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ANALYSIS OF JOULE AND VISCOUS HEATING IN COMBINED PRESSURE AND ELECTROOSMOTIC DRIVEN FLOWS IN ISOTHERMAL MICROTUBES

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Abstract. *This paper presents solutions to heat transfer problems that occur in flow microtubes driven by the combined effect of electroosmosis and a pressure gradient. Fully developed velocity profiles are considered, leading to a Graetz-type problem for the thermally developing flow. The case without axial diffusion and a isothermal wall condition is analyzed. The solution methodology is based on the Generalized Integral Transform Technique, which is shown to lead to a fully analytical solution for the problem in terms of a matrix exponential. With the solution of the temperature fields, the behavior of the Nusselt number is investigated for different test-cases.*

Keywords: *electrokinetics, electroosmosis, laminar flow, forced convection, duct flow*

1. NOMENCLATURE

$A_{m,n}$	transformed equation coefficients
b_n	transformed inlet condition coefficients
$\boldsymbol{A}$, $\boldsymbol{D}$, $\boldsymbol{M}$	integral transform coefficients matrices
$\boldsymbol{b}$	integral transform coefficients vector
D	diameter
E_x	electric field in main flow direction
F	filter function
k_B	Boltzmann constant
N	norm
$n_{\max}$	truncation order
p	pressure
T	temperature
T_w	wall temperature
T_0	inlet temperature
u	velocity
u^*	dimensionless velocity
$\bar{u}$	average velocity
x, r	spatial coordinates

Greek symbols

α	thermal diffusivity
γ	eigenvalues
Φ	eigenfunctions
λ	Debye length
ρ_e	charge density
ξ, η	dimensionless coordinates
Θ	dimensionless temperature
Θ_m	dimensionless mean stream temperature
θ	filtered dimensionless temperature
$\bar{\theta}$	transformed dimensionless temperature

Dimensionless numbers

Br	Brinkman number
Jo	dimensionless Joule heating parameter
Nu	Nusselt number (based on diameter)
Pe	Peclet number (based on half diameter)

Subscripts

n, m	summation indexes

2. INTRODUCTION

Driving fluid flow through extremely small passages can become quite cumbersome if one relies sole on pressure gradient effects. This occurs due to the increased pressure drop associated with micro-scale tube diameters, and hence alternative means of flow actuation become necessary. One such alternative is by applying an electrical field, leading to the so-called electrokinetic flows. One type of electrokinetic flow is that given by electroosmotic effects. Horiuchi and Dutta (2004) examined the Nusselt number in electroosmosis driven flow in parallel plates channels. A fully analytical solution was presented for the simplified case with a thin EDL, leading to a plug-like velocity profile is analyzed for the thermally developing region. Chakraborty (2006) developed analytical solutions for fully developed flow in circular micro-tubes with both pressure and electroosmotic driven flow. Chen (2009) also analyzed the thermally developed flow actuated by both pressure gradient and electroosmotic effects; however, parallel plates micro-channels were considered, and the effects of variable fluid properties were also analyzed. Dey *et al.* (2011) also analyzed the fully developed Nusselt number using analytical solutions, but the authors considered the case with thick EDL, in which the electroosmotic velocity profile is no longer plug-like. The work of Maynes and Webb (2004) also analyzed the fully-developed heat transfer in micro channel flow driven by electroosmosis. The work of Sharma and Chakraborty (2008) presented an analytical solution for the thermally-developing pressure and electroosmotic driven flow with a step-change in wall temperature, considering the thin EDL case.

Several recent heat transfer studies in micro-channels involve analytical solutions for calculating Nusselt numbers in both developing and thermally developed flow. Nevertheless, if more complicated problems are considered, such as those with non-linear effects and complex geometries, traditional numerical methods are required.An alternative to traditional numerical methods, is the hybrid analytical-numerical method known as the Generalized Integral Transform Technique (GITT) (Cotta, 1994), which is based on seeking solutions in terms of orthogonal eigenfunction expansions. Among applications of the GITT in micro-channel problems, one should mention study (Mikhailov and Cotta, 2005), which investigates heat transfer solutions for slip-flow in parallel plate channels, and (Castellões *et al.*, 2010) which examines flows within wavy walls. Very recently, two studies presented GITT solutions to extended Graetz problems in flows driven by the combined effect of electroosmosis and a pressure gradient in microchannels (Sphaier, 2012a,b); nevertheless these works were strictly focused on parallel-plates ducts. Other recent studies also present solutions for a similar problem within parallel plates channels, but using different methodologies which require a finite-volume solution (Dey *et al.*, 2012) and employ non-orthgonal eigenfunction bases (Sadeghi *et al.*, 2012).

