STATIONARY AND TRANSIENT SOURCE RECONSTRUCTION IN CARTESIAN GEOMETRY

Nilson Costa Roberty, nilson@con.ufrj.br
Programa de Engenharia Nuclear-Coppe/UFRJ/RJ/Brasil

Marcelo José Colaço, colaco@mecanica.ufrj.br
Programa de Engenharia Mecanica-Coppe/UFRJ/RJ/Brasil

Carlos José dos Santos Alves, carlos.alves@math.ist.utl.pt
Centro de Matematica e Aplicações-Ist/Utl/Lisboa/Pt

Abstract. *In this work we present a methodology to reconstruct rectangular sources in the transient heat conduction equation model which is based on the minimization of the difference between the solutions of the model consistent with the inverse problem Cauchy data in the boundary. The Nelder-Mead simplex direct search algorithm starting with random initial values is used to reconstruct the given rectangular source by using only the Cauchy data in the boundary determined with a different solution of the direct problem.*

Keywords: *rectangular source reconstruction, heat equation, transient problem, stationary problem, Green's function*

1. INTRODUCTION

Stationary and transient source reconstruction from Cauchy boundary data has been investigate in previous works (Roberty and Rainha (2010)) and (Roberty and Rainha (2011)). In this work we present a new methodology based on Green's and Neumann functions for the first and the second initial boundary value problem (Friedman, 1964) for the second order parabolic partial differential equation related with the heat equation. Source reconstruction with various frequencies was investigated in (Alves *et al.* (2009)). and with the methods of fundamental solution in (Alves *et al.* (2008)).

2. DIRECT TRANSIENT HEAT EQUATION PROBLEM

By $\Omega \subset \Re^d$, $d = 1, 2, 3$ we denote a bounded domain with smooth boundary $\Gamma = \partial\Omega$, which means that it will be locally parametrized with C^∞ functions and that Ω is locally on one side of its connected boundary. In the spatial surface Γ the normal ν is defined almost everywhere and the induced measure on the surface is denoted by $d\sigma$. In the time-space $\Re^{d+1}$, we consider the time interval $I := (0, T)$, $T > 0$ to form the bounded cylinder $Q := I \times \Omega$, whose lateral time-space surface is $\Sigma := I \times \Gamma$. A section in this cylinder is $\Omega_t := \{t\} \times \Omega$, $t \in I$, and the complete cylinder boundary is

$$\partial Q = \overline{\Sigma} \cup \Omega_0 \cup \Omega_T,$$

where Ω_0 and Ω_T are, respectively, the cylinders' bottom and top sections. At cylinder top and bottom there exist the corners $\Gamma_0 = \overline{\Omega_0} \cap \overline{\Sigma} \subset \Re^{d-1}$ and $\Gamma_T = \overline{\Omega_T} \cap \overline{\Sigma} \subset \Re^{d-1}$, respectively.

The direct transient heat source initial boundary value problem consists in to find $u(t, x)$, with $(t, x) \in Q$, given a boundary input $g(t, x)$ with $(t, x) \in \overline{\Sigma}$, an initial input $u_0(x)$, with $(t, x) \in \Omega_0$, and a source distribution $f(t, x)$ with

$(t,x) \in Q$ that verifies the problem :

$$(P_{u_0,g,f}) \begin{cases} \frac{1}{\alpha}\partial_t u - \Delta u = f & \text{in } Q, \\ u = u_0 & \text{in } \Omega_0, \\ u = g & \text{on } \Sigma. \end{cases} \tag{1}$$

and Dirichlet data compatibility condition, $u_0 = g$ at the time-space cylinder corner Γ_0.

The following Hilbert space norm:

$$||v||_{L^2(I;X)} = (\int_I ||v||_X^2 dt)^{\frac{1}{2}} < \infty.$$

is appropriated to control errors in the space time cylinder. Simplification for the stationary case can be obtaining by considering time independent fields.

