ACCURACY ASSESSMENT OF THE SPECTRAL DIFFERENCE SCHEME FOR CONSERVATION LAWS

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Abstract. *The Spectral Difference method is designed to support high-order accuracy on unstructured grids. It uses a domain transformation formulation that eases the implementation while improving computational efficiency. The authors investigate various solution and flux point distributions of different orders of accuracy for model problems. The method is found to achieve the expected order of accuracy and its one-dimensional reconstruction procedure will be used for 2-D simulations on the final paper as reported here.*

Keywords: *High-Order, Spectral Difference, Unstructured Meshes, CFD*

1. INTRODUCTION

The Spectral Difference (SD) method represents a recent development on high-order schemes for unstructured meshes, initially developed by Liu *et al.* (2006); Wang *et al.* (2007). The SD method has shown favorable results for literature test cases regarding solution accuracy and performance in favor of the Spectral Volume and Discontinuous Galerkin schemes, for instance in the work of Sun *et al.* (2007) and Liang *et al.* (2011). Moreover, it is based on a simpler formulation that leads to an easier numerical implementation and extension to two and three dimensions. The SD scheme is very popular among high-order research groups throughout the world. The CFD group at DCTA/IAE has an extensive experience with numerical methods for aerodynamic applications, including high-order methods such as the ENO and WENO schemes and Spectral Finite Volume method for example. The authors expected that the SD method is capable to outperform these methods in computational efficiency and improve the resolution capabilities for the problems of interest of the institute, mainly external compressible aerodynamic analysis. The present paper presents an effort towards the implementation of the SD method for conservation laws in one dimension, in order to assess its order of accuracy and efficiency for model problems. The solver formulation is currently being extended for two dimensions and the final paper will consider solutions of the Euler equations for compressible flow applications.

2. SPECTRAL DIFFERENCE METHOD

The SD method is designed to work with unstructured mesh on a transformed space, referred to as the computational space, to enable an efficient computational performance. The standard element in such space is made up with two sets of collocation points for properties storage. These sets are known as Solution Points (SP) and Flux points (FP). The amount of SP and FP in the standard element defines the order of the method, as illustrated in Fig. 1. The conserved variables at the SP are used to create an interpolation function for the flux computed at the FP. Across the element interface a numerical flux is required that brings the dissipation and upwind characteristic of the scheme. This computation requires that two neighboring mesh elements share the same points distribution. Therefore, the SD formulation is better suited for meshes with a unique type of elements. The literature recommends the adoption of quadrilateral elements. Section 4.presents the 2-D SD method formulation in greater detail as currently being implemented by the authors. The time integration of these equations, in the present work, is computed with an explicit Runge-Kutta type scheme, as indicated by Wang *et al.* (2007).

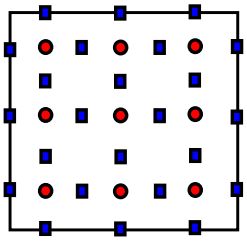


Figure 1. Possible flux points (blue squares) and solution points (red circles) distribution for the SD method.

The SD method formulation for quadrilateral elements follows a one-dimensional reconstruction procedure. That is, the required computations are applied at one dimension at a time. In order to better understand and develop the tools required for high-order calculations, the authors initially considered a 1-D solver for scalar laws of conservation. It was found that the positioning of both SP and FP bear a large influence on the numerical solution accuracy and stability. Moreover, the expected order of accuracy was achieved for the model convection equation on sample problems and those were found to be in good agreement with the literature.

3. ACCURACY ASSESSMENT

The 1-D study showed that the overall order of accuracy of the method was achieved for various initial conditions. The model convection equation is used as the model problem with periodic boundary conditions. The points distribution within the standard element follows that of den Abeele *et al.* (2007). Figures 2(b) and 2(a) show that for the appropriate element size, or mesh spacing, the nominal order of accuracy is achieved. The order is measured as the slope of the curve fitted within the points of the figures, as in Table 1 for the L_∞ and L_1 error norms. However, for larger sizes of single cells, it was found that there are not enough SP to represent the problem properly. The 1-D results, shown in the figures also indicate another issue with respect to the machine precision that tends to level-off the 5th and 6th-order curves. The same effect is observed on Table 1 for the sine wave distribution. In order to compute the order of accuracy, those points near machine precision were not considered on the curve fit. A comparison with two time integration schemes was also considered to insure that the numerical error is dominated by the spatial discretization method. Both 3rd and 4th-order RK schemes were employed with a very restricted time step.