The purpose of this work is to provide a GITT solution for thermally developing flow within micro-tubes, i.e. circular ducts, driven by both electroosmosis and a pressure gradient. The effects of Joule and viscous dissipation heating are included in the analysis and flows with both thin and thick EDL are considered; however, the simplified case without axial diffusion will be considered.

3. PROBLEM FORMULATION

The studied problem is that of steady incompressible laminar flow inside a micro-tube, driven by both pressure and electroosmotic effects. The flow is considered dynamically developed, but thermally developing. Under this assumptions, the momentum equations are simplified to:

$$\mu \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) + \rho_e\, E_x = \frac{\mathrm{d}p}{\mathrm{d}x} \qquad \text{for} \qquad 0 \le r \le D/2, \qquad \left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)_{r=0} = 0, \qquad u(D/2) = 0, \tag{1}$$

in which ρ_e is the distribution of excess charge density, which is obtained from a simultaneous solution of Poisson's equation of potential distribution and Boltzmann equation of charge density distribution. For the current problem assumptions, this corresponds to solving the following system:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) = -\frac{\rho_e}{\epsilon\,\epsilon_0}, \qquad \psi'(0) = 0, \qquad \psi(D/2) = \zeta, \tag{2a}$$

in which the charge density per unit volume is given by:

$$\rho_e \approx -2\,\frac{n_0\, z^2\, e^2\, \psi}{k_B\, T} = -\omega^2\, \epsilon\, \epsilon_0, \tag{2b}$$

where z is the valence of the concerned charge, e is the electronic charge, n_0 is the average number of positive or negative ions in the buffer, and ψ is the electroosmotic potential. k_B is the Boltzmann constant and T is the absolute temperature.

The adopted expression also assumes that the wall zeta potential is constant and small enough ($\zeta < 3\,k_B\,T$) so that the Debey-Hückel linearization approximation may be applied. The parameter ω is the Debye-Hückel parameter (Dutta and Beskok, 2001), also related to the characteristic thickness of the EDL, which is also known as the Debye length (λ):

$$\omega \,=\, \frac{1}{\lambda} \,=\, \sqrt{\frac{2\,n_0\,z^2\,e^2}{\epsilon\,\epsilon_0\,k_B\,T}}. \tag{3}$$

The solution of system (2) yields the following potential and charge distributions:

$$\psi \,=\, \zeta\,\frac{I_0(r/\lambda)}{I_0(D/(2\,\lambda))}, \qquad \rho_e \,=\, -\frac{\epsilon\,\epsilon_0\,\zeta}{\lambda^2}\,\frac{I_0(r/\lambda)}{I_0(D/(2\,\lambda))}, \tag{4}$$

which leads to the following velocity profile:

$$u(r) = u_{HP}\left(1-\left(\frac{2\,r}{D}\right)^2\right) + u_{HS}\left(1-\frac{I_0(r/\lambda)}{I_0(D/(2\,\lambda))}\right), \qquad u_{HP} = -\frac{D^2}{16\,\mu}\,\frac{\mathrm{d}p}{\mathrm{d}x}, \qquad u_{HS} = -\frac{\epsilon\,\epsilon_0\,\zeta\,E_x}{\mu}. \tag{5}$$

where I_0 is the zeroth-order modified Bessel function of the first kind and u_{HP} and u_{HS} are the reference Hagen-Poiseiulle and Helmholtz-Smoluchowski velocities, corresponding, respectively, to characteristic velocities for pure pressure-diven flow and pure electroosmotic-driven flow.