3. THE INVERSE SOURCE PROBLEM

The inverse source problem that we address consists in the recovery of the source f, knowing the Extended Dirichlet to Neumann map, which is a combination of the Cauchy data including values of the temperature field at the space time boundary and values of the heat flux at the space boundary (Roberty and Rainha (2011)). It is proved that only one set of Cauchy data contains all information to be used in source reconstruction and a reciprocity gap based formulation permit us to present a reconstruction methodology which is independent of the direct problem. We alternatively can formulate the inverse problem in a way which is dependent on the direct problem and that suggests another methodology, based on a minimization problem. That is, the inverse problem will be: To find the temperature and the source distribution $u(t,x)$ and $f(t,x)$ with $(t,x) \in Q$, given a boundary input for the temperature field $g(t,x)$ and for the normal derivative $g_\nu(t,x)$ with $(t,x) \in \overline{\Sigma}$, an initial input $u_0(x)$ with $(t,x) \in \Omega_0$ with $(t,x) \in Q$ that verifies the problem :

$$(IP_{u_0,g,g_\nu}) \begin{cases} \frac{1}{\alpha}\partial_t u - \Delta u = f & \text{in } Q, \\ u = u_0 & \text{in } \Omega_0, \\ u = g \text{ and } u_\nu = g_\nu & \text{on } \Sigma. \end{cases} \tag{2}$$

3.1 Optimization problem

Since there exists an over determination of Cauchy data in the space boundary, we can partition (2) in two well posed direct problems, in a way that is similar to the method proposed by (Kohn and Vogelius (1987)) for eletrical impedance tomography problems and has been recently investigated by (Machado (2012)) in the context of topological derivative application to inverse potential problems. For this, let us split the boundary in two disjoint parts, $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then we can alternate the Dirichlet and the Neumann data over these two parts of the boundary, in which two well posed problems can be used to try different sources in order to verify if the source is compatible with the Cauchy data of the inverse problem. Giving a source guess $f(t,x)$, the auxiliary direct transient heat source initial boundary value problem consists in to find $u^{(l)}(t,x)$, for $l = 1,2$, with $(t,x) \in Q$, given a boundary Cauchy input $(g(t,x), g_\nu(t,x))$, with $(t,x) \in \overline{\Sigma}$, an initial input $u_0(x)$, with $(t,x) \in \Omega_0$, with $(t,x) \in Q$ that verifies the problem :

$$(P^{(l)}_{u_0,g,g_\nu,f}) \begin{cases} \frac{1}{\alpha}\partial_t u^{(l)} - \Delta u^{(l)} = f & \text{in } Q, \\ u^{(l)} = u_0 & \text{in } \Omega_0, \\ u^{(l)}|_{\Gamma_l} = g|_{\Gamma_l} \text{ and } u_\nu^{(l)}|_{\Gamma_{\neg l}} = g_\nu|_{\Gamma_{\neg l}} & \text{on } \Sigma. \end{cases} \tag{3}$$

Obviously, it is quite simple to understand that when the source is consistent with the Cauchy pair characterizing the inverse problem, the difference of the two solutions will be equal if data do not have noise, $u^{(1)}(t,x,f) \approx u^{(2)}(t,x,f)$. Based on this, we formulate the following Optimization Problem based on the following discrepancy functional:

To find $f(t,x)$ such that

$$f(t,x) = arg\inf\{||u^{(1)}(t,x,h) - u^{(2)}(t,x,h)||_{L^2(I;X)}, h \in \mathcal{C}^f\} \tag{4}$$

The a priori information related with the class of functions C^f in with the search must be done cannot be avoided, since the central question in the inverse source problem is non uniqueness. Meanwhile, it has being proved (Isakov (1990)) that the class of characteristic sources with known intensities presents uniqueness. In this work, in order to introduce an analytic solution to the direct problem, we will restrict our class to rectangular characteristic source with unitary intensity.

