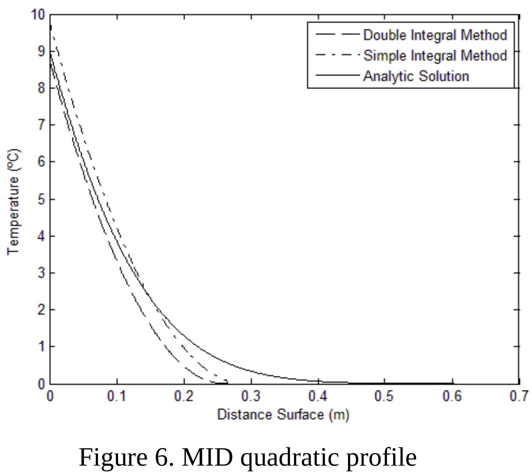

Figure 6. MID quadratic profile

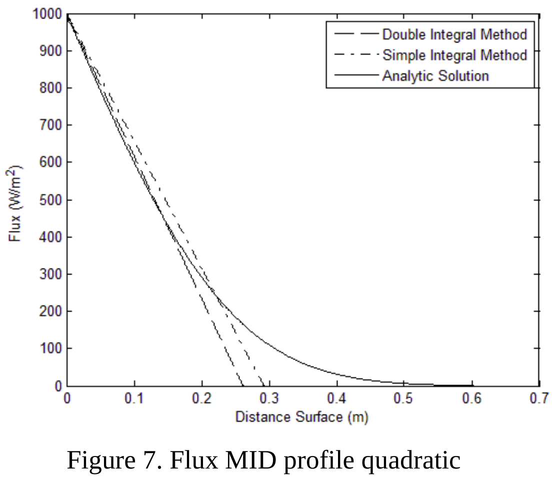

Figure 7. Flux MID profile quadratic

Figure 8 shows the temperature distribution obtained with the similarity method, as well as the approximations performed by the integral methods with cubic profile. By analyzing this figure it is noticeable that significant improvement in the temperature profile results when compared with Fig. 5 where the integral methods were used for a quadratic profile. As shown in Fig. 8, the approximation of the heat flux is described by a parabolic profile when using a temperature cubic profile, which is reflected in a significant increase in the description of heat flow along the body and hence a better precision in the description of the temperature profile.

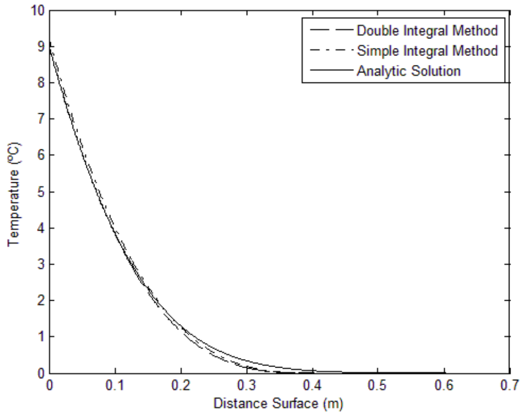

Figure 8. DIM cubic profile

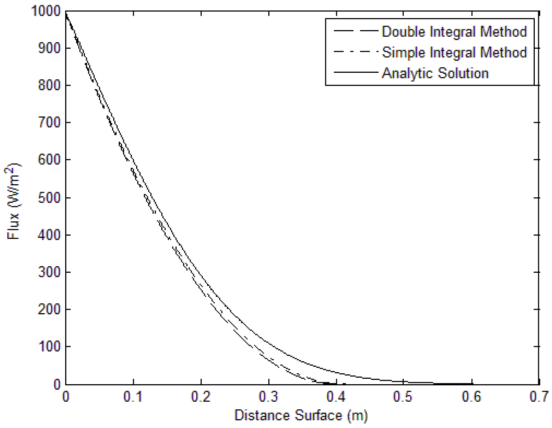

Figure 9. DIM flux cubic profile

2.3 Application of the double integral method with Robin's boundary condition

Here the double integral method is applied to solve a problem involving one-dimensional thermal conduction in a semi-infinite body which is initially at zero degrees Celsius and with a boundary condition at $x = 0m$ of Robin type. Assuming that initially the boundary condition has its most generic form which is convenient when particularized, has the problem has the complete physical description, and its mathematical model can be given by heat conduction equation and their boundary conditions described below.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{26}$$