The temperature distribution, already given in dimensionless form, is governed by the following equation and boundary conditions:

$$u^*\,\frac{\partial\Theta}{\partial\xi} \,=\, \mathrm{Pe}^{-2}\frac{\partial^2\Theta}{\partial\xi^2} \,+\, \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\,\frac{\partial\Theta}{\partial\eta}\right) \,+\, \mathrm{Br}\left(\frac{\partial u^*}{\partial\eta}\right)^2 \,+\, \mathrm{Jo}, \tag{6a}$$

$$\Theta(\xi,1) \,=\, 0, \qquad \left(\frac{\partial\Theta}{\partial\eta}\right)_{\eta=0} \,=\, 0, \qquad \Theta(0,\eta) \,=\, 1, \qquad \left(\frac{\partial\Theta}{\partial\xi}\right)_{\xi\to\infty} \,=\, 0. \tag{6b}$$

where u^* is the the normalized velocity profile, given by:

$$u^* \,=\, \frac{u}{\bar{u}} \,=\, \frac{\left(\left(1-\eta^2\right)\Omega+1\right)\,I_0(\kappa_\lambda)-I_0(\eta\kappa_\lambda)}{\frac{\Omega}{2}\,I_0(\kappa_\lambda)+I_2(\kappa_\lambda)}. \tag{7}$$

The dimensionless parameters are given by:

$$\mathrm{Br} \,=\, \frac{\mu\,\bar{u}^2}{k\,\Delta T}, \qquad \mathrm{Jo} \,=\, \frac{E_x^2\,\sigma}{k\,\Delta T}\left(\frac{D}{2}\right)^2, \qquad \mathrm{Pe} \,=\, \frac{\bar{u}\,D/2}{\alpha}, \qquad \kappa_\lambda \,=\, \frac{D/2}{\lambda}, \qquad \Omega \,=\, \frac{1}{4}\,\frac{\mathrm{d}p}{\mathrm{d}x}\,\frac{(D/2)^2}{\epsilon\,\epsilon_0\,\zeta\,E_x}, \tag{8}$$

where $\Delta T = T_0 - T_w$ and the dimensionless variables are defined as:

$$\Theta(\xi,\eta) = \frac{T(x,y)-T_w}{T_0-T_w}, \qquad \eta \,=\, \frac{r}{D/2}, \qquad \xi \,=\, \frac{x}{D/2\,\mathrm{Pe}}. \tag{9}$$

Finally, the Nusselt number, in terms of the dimensionless variables, is calculated from:

$$\mathrm{Nu} \,=\, \frac{2\,(\partial\Theta/\partial\eta)_{\eta=1}}{\Theta_w-\Theta_m}, \qquad \Theta_w \,=\, 0, \qquad \Theta_m \,=\, 2\int_0^1 \eta\,u^*\,\Theta\,\mathrm{d}\eta. \tag{10}$$

4. INTEGRAL TRANSFORM SOLUTION

As usual in GITT solutions, a filter problem is proposed for removing non-homogenities from the original system. The filter is based on the following solution separation:

$$\Theta(\xi,\eta) \;=\; \theta(\xi,\eta) + F(\eta), \tag{11}$$

in which θ is the filtered variable and F is the filter function. The filter function is obtained from solving the following filter problem:

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\, F'(\eta)\right) \;+\; \mathrm{Br}\left(\frac{\partial u^*}{\partial \eta}\right)^2 \;+\; \mathrm{Jo} \;=\; 0 \qquad F'(0) \;=\; 0, \qquad F(1) \;=\; 0. \tag{12}$$

With the filter problem, the a homogeneous problem (governing equations and boundary conditions in transversal direction) is obtained:

$$u^*\frac{\partial \theta}{\partial \xi} = \mathrm{Pe}^{-2}\frac{\partial^2\theta}{\partial \xi^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right), \tag{13a}$$

$$\theta(\xi,1) = 0, \qquad \left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0} = 0, \qquad \theta(0,\eta) = 1 - F(\eta), \qquad \left(\frac{\partial \theta}{\partial \xi}\right)_{\xi\to\infty} = 0. \tag{13b}$$

The solution of the considered problem is accomplished employing the Generalized Integral Transform Technique (Cotta, 1993). The solution process is started by defining the transformation pair:

$$\text{Transform} \quad \Rightarrow \quad \bar\theta_n(\xi) = \int_0^1 \eta\,\theta(\xi,\,\eta)\Phi_n(\eta)\,\mathrm{d}\eta, \qquad\qquad \text{Inversion} \quad \Rightarrow \quad \theta(\xi,\,\eta) = \sum_{n=1}^{\infty}\frac{\bar\theta_n(\xi)\,\Phi_n(\eta)}{N_n}, \tag{14}$$

where Φ_n's are orthogonal solutions of a Sturm-Liouville problem. For the current application the one-dimensional Helmholtz problem is selected:

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta\,\Phi_n'(\eta)) \;+\; \gamma_n^2\,\Phi_n(\eta) \;=\; 0, \qquad \text{for} \qquad 0 \le \eta \le 1, \qquad \Phi_n'(0) \;=\; 0, \qquad \Phi_n(1) \;=\; 0. \tag{15}$$

which leads to infinite nontrivial solutions in the form:

$$\Phi_n(\eta) \;=\; J_0(\gamma_n\,\eta), \qquad \text{with} \qquad J_0(\gamma_n) \;=\; 0, \qquad \text{and} \qquad N_n \;=\; \int_0^1 \eta\,\Phi_n^2(\eta)\,\mathrm{d}\eta. \tag{16a}$$

where n is a positive integer.

The transformation of the given problem is accomplished by multiplying eq. (13a) by Φ_n, integrating within $0 \le \eta \le 1$, and applying the inversion formula (14) to the non-transformable terms, which yields:

$$\mathrm{Pe}^{-2}\,\theta_n''(\xi) \;-\; \sum_{m=1}^{\infty} A_{n,m}\,\bar\theta_m'(\xi) - \gamma_n^2\,\bar\theta_n(\xi) \;=\; 0, \qquad \bar\theta_n(0) \;=\; b_n, \qquad \bar\theta_n'(\infty) \;=\; 0, \tag{17}$$

for $n = 1, 2, \ldots, \infty$. The $A_{m,n}$ and b_n coefficients are given by:

$$A_{n,m} = \frac{1}{N_m}\int_0^1 \eta\,u^*(\eta)\,\Phi_m(\eta)\,\Phi_n(\eta)\,\mathrm{d}\eta, \qquad b_n \;=\; \int_0^1 \eta\,(1 - F(\eta))\,\Phi_n(\eta)\,\mathrm{d}\eta. \tag{18}$$

In order to solve system (17), the infinite system representation must be truncated to a finite number of terms $n_{\max}$, which is denoted the truncation order. After truncating these equations the following vectorial form is introduced:

$$\mathrm{Pe}^{-2}\,\bar{\boldsymbol{T}}''(\xi)\,-\,\boldsymbol{A}\cdot\bar{\boldsymbol{T}}'(\xi)\,-\,\boldsymbol{D}\cdot\bar{\boldsymbol{T}}(\xi)\,=\,\boldsymbol{0},\qquad \bar{\boldsymbol{T}}(0)\,=\,\boldsymbol{b},\qquad \bar{\boldsymbol{T}}'(\infty)\,=\,\boldsymbol{0},\tag{19}$$

where $\bar{\boldsymbol{T}}$ is a vector with the unknown transformed potentials, the matrix $\boldsymbol{A}$ is given by the coefficients $A_{m,n}$ and $\boldsymbol{D}$ is a diagonal matrix, whose coefficients are given by $D_{n,n} = \gamma_n^2$.

For situations in which axial heat diffusion could be discarded, as considered in this study, the boundary value problem represented by equations (19) is reduced to an initial value problem and a fully analytical solution can be obtained:

$$\bar{\boldsymbol{T}}(\xi)\,=\,\boldsymbol{C}(\xi)\cdot\boldsymbol{b},\qquad \text{with}\qquad \boldsymbol{C}(\xi)\,=\,\exp(\boldsymbol{M}\,\xi)\qquad \text{and}\qquad \boldsymbol{M}\,=\,-\boldsymbol{A}^{-1}\cdot\boldsymbol{D}.\tag{20}$$

where the coefficients of $\boldsymbol{b}$ are given by eq. (18), and $\boldsymbol{C}$ is a matrix exponential.