3.2 Analytical formulation for the auxiliary problems

Let us consider the time space domain $(0,T)\times\Omega := (0,T)\times\prod_{i=1}^{d}\{-1 < x_i < +1\}$. It has space boundary

$$\partial\Omega = \Gamma = \prod_{i=1}^{d}\Gamma_i^- \cup \Gamma_i^+ := \prod_{i=1}^{d}\{-1 = x_i\}\cup\{x_i = +1\} \tag{5}$$

over which we proceed like (Haji-Sheikh and Beck (1994)) to define the following spatial auxiliary eigenvalue problem

$$-\Delta v_\lambda = \lambda v_\lambda. \tag{6}$$

Different sets of spatial (eigenvalue, eigenfunctions) pairs (λ, v_λ) are consistent with this box domain

1. For all $i = 1,...,d$; $v_\lambda|_{x_i=-1} = 0$ and $\frac{\partial v_\lambda}{\partial x_i}|_{x_i=+1} = 0$ if

$$v_\lambda(x_1,...,x_d) = \prod_{i=1}^{d}\sin(\frac{2k_i+1}{4}\pi(x_i+1)) \text{ with } \lambda = \sum_{i=1}^{d}(\frac{2k_i+1}{4}\pi)^2; \tag{7}$$

2. for all $i = 1,...,d$; $\frac{\partial v_\lambda}{\partial x_i}|_{x_i=-1} = 0$ and $v_\lambda|_{x_i=+1} = 0$ if

$$v_\lambda(x_1,...,x_d) = \prod_{i=1}^{d}\sin(\frac{2k_i+1}{4}\pi(x_i-1)) \text{ with } \lambda = \sum_{i=1}^{d}(\frac{2k_i+1}{4}\pi)^2; \tag{8}$$

3. for all $i = 1,...,d$; $v_\lambda|_{x_i=\pm1} = 0$ if

$$v_\lambda(x_1,...,x_d) = \prod_{i=1}^{d}\sin(\frac{k_i\pi}{2}(x_i+1)) \text{ with } \lambda = \sum_{i=1}^{d}(\frac{k_i\pi}{2})^2. \tag{9}$$

where k_i is a multiindex. An straightforward calculation based on the second Green's formula shows that the analytical solution to the problem

$$-\Delta u + \kappa^2 u + \frac{\partial u}{\partial t} = f(t,x) \in Q \tag{10}$$

with arbitrary but consistent boundary conditions satisfies the equation:

$$u(t,x) = \sum_{\lambda} u_\lambda(t) v_\lambda(x_1,...,x_d) \tag{11}$$

where the transient spatial projection associated with the spatial basis is

$$u_\lambda(t,x) = u_\lambda(0,x)\exp(-(\lambda+\kappa^2)t) + \int_0^t (f_\lambda(\tau,x) + J_\lambda(\tau,x)\exp(-(\lambda+\kappa^2)(t-\tau))d\tau \; ; \tag{12}$$

the transient source projection is

$$f_\lambda(t,x) = \int_\Omega f(t,x)v_\lambda(t,x)dx \tag{13}$$

and the boundary terms are

$$J_\lambda(t,x) = \sum_{i=1}^{d}\{-\int_{\Gamma_i^-}[v_\lambda\frac{\partial u}{\partial x_i} - u\frac{\partial v_\lambda}{\partial x_i}]_{x_i=-1}\frac{dx}{dx_i} + \int_{\Gamma_i^+}[v_\lambda\frac{\partial u}{\partial x_i} - u\frac{\partial v_\lambda}{\partial x_i}]_{x_i=+1}\frac{dx}{dx_i}\}. \tag{14}$$

The spatial basis given by (7), (8) and (9) can be used to construct the three solutions different solutions for the transient heat equation

1. The auxiliary problem (1)

$$(P^{(1)}_{u_0,g^-,g^+_\nu,f})\begin{cases} \frac{1}{\alpha}\partial_t u^{(l)} - \Delta u^{(l)} = f & \text{in } Q, \\ u^{(1)} = u_0 & \text{in } \Omega_0, \\ u^{(1)}(t,.)|_{\Gamma_i^-} = g_i^- \text{ and } u_\nu^{(1)}|_{\Gamma_i^+} = g^+_{\nu_i} & . \end{cases} \tag{15}$$