4. EXTENDED FORMULATION

At the time of this writing, the authors are working on an extend formulation for 2-D problems, mainly to solve the Euler equations. The implementation follows the formulation presented in Refs. Wang *et al.* (2007); May and Jameson (2006). In order to achieve an efficient implementation, all elements in the physical domain (x, y) are transformed into a unit square element in the computational domain. Such transformation can be written as

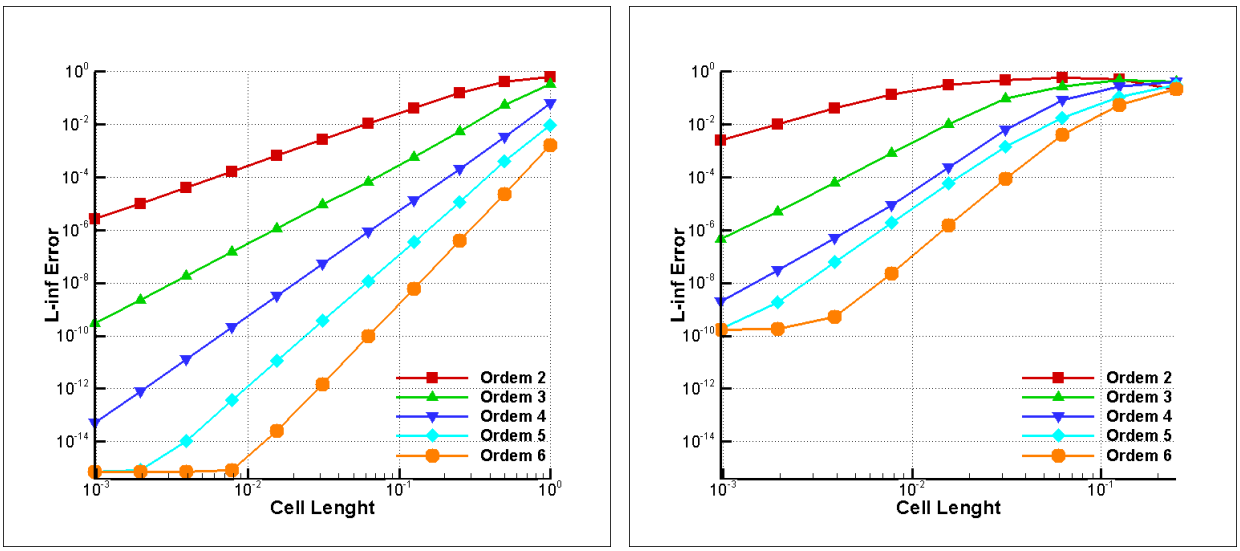
$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{i=1}^{K} M_i(\xi, \eta) \begin{pmatrix} x_i \\ y_i \end{pmatrix} \tag{1}$$

where K is the number of points used to define the physical element, (x_i, y_i) are the Cartesian coordinates of those points, and $M_i(\xi, \eta)$ are the shape functions of the geometric transformation. The governing equations in the physical domain are then transferred into the computation domain , and can be rewritten as

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{E}}{\partial x} + \frac{\partial \tilde{F}}{\partial y} = 0 \tag{2}$$

Table 1. Accuracy assessment of the 1-D SD method for the wave equation.