5. RESULTS AND DISCUSSION

After the presentation of the adopted methodology, illustrative results are provided and discussed, starting with cases that involve no flow heating effects. The first results examine the effects of the variation of the EDL thickness on the Nusselt number, as presented in table 1 for different truncation orders ($n_{\max}$) and different κ_λ values, including the limiting case with $\kappa_\lambda = \infty$, corresponding to the simple plug-flow solution. As can be seen, larger Nusselt values are obtained for thiner EDLs due to the larger velocities near the walls, which will lead to better heat transfer rates at these locations. The maximum values naturally occur in the limiting plug-flow situation, and as the κ_λ values are increased the Nu values approach this limit, as expected. When comparing to traditional literature results, one notices that for $\kappa_\lambda = \infty$, the fully-developed Nusselt values are in agreement with the well-known plug-flow value of 5.78.When looking into the local convergence behavior of the solutions, one notices that better convergence rates are generally seen for positions upstream. Moreover, the solution with $\kappa_\lambda = \infty$ yields the best convergence behavior: as little as 40 terms are sufficient to ensure a converged solution with six significant figures at the worst position ($\xi = 0.001$) and only 5 terms yield the same converge behavior for upstream positions. When examining the other cases, one can observe that the worst convergence rate is seen for $\kappa_\lambda = 100$, which may be attributed to the higher velocity gradient near the wall.

Table 1. Nuseelt values for different EDL thicknesses without flow heating in purely electroosmotic flow.

$n_{\max}$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=0$, $\mathrm{Jo}=0$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=0$, $\mathrm{Jo}=0$				
5	17.6150	8.28696	4.05503	3.69963	3.69963	18.5656	10.8926	5.21033	4.80779	4.80779
10	19.5531	7.57838	4.04522	3.69510	3.69510	26.1289	10.0253	5.18070	4.79295	4.79295
20	16.5517	7.55356	4.04386	3.69448	3.69448	22.6511	9.92775	5.17564	4.79031	4.79031
40	16.4669	7.55030	4.04368	3.69440	3.69440	22.3612	9.91506	5.17493	4.78993	4.78993
60	16.4599	7.54996	4.04366	3.69439	3.69439	22.3336	9.91373	5.17486	4.78989	4.78989
80	16.4582	7.54988	4.04365	3.69438	3.69438	22.3267	9.91341	5.17484	4.78988	4.78988
100	16.4575	7.54985	4.04365	3.69438	3.69438	22.3243	9.91329	5.17483	4.78988	4.78988
	$\Omega=0$, $\kappa_\lambda=100$, $\mathrm{Br}=0$, $\mathrm{Jo}=0$					$\Omega=0$, $\kappa_\lambda=\infty$, $\mathrm{Br}=0$, $\mathrm{Jo}=0$				
5	19.8513	12.6537	6.07839	5.67086	5.67086	20.1888	12.8103	6.17893	5.78319	5.78319
10	31.2008	12.8958	6.07732	5.67027	5.67027	31.6729	13.0686	6.17893	5.78319	5.78319
20	35.9207	12.8594	6.07576	5.66939	5.66939	37.1297	13.0687	6.17893	5.78319	5.78319
40	35.0840	12.8292	6.07439	5.66862	5.66862	37.2975	13.0687	6.17893	5.78319	5.78319
60	34.8392	12.8204	6.07399	5.66839	5.66839	37.2975	13.0687	6.17893	5.78319	5.78319
80	34.7585	12.8173	6.07385	5.66831	5.66831	37.2975	13.0687	6.17893	5.78319	5.78319
100	34.7257	12.8161	6.07379	5.66827	5.66827	37.2975	13.0687	6.17893	5.78319	5.78319

Next, table 2 examines the effects of the flow driving mechanism parameter (Ω) for the thin EDL limit, again, presented for different truncation orders. Naturally, there is no need to present the limiting case of $\Omega = 0$ for this case, since these results have already been displayed in table 1. By observing the Nusselt values presented in this table, one clearly sees that for small and large Ω values, respectively, the solution approaches the traditional Nusselt values seen in plug-flow

Table 2. Nusselt values for different values of Ω with thin EDLs and no flow heating.