2. The auxiliary problem (2)

$$(P^{(2)}_{u_0,g^+,g^-_\nu,f})\begin{cases} \frac{1}{\alpha}\partial_t u^{(2)} - \Delta u^{(2)} = f & \text{in } Q, \\ u^{(2)} = u_0 & \text{in } \Omega_0, \\ u^{(2)}(t,.)|_{\Gamma_i^+} = g_i^+ \text{ and } u_\nu^{(2)}|_{\Gamma_i^-} = g^-_{\nu_i} & . \end{cases} \tag{16}$$

3. The auxiliary problem for synthetic data generation

$$(P^{(2)}_{u_0,g^+,g^-_\nu,f})\begin{cases} \frac{1}{\alpha}\partial_t u^{(2)} - \Delta u^{(2)} = f & \text{in } Q, \\ u^{(2)} = u_0 & \text{in } \Omega_0, \\ u^{(2)}(t,.)|_{\Gamma_i^+} = g_i^+ \text{ and } u^{(2)}|_{\Gamma_i^-} = g^-_i & . \end{cases} \tag{17}$$

by direct substitution of the respective boundaries values in the boundary term (14). Note that if data are consistent with the same source problem, that is the Cauchy data

$$\mathcal{C} = \{\left(g_i^-, g_i^+, g_{\nu i}^-, g_{\nu i}^+\right), i = 1, ..., d\} \tag{18}$$

are the same in the three problems, then the source term is the same. In this work we are using the problem (3) to generates the data to be used in the source reconstruction problem. Problems (1) and (2) are using to define the discrepancy functional (4).

3.3 The Nelder-Mead Simplex Algorithm

The discrepancy functional (4) is a non-linear functional of the source function f. The minimization problem can be solved with different optimization algorithms, but the cost of determining derivatives with respect to parameters defining the source makes the utilization of gradient based algorithms more difficult to implement than those algorithms based only on functional evaluation. Such is the case of the Nelder-Mead method introduced by (Lagarias *et al.* (1998)). This algorithm attempt only to minimizes the scalar-value non linear discrepancy function of characteristic sources parameters that can be obtained from truncated Fourier series solutions of problems (1) and (2). It falls in the general class of direct search methods. A non degenerated simplex with the same dimension of the number of parameters to be determined is established at each step. Each iteration begins with this simplex, which is the hull of $n + 1$ vertices in the n dimensional parameters space.The Nelder-Mead simplex algorithm is based on four operations:

1 -Reflection with algorithm parameter $\rho > 0$;

2 -Expansion with algorithm parameter $\chi > 1$;

3 -Contraction with algorithm parameter $0 < \gamma < 1$:

  [i] outside ;

  [ii] inside ;

4 -Shrink with algorithm parameter $0 < \sigma < 1$.

Steps [1]-[3] are used to create a new simplex by attempting to replace the vertex with the highest functional values with a smaller. If this attempt is unsuccessful, then the current simplex is reduced in size using step [4], and the entire procedure is repeated. We adopted here the universal typical algorithm parameters values: $\rho = 1$, $\chi = 2$, $\gamma = \frac{1}{2}$ and $\sigma = \frac{1}{2}$.

4. RESULTS FOR THE TWO DIMENSIONAL STATIONARY CASE

We have implemented the minimization of the discrepancy functional (4) and reconstructed the exact source as given in Tab. 1. The Nelder-Mead simplex direct search algorithm has been started with random generated data. Experimental data has been synthetically produced by using the analytical solution of the Dirichlet problem obtained with basis given by (9). Problems $l = 1$ and $l = 2$ in the optimization functional are obtained with basis (7) and (8), respectively. In this example the Fourier series has been truncated with 100 and the number of collocations points is 40. Figure (1) shows

Table 1. Characteristic source dimensions

Type of source	a_1	b_1	a_2	b_2
Exact source	-0.5000	+0.5000	-0.5000	+0.5000
Random source	-0.4074	+0.0635	-0.4529	+0.4576
Recosntructed 80 iter	-0.4975	+0.0962	-0.7446	+0.9357
Reconstructed 286 iter	-0.4970	+0.4970	-0.4968	+0.4968

the solution of the Dirichelet problem and its Neumann data use in the mixed problems (1) and (2). The discrepancy functional value for this model, with exact source parameters, is $4.2047e - 006$. Convergence is quite satisfactory when the Discrepancy become close to this value for 286 iterations in the Nelder-Mead algorithm.