$$T(\delta, t) = 0 \tag{27}$$

$$\frac{\partial T}{\partial x}(\delta, t) = 0 \tag{28}$$

$$\frac{\partial T}{\partial x}(0, t) = -f(z, t) \tag{29}$$

$$\frac{\partial^2 T}{\partial x^2}(\delta(t), t) = 0 \tag{30}$$

Considering the quadratic and cubic polynomial profiles for the temperature distribution given by Eq. (31) and Eq. (32)

$$T(x,t) = \frac{f(z,t)}{2\delta}(\delta - x)^2 \tag{31}$$

$$T(x,t) = \frac{f(z,t)}{3\delta^2}(\delta - x)^3 \tag{32}$$

Making a change of variable indicated by Eq. (33) and Eq. (34), respectively, and substituting the quadratic and cubic profile it is possible to rewrite these equations as a function of the surface temperature:

$$z(t) = T(0,t) \tag{33}$$

$$\delta(t) = \frac{2z}{f(z,t)} \tag{34}$$

$$\delta(t) = \frac{3z}{f(z,t)} \tag{35}$$

Therefore,

$$T(x,t)=\frac{z(t)}{\delta(t)^2}(\delta-x)^2=T(0,t)\left(1-\frac{x}{\delta}\right)^2 \tag{36}$$

$$T(x,t)=\frac{z(t)}{\delta(t)^3}(\delta-x)^3=T(0,t)\left(1-\frac{x}{\delta}\right)^3 \tag{37}$$

Applying the double integral method in the heat conduction Eq. (26)

$$\int_0^{\delta(t)}\int_0^x \alpha\frac{\partial^2 T}{\partial x^2}=\int_0^{\delta(t)}\int_0^x \frac{\partial T}{\partial t}dxdx \tag{38}$$

After manipulating the previous expression results

$$\int_0^{\delta(t)} \alpha\frac{\partial T}{\partial x}(x,t)dx+\int_0^{\delta(t)}\int_0^{\delta(t)}\frac{\partial T}{\partial t}=\int_0^{\delta(t)}\int_0^x\frac{\partial T}{\partial t} \tag{39}$$

Substituting the quadratic profile Eq. (36) and the cubic profile Eq. (37) into Eq. (36) yields:

$$\frac{d}{dt}\left(\frac{z^3}{f^2(z,t)}\right)=3\alpha z \tag{40}$$

$$\frac{d}{dt}\left(\frac{9z^3}{f^2(z,t)}\right)=20\alpha z \tag{41}$$

Considering that the function $f(z,t)$ given by the boundary condition Eq.(29) is defined as shown in Eq.(39) and substituting it in Eq.(40) and Eq.(41) the results are their ordinary differential equations.

$$f(z_1)=\left(\frac{h}{k}\right).(z_0-z_1) \tag{42}$$

$$\frac{3z_1 f^2(z_1)\frac{dz_1}{dt}-2z_1^3 f(z_1)\frac{df}{dz_1}\frac{dz_1}{dt}}{\left[f^2(z_1)\right]^2}=3\alpha z_1 \tag{43}$$

$$\frac{\left[27z_1^2\frac{dz_1}{dt}f^2(z_1)-18z_1^3\frac{df}{dz_1}\frac{dz_1}{dt}\right]}{f_1^4(z_1)}=20\alpha z_1 \tag{44}$$