$n_{\max}$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$
	$\Omega=0.1, \kappa_\lambda=\infty, \mathrm{Br}=0, \mathrm{Jo}=0$					$\Omega=1, \kappa_\lambda=\infty, \mathrm{Br}=0, \mathrm{Jo}=0$				
5	20.0762	12.5985	6.03041	5.63218	5.63218	19.3873	11.3115	5.25830	4.86027	4.86027
10	31.3389	12.7711	6.02945	5.63169	5.63169	29.0219	11.0304	5.25305	4.85771	4.85771
20	36.3175	12.7673	6.02933	5.63162	5.63162	31.0149	11.0116	5.25237	4.85736	4.85736
40	36.4156	12.7669	6.02931	5.63161	5.63161	30.8605	11.0095	5.25228	4.85732	4.85732
60	36.4144	12.7669	6.02931	5.63161	5.63161	30.8536	11.0093	5.25227	4.85731	4.85731
80	36.4141	12.7669	6.02931	5.63161	5.63161	30.8521	11.0092	5.25227	4.85731	4.85731
100	36.4140	12.7669	6.02931	5.63161	5.63161	30.8515	11.0092	5.25227	4.85731	4.85731
	$\Omega=10, \kappa_\lambda=\infty, \mathrm{Br}=0, \mathrm{Jo}=0$					$\Omega=\infty, \mathrm{Br}=0, \mathrm{Jo}=0$				
5	18.0710	8.97917	4.27116	3.90437	3.90437	17.5711	8.18315	4.01542	3.66175	3.66175
10	22.6659	8.28440	4.26239	3.90030	3.90030	19.2766	7.49769	4.00611	3.65747	3.65747
20	20.0844	8.26008	4.26119	3.89974	3.89974	16.3581	7.47393	4.00482	3.65688	3.65688
40	19.9910	8.25693	4.26103	3.89967	3.89967	16.2736	7.47084	12.2062	3.65680	3.65680
60	19.9826	8.25661	4.26101	3.89966	3.89966	16.2669	7.47052	4.00463	3.65680	3.65680
80	19.9806	8.25653	4.26101	3.89966	3.89966	16.2652	7.47044	4.00463	3.65679	3.65679
100	19.9798	8.25650	4.26101	3.89966	3.89966	16.2647	7.47041	4.00463	3.65679	3.65679

(5.78) and Hagen-Poiseuille flow (3.66), in the thermally developed region ($\xi = 10$), as expected. Also, since a plug-flow configuration yields better heat transfer rates, higher Nusselt values are seen for smaller Ω values. When analyzing the convergence behavior of the presented results, one notices, again, that better convergence rates are generally seen for positions upstream. Furthermore, if the convergence between the different cases are compared, it is seen that better results are obtained for smaller values of Ω, as these approach the plug-flow situation without increasing the wall velocity gradient, differently than what occurred for $\kappa_\lambda = 100$ in table 1.

The next results examine the effects of Joule heating for purely electroosmotic-driven flow with different EDL thicknesses. Table 3 displays Nusselt values calculated for different truncation orders for $\kappa_\lambda = 1$ and $\kappa_\lambda = 10$ and different values of the Joule heating parameter. As observed from this table, the parameter Jo has the effect of increasing the Nusselt number towards the fully developed region when compared to the cases without heating. It is also interesting to note that, towards the fully developed region, the Nusselt values become independent of Jo, regardless of its magnitude. With regards to convergence behavior, one notices that in downstream positions, the convergence rates are similar to those seen in cases without flow heating effects; however, in upstream positions, the convergence rate is improved in cases with $\mathrm{Jo} \neq 0$. This is expected to occur since the employed filter solution is actually the fully developed solution for these cases.

After Joule heating, table 4 presents local Nu values calculated with different truncation orders for different values of the EDL thickness parameter κ_λ and different values of the Brinkman number. As can be seen, reducing the EDL thickness increases the Nusselt number significantly, due to the higher velocity gradients at the wall. Nevertheless, this augmentation is seen for all ξ values only for the case with $\mathrm{Br} = 1$; for lower Brinkman values the significant augmentation in Nu is only seen for positions upstream, as also observed in the Joule heating cases. In fact, as the flow enters the fully developed region, the calculated Nusselt value becomes independent of the value of Br, similarly to what was observed in cases with Joule heating. This effect is in accordance with literature results for fully developed solutions. Finally, when looking into the convergence behavior of the solution, similar observations made for the Nusselt results calculated with Joule heating apply: in the downstream region, the convergence is similar to that of no-heating cases, while in upstream region much better convergence rates are seen, which is again due to the fact that the filter solution for $\mathrm{Br} \neq 0$ is the actual fully-developed solution.

6. CONCLUSIONS

This paper presented a formal solution by the GITT to the extended Graetz problem, including the effects of viscous heating, Joule heating, and flow actuation by means of pressure gradient and electroosmosis. The case of isothermal walls was considered. Fully analytical solutions in terms of a matrix exponential are obtained for cases with negligible axial

Table 3. Nusselt values in purely electroosmotic flow with Joule heating for different EDL thicknesses.