Figure (2) shows the exact source and the random generated source used to start the algorithm. Figure (3) shows two reconstructions. The first with insufficient number of algorithm iterations but the second with a satisfactory reconstruction obtained with more iterations.

4.1 The case of sources with non connected support

A second experiment has been implemented for a characteristic source with a non connected support as shown in Fig.(4). Contrary of the characteristic source with connected support, for this class of source it is not known a mathematical proof of an uniqueness theorem for the reconstruction from Cauchy boundary data. Here, we have doubled the number of terms in the Fourier series, that is, 200 and also doubled the number of collocation points, 80. The reconstruction is conducted under the information that the source support is bi connected and initial values are randomized as shown in

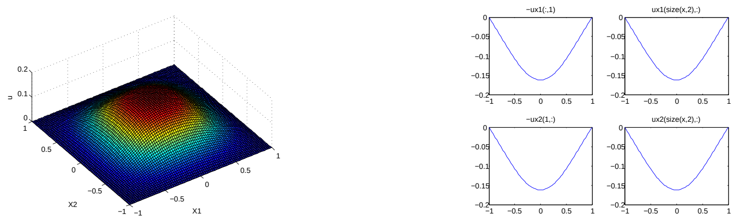

Figure 1. Homogeneous dirichlet model solution with boundary data.

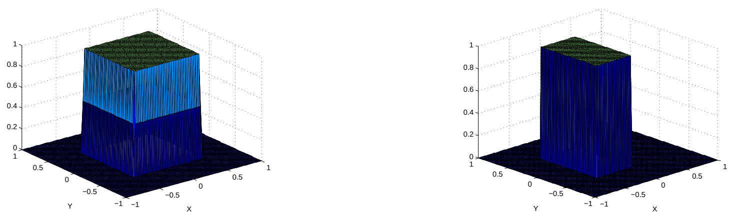

Figure 2. Exact and Random generate source support.

Tab.(2). The discrepancy for the exact source data is $9.1945e-007$. The algorithm stops of improve the reconstruction at discrepancy value $1.6126e-006$. Results are not so good as they were in the connected source case, but it is improved by the algorithm until the numerical precision of the discrepancy function is reached. After this limit value, no further improvement can be reached. From experiment, due to the presence of numerical noise, it is not possible to conclude if there exist only one local minimum of discrepancy function, related with the exact source data, or there exists others bi connected sources that satisfies the same Cauchy data pair. In both case, of course, we can conclude that if there exist more than one source, they are very close.

5. CONCLUSIONS

The present methodology gives an alternative way to determine stationary and transient sources from over determination of boundary conditions in the heat equation model. It is based on the minimization of the difference between two direct well posed problems. Analytical solutions using Fourier series are used in the evaluation of the discrepancy

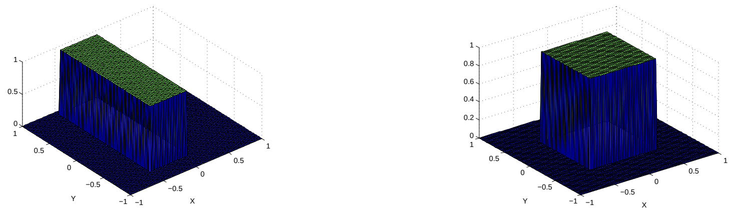

Figure 3. Reconstructed source for 80 and 286 iterations.

Table 2. Characteristic biconnected source dimensions

Type of source	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	Discrepancy
Exact source	-0.7000	-0.1000	-0.7000	-0.1000	0.7000	+0.1000	0.7000	+0.1000	9.1945 e-007
Random source	-0.7156	-0.0689	-0.6169	-0.1929	0.7284	+0.1698	0.7919	+0.1868	3.9987 e-005
Reconstructed 82 iter	-0.7759	-0.0949	-0.6867	-0.1693	0.7359	+0.1774	0.7575	+0.1792	4.0551 e-006
Reconstructed 675 iter	-0.7588	-0.0949	-0.6867	-0.1931	0.6917	+0.0520	0.6715	+0.0698	1.6126 e-006
Reconstructed 1335 iter	-0.7587	-0.0949	-0.6867	-0.1931	0.6919	+0.0520	0.6730	+0.0697	1.6106 e-006