Order	NDOF	h	L_∞ error	L_∞ order	L_1 error	L_1 order
2	4	1.000E+00	6.021E-01	–	5.413E-01	–
	8	5.000E-01	4.215E-01	0.51	3.108E-01	0.80
	512	7.812E-03	1.637E-04	2.00	1.041E-04	2.00
	2048	1.953E-03	1.023E-05	2.00	6.510E-06	2.00
3	6	1.000E+00	3.387E-01	–	2.029E-01	–
	12	5.000E-01	5.234E-02	2.69	1.937E-02	3.39
	48	1.250E-01	5.535E-04	3.27	2.402E-04	3.12
	3072	1.953E-03	2.338E-09	2.99	9.047E-10	3.00
4	8	1.000E+00	6.080E-02	–	2.709E-02	–
	32	2.500E-01	2.073E-04	3.94	8.377E-05	4.13
	512	1.562E-02	3.260E-09	4.00	1.224E-09	4.00
	2048	3.906E-03	1.263E-11	4.01	4.808E-12	3.99
	8192	9.766E-04	5.157E-13	0.57	3.065E-13	0.27
5	10	1.000E+00	9.320E-03	–	3.630E-03	–
	40	2.500E-01	1.153E-05	5.10	6.207E-06	4.86
	160	6.250E-02	1.142E-08	4.97	6.103E-09	5.00
	640	1.562E-02	1.127E-11	4.99	5.982E-12	4.99
	2560	3.906E-03	2.104E-13	1.02	1.298E-13	0.89
	10240	9.766E-04	7.930E-13	-0.99	4.995E-13	-1.00
6	12	1.000E+00	1.672E-03	–	6.279E-04	–
	48	2.500E-01	4.068E-07	5.80	1.415E-07	6.02
	768	1.562E-02	1.679E-13	3.10	9.192E-14	2.65
	3072	3.906E-03	6.010E-13	-0.98	3.786E-13	-1.00
	12288	9.766E-04	2.386E-12	-0.99	1.515E-12	-1.00



(a) Sine wave (b) Gaussian pulse

Figure 2. Spectral Difference error versus mesh spacing plot for order of accuracy measurement.

where $\tilde{Q}$, $\tilde{E}$ and $\tilde{F}$ are the conserved variables vector and inviscid fluxes, respectively, for the Euler equations in 2-D on the computational domain.

Using the solution at N solution points, a degree $(N-1)$ polynomial can be built using the following Lagrange basis defined as

$$h_i(X) = \prod_{s=0, s\neq i}^{N} \left(\frac{X - X_s}{X_i - X_s} \right) \tag{3}$$

similarly, using the fluxes at $(N+1)$ flux points, a degree N polynomial can be built for the flux using a similar Lagrange basis defined as

$$l_{i+\frac{1}{2}}(X) = \prod_{s=0, s\neq i}^{N} \left(\frac{X - X_{s+\frac{1}{2}}}{X_{i+\frac{1}{2}} - X_{s+\frac{1}{2}}} \right) \tag{4}$$

The reconstructed solution for the conserved variables in the standard element is just the tensor products of the two one-dimensional polynomial, as well as for the reconstructed flux polynomials,

$$\begin{aligned} Q(\xi,\eta) &= \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\tilde{Q}_{i,j}}{|J_{i,j}|} h_i(\xi)\cdot h_j(\eta), \\ \tilde{E}(\xi,\eta) &= \sum_{j=1}^{N}\sum_{i=0}^{N} \tilde{E}_{i+\frac{1}{2},j} l_{i+\frac{1}{2}}(\xi)\cdot h_j(\eta), \\ \tilde{F}(\xi,\eta) &= \sum_{j=0}^{N}\sum_{i=1}^{N} \tilde{F}_{i,j+\frac{1}{2}} h_i(\xi)\cdot l_{j+\frac{1}{2}}(\eta). \end{aligned} \tag{5}$$

The reconstructed fluxes are only element-wise continuous, but discontinuous across cell interfaces. In the present work, the Roe approximate Riemann solver is considered.

At the moment of writing, the SD method implementation is currently under way on the available numerical framework. A second, third and fourth-order reconstruction procedure will be considered for benchmark against literature test cases.

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6. RESPONSIBILITY NOTICE

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