$n_{\max}$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=0$, $\mathrm{Jo}=10^{-6}$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=0$, $\mathrm{Jo}=10^{-6}$				
5	17.6150	8.28696	4.05503	3.69965	6.03040	18.5656	10.8926	5.21033	4.80784	6.93156
10	19.5531	7.57838	4.04522	3.69512	6.03040	26.1289	10.0253	5.18070	4.79300	6.93156
20	16.5517	7.55356	4.04386	3.69450	6.03040	22.6511	9.92775	5.17564	4.79035	6.93156
40	16.4669	7.55030	4.04368	3.69441	6.03040	22.3612	9.91506	5.17493	4.78998	6.93156
60	16.4599	7.54996	4.04366	3.69441	6.03040	22.3336	9.91373	5.17486	4.78994	6.93156
80	16.4582	7.54988	4.04365	3.69440	6.03040	22.3267	9.91341	5.17484	4.78993	6.93156
100	16.4575	7.54985	4.04365	3.69440	6.03040	22.3243	9.91329	5.17483	4.78993	6.93156
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=0$, $\mathrm{Jo}=10^{-4}$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=0$, $\mathrm{Jo}=10^{-4}$				
5	17.6150	8.28698	4.05509	3.70153	6.03040	18.5656	10.8926	5.21038	4.81253	6.93156
10	19.5531	7.57841	4.04528	3.69700	6.03040	26.1289	10.0253	5.18075	4.79773	6.93156
20	16.5517	7.55359	4.04392	3.69638	6.03040	22.6511	9.92777	5.17570	4.79509	6.93156
40	16.4669	7.55032	4.04374	3.69630	6.03040	22.3612	9.91508	5.17499	4.79472	6.93156
60	16.4599	7.54999	4.04372	3.69629	6.03040	22.3336	9.91375	5.17491	4.79468	6.93156
80	16.4582	7.54991	4.04372	3.69629	6.03040	22.3267	9.91343	5.17489	4.79467	6.93156
100	16.4575	7.54988	4.04371	3.69629	6.03040	22.3243	9.91331	5.17489	4.79467	6.93156
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=0$, $\mathrm{Jo}=10^{-2}$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=0$, $\mathrm{Jo}=10^{-2}$				
5	17.6159	8.28926	4.06118	3.87551	6.03040	18.5664	10.8942	5.21594	5.19601	6.93156
10	19.5542	7.58081	4.05139	3.87153	6.03040	26.1296	10.0270	5.18639	5.18487	6.93156
20	16.5529	7.55600	4.05003	3.87098	6.03040	22.6519	9.92954	5.18135	5.18289	6.93156
40	16.4681	7.55274	4.04985	3.87091	6.03040	22.3621	9.91685	5.18064	5.18261	6.93156
60	16.4611	7.55240	4.04984	3.87090	6.03040	22.3344	9.91553	5.18057	5.18258	6.93156
80	16.4593	7.55232	4.04983	3.87090	6.03040	22.3275	9.91520	5.18055	5.18257	6.93156
100	16.4587	7.55229	4.04983	3.87090	6.03040	22.3251	9.91509	5.18054	5.18257	6.93156
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=0$, $\mathrm{Jo}=1$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=0$, $\mathrm{Jo}=1$				
5	17.7122	8.51449	4.61566	5.82036	6.03040	18.6499	11.0481	5.71264	6.85616	6.93156
10	19.6615	7.81914	4.60836	5.82024	6.03040	26.1982	10.2002	5.69032	6.85591	6.93156
20	16.6691	7.79508	4.60734	5.82022	6.03040	22.7325	10.1051	5.68654	6.85586	6.93156
40	16.5847	7.79192	4.60721	5.82022	6.03040	22.4435	10.0927	5.68601	6.85586	6.93156
60	16.5777	7.79160	4.60719	5.82022	6.03040	22.4159	10.0914	5.68595	6.85586	6.93156
80	16.5760	7.79152	4.60719	5.82022	6.03040	22.4091	10.0911	5.68594	6.85586	6.93156
100	16.5753	7.79149	4.60719	5.82022	6.03040	22.4066	10.0910	5.68593	6.85586	6.93156

diffusion. The results demonstrated that the convergence behavior of the GITT is such that, for positions upstream, a high convergence rate is seen, whereas near the channel entrance more terms are required in the truncated series.

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Table 4. Nusselt values with viscous heating in purely electroosmotic flow for different EDL thicknesses.