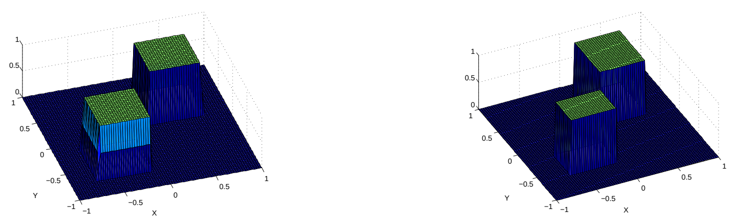

Figure 4. Exact and Random generate source biconnected support.

functional with guessed source parameters. We had used the analytical solution of the direct problems for rectangular geometry, but the methodology can also be implemented for arbitrary geometry if direct problems are solved with numerical solvers based on finite element method. The adopted Nelder-Mead simplex algorithm avoid the necessity of compute

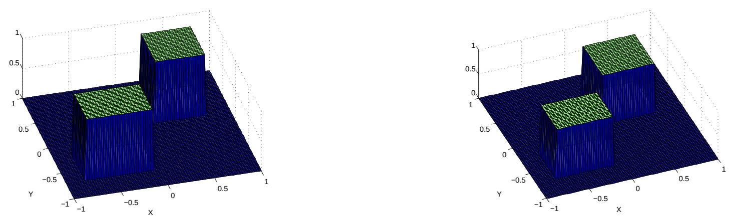

Figure 5. Reconstructed biconnected source for 82 and 675 iterations.

derivatives, which is adequate to the necessity of iterate a huge numbers of solutions of the direct problems. Results show that this is a new way to reconstruct sources for Cauchy boundary data. For characteristic sources with simply connected support, the reconstruction is very good, and in the non connected case, we arrive very close to one positive reconstruction.

6. ACKNOWLEDGEMENTS

Authors want to thanks CNPq-Br, CAPES-Br and FCT-Pt for the partial support. This work is part of a CAPES/FCT project.

7. REFERENCES

Alves, C., Martins, N., Colaco, M., Olande, H. and Roberty, N., 2008. “Recovering the source term in a linear diffusion problem by the method of fundamental solutions”. *Inverse Problems in Science and Engineering*, Vol. 16, No. 8, pp. 1005–1021.

Alves, C., Martins, N. and Roberty, N., 2009. “Full identification of acoustic sources with multiples frequencies and boundary measurements”. *Inverse Problems and Imaging*, Vol. 3, No. 3, pp. 275–294.

Friedman, A., 1964. *Partial Differential Equations of Parabolic Type*. Prentice Hall, INC, Northwester University.

Haji-Sheikh, A. and Beck, J.V., 1994. “Green’s function solution for thermal wave eqaution in finite bodies”. *Int. J. Heat Mass Transfer*, Vol. 37, No. 17, pp. 2615–2626.

Isakov, V., 1990. *Inverse Source Problems*. American Mathematical Society.

Kohn, R. and Vogelius, M., 1987. “Relaxation of a variational method for impedance computed tomography”. *Comm. Pure Appl. Math.*, Vol. 6, No. 40, pp. 745–777.

Lagarias, J., Reeds, J., Wright, M. and Wright, P., 1998. “Convergence properties of the nelder-mead simplex method in low dimensions”. *Siam J. Optim.*, Vol. 9, No. 1, pp. 112–147.

Machado, T.J., 2012. “Derivada topológica no problema inverso do potencial”. *MsC Dissertation, Laboratório Nacional de Computacão Científica*.

Roberty, N.C. and Rainha, M.J., 2011. “Moving heat source reconstruction from cauchy boundary data: The cartesian coordinates case”. *Mathematical Problems in Engineering*, Vol. 2011, pp. 1–19.

Roberty, N.C. and Rainha, M.L., 2010. “Moving heat source reconstruction from the cauchy boundary data.” *Mathematical Problems in Engineering*, Vol. 2010, pp. 1–22.