The above differential equations are separable and the solution of each one takes the following forms:

$$\frac{4}{3}\left(\frac{h}{k}\right)^2\alpha t=\frac{4}{9}\left(1-\frac{1}{1-\frac{z}{z_0}}\right)+\frac{4}{9}\left(\frac{1}{\left(1-\frac{z}{z_0}\right)}-1\right)+\frac{4}{9}\ln\left(1-\frac{z}{z_0}\right) \tag{45}$$

$$\frac{4}{3}\left(\frac{h}{k}\right)^2\alpha t=0{,}6\left(\frac{1}{\left(1-\frac{z}{z_0}\right)^2}-1\right)+0{,}6\ln\left(1-\frac{z}{z_0}\right)+0{,}6\left(1-\frac{1}{1-\frac{z}{z_0}}\right) \tag{46}$$

The exact analytical solution presented by Carslaw and Jaeger (1959) is given by:

$$\frac{z}{z_0} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) - e^{\frac{h}{k}x + h^2\alpha t} erfc\left\{\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right\} \quad (47)$$

The equations Eq. (45) and Eq. (46) can be drawn in terms of two of the three dimensionless parameters

$$\frac{x}{2\sqrt{\alpha t}} \quad (48)$$

$$\frac{h}{k}\sqrt{\alpha t} \quad (49)$$

$$\frac{h}{k}.x \quad (50)$$

Figs. (10) and (11) show the graphs of the exact analytical solution, as well as the solutions obtained by the double integral and simple integral method. These graphs were drawn with $\frac{z}{z_0}$ versus $\log_{10}\left(\frac{h}{k}\sqrt{\alpha t}\right)$ with $\frac{x}{2\sqrt{\alpha t}} = 0$.

In Fig.10 both integral methods are using the quadratic profiles in their approaches. As it can be seen, the of the double integral method has low sensitivity to the choice of the profile, whereas the approximation made by simple integral method was found to be a bit coarse when compared with the other two methods. Fig. 11 shows both integral methods using cubic profiles. It may be noticed a significant improvement in the approximation of the simple integral method while the double integral method presented only marginal improvements. It is practically impossible to distinguish each of the integral solutions in this figure.

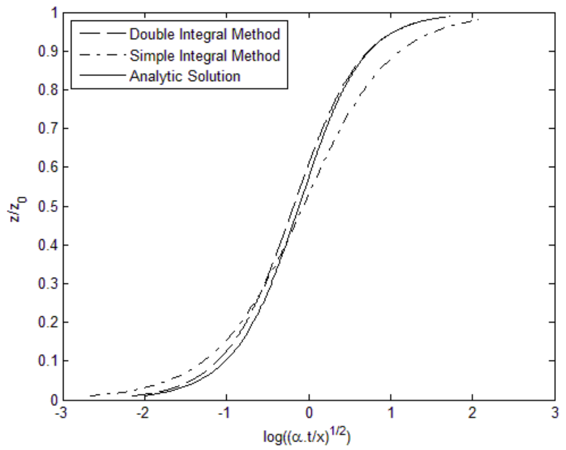

Figure 10. DIM quadratic profile

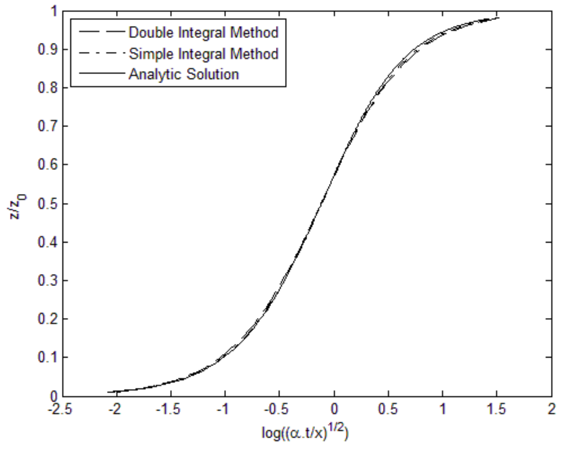

Figure 11. DIM cubic profile

3. ACKNOWLEDGEMENTS

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