$n_{\max}$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$	$\xi=10$
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=10^{-6}$, $\mathrm{Jo}=0$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=10^{-6}$, $\mathrm{Jo}=0$				
5	17.6150	8.28696	4.05504	3.69988	10.0804	18.5656	10.8926	5.21035	4.80965	39.6628
10	19.5531	7.57839	4.04523	3.69535	10.0804	26.1289	10.0253	5.18072	4.79481	39.6628
20	16.5517	7.55356	4.04386	3.69473	10.0804	22.6511	9.92777	5.17566	4.79217	39.6628
40	16.4669	7.55030	4.04369	3.69464	10.0804	22.3612	9.91507	5.17495	4.79179	39.6628
60	16.4599	7.54997	4.04367	3.69464	10.0804	22.3336	9.91375	5.17488	4.79176	39.6628
80	16.4582	7.54989	4.04366	3.69463	10.0804	22.3267	9.91342	5.17486	4.79175	39.6628
100	16.4575	7.54986	4.04366	3.69463	10.0804	22.3243	9.91330	5.17485	4.79174	39.6628
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=10^{-4}$, $\mathrm{Jo}=0$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=10^{-4}$, $\mathrm{Jo}=0$				
5	17.6151	8.28732	4.05587	3.72443	10.0804	18.5664	10.8938	5.21275	4.99235	39.6628
10	19.5533	7.57876	4.04606	3.71995	10.0804	26.1297	10.0265	5.18312	4.97816	39.6628
20	16.5519	7.55394	4.04470	3.71933	10.0804	22.6519	9.92896	5.17807	4.97563	39.6628
40	16.4671	7.55068	4.04452	3.71925	10.0804	22.3620	9.91627	5.17736	4.97527	39.6628
60	16.4601	7.55034	4.04450	3.71924	10.0804	22.3343	9.91494	5.17728	4.97523	39.6628
80	16.4584	7.55026	4.04450	3.71924	10.0804	22.3275	9.91462	5.17727	4.97522	39.6628
100	16.4577	7.55023	4.04449	3.71924	10.0804	22.3250	9.91450	5.17726	4.97522	39.6628
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=10^{-2}$, $\mathrm{Jo}=0$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=10^{-2}$, $\mathrm{Jo}=0$				
5	17.6318	8.32321	4.13859	5.50185	10.0804	18.6491	11.0095	5.45125	16.9440	39.6628
10	19.5713	7.61595	4.12899	5.50019	10.0804	26.2015	10.1455	5.42262	16.9589	39.6628
20	16.5711	7.59120	4.12766	5.49997	10.0804	22.7287	10.0484	5.41774	16.9616	39.6628
40	16.4863	7.58795	4.12749	5.49994	10.0804	22.4391	10.0358	5.41706	16.9619	39.6628
60	16.4793	7.58762	4.12747	5.49993	10.0804	22.4115	10.0344	5.41699	16.9620	39.6628
80	16.4776	7.58753	4.12746	5.49993	10.0804	22.4047	10.0341	5.41697	16.9620	39.6628
100	16.4770	7.58750	4.12746	5.49993	10.0804	22.4022	10.0340	5.41696	16.9620	39.6628
	$\Omega=0$, $\kappa_\lambda=1$, $\mathrm{Br}=1$, $\mathrm{Jo}=0$					$\Omega=0$, $\kappa_\lambda=10$, $\mathrm{Br}=1$, $\mathrm{Jo}=0$				
5	19.2871	11.7211	9.60944	10.0594	10.0804	26.8629	22.0670	23.1510	39.2450	39.6628
10	21.3626	11.1360	9.61153	10.0595	10.0804	33.3383	21.5057	23.1736	39.2465	39.6628
20	18.4768	11.1182	9.61183	10.0596	10.0804	30.3546	21.4427	23.1783	39.2468	39.6628
40	18.3963	11.1158	9.61187	10.0596	10.0804	30.0947	21.4345	23.1791	39.2468	39.6628
60	18.3897	11.1156	9.61187	10.0596	10.0804	30.0702	21.4336	23.1791	39.2469	39.6628
80	18.3881	11.1155	9.61187	10.0596	10.0804	30.0641	21.4334	23.1792	39.2469	39.6628
100	18.3875	11.1155	9.61187	10.0596	10.0804	30.0619	21.4333	23.1792	39.2469	39.6628

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