SOLUTIONS FOR INCOMPRESSIBLE ARBITRARY UNSTEADY FLOWS BY INDICIAL RESPONSES AND FREQUENCY ANALYSIS

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Abstract. *In order to understand the compressible flow around slender wings in unsteady arbitrary motion, it is necessary first to study the tools used for solving the incompressible linearized problem. There are two approaches commonly used to obtain the complete solutions: indicial responses and frequency analysis. The former is based in an infinitesimal variation of the body's position using indicial admittance and Duhamel's integral to compose arbitrary motions. The later employs the assumption of harmonic movement and uses Theodorsen's solution to calculate the circulation in any time according to reduced frequency. Through a Fourier transform the arbitrary motion can be composed. This work presents the application of both methods to solve two general sinusoidal movements of pitching and heaving motion and, after that, a superposition of them. It has been calculated the lift and moment coefficient for the thin airfoil and the results are analyzed by comparing these methods and observing some physical characteristics of each motion.*

Keywords*: airfoil, unsteady, incompressible, indicial responses, reduced frequency*

1. INTRODUCTION

The study of the unsteady aerodynamics dates from the period in which were necessary analysis of the famous phenomena called *flutter* phenomena. From some historical notes, the first occurrence of this phenomena was in the World War I, when a bomber's tail suffered instability problems of structural feature. Thus, by the year of 1935, some researchers like Glauert, Frazer, Dücan, Küssner and Theodorsen have been done works about the subject and became it viable for design applications (Bisplinghoff et al, 1955).

However, not only the effects of structural instability have stimulated the study of unsteady flows. Others fields like flows around helicopters' blades and prediction of aeroelastics and aeroacustics effects of vibrations in turbine vanes or compressors need a greater comprehension about unsteady aerodynamics and its applications.

Finally, the attempt of understanding the unsteady incompressible flows as well as the analytical solution of them allows a great view of the flow and helps to predict important features that can be applied in several aerodynamic designs. This work aims to study the analytical solutions by two methods and to compare its results. The methods solve the incompressible unsteady flow about a thin airfoil in arbitrary motion. The former uses a harmonic decomposition of the motion and the other integrates the indicial responses by a convolution product between the response and the motion. The results presented are the lift and the moment coefficient for three sinusoidal movements.

2. MATHEMATICAL MODEL

Leading with the mass balance and the hypothesis of irrotational and incompressible flow, Laplace Equation (Eq. 1a) describes the velocity potential or the acceleration potential. The simplified frame of an irrotational flow is based on Kelvin circulation theorem. Furthermore, the hypothesis of an inviscid flow is assumed, and the momentum balance leads to a relation between pressure and velocity known as Euler Equation, which can lead to Kelvin Equation (or Unsteady Bernoulli Equation), reported as Eq. (1b), valid for flow field with undisturbed velocity U.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{1a}$$

$$\frac{\partial\phi}{\partial t}+\frac{q^2}{2}+\frac{p}{\rho}=\frac{U^2}{2}+\frac{p_\infty}{\rho} \tag{1b}$$

In Eqs. (1a) and (1b), ϕ represents the velocity potential, U is the undisturbed (freestream) velocity, q is the intensity of velocity vector, p represents pressure, p_∞ is the undisturbed pressure and ρ is local density (assumed constant).

The problem is to solve the flow field for an airfoil oscillating in angle of attack and vertical position in an arbitrary way. An additional hypothesis is that of small velocity perturbation compared with freestream velocity. To accomplish this need, this formulation deals with thin airfoils, only. In this way, the velocities at the upper and lower surfaces of the airfoil are approximated as the upper and lower limits ($z \to 0^+$ and $z \to 0^-$, respectively) at corresponding x position. The coordinates system is that shown in Fig.1. The airfoil follows a vertical movement (heaving motion) given by "h" position, which is opposite to z direction, and the angle of attack is α . The reference axis on the airfoil of chord $2b$ passes by x-position $a \cdot b$.

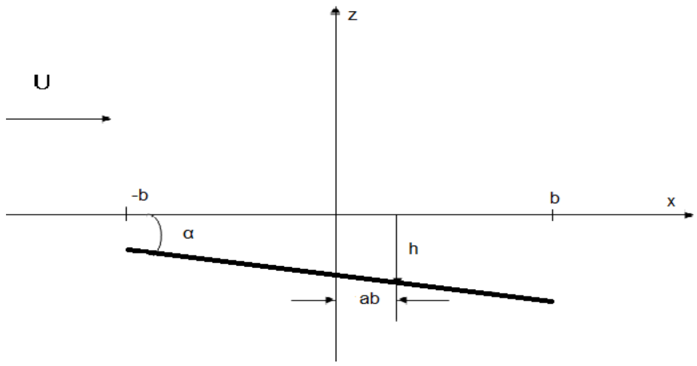

Figure 1. Coordinate system: airfoil in heaving motion and with angle of attack α

The boundary conditions are that of tangential velocity at the surface and zero perturbation at infinity. Another consideration necessary to represent the airfoil is the Kutta condition at trailing edge. It imposes finite velocity at that point. Equations (2) and (3) express the vertical velocity condition at surface, where subscript U and L are for upper and lower side, respectively.

$$w = \frac{\partial z_U}{\partial t} + U \frac{\partial z_U}{\partial x}, \; z = z_U \tag{2}$$

$$w = \frac{\partial z_L}{\partial t} + U \frac{\partial z_L}{\partial x}, \; z = z_L \tag{3}$$

This work follows a method of imposing elementary flows, locating sources, sinks and vortices at prescribed positions in order to solve Laplace Equation with boundary conditions.

The linearity of Laplace Equation permits a separate treatment of thickness distribution and camber distribution. Distributing sources and sinks along x-axis in Fig. 1, thickness distribution is satisfied in terms of boundary conditions. The problem of camber, on the other hand, can be separated in a static skeleton and an oscillating flat plate. The static problem is not the objective of the work, and can be treated by distributing vortices, just like Anderson (1984) does. The remaining problem to complete the frame of an arbitrary thin airfoil oscillating is that of a flat plate, treated by Theodorsen (1979).

A tool used to simplify the solution is the conformal mapping. It involves the transformation of the traditional coordinate system of Fig. 1 which represents a flat plate as a segment of chord 2b in another plane in which the flat plate is a circle of diameter b. The expression of this conformal mapping, known as Kutta-Joukowski transformation, is given by Eq. (4), in which X and Z are the spatial coordinates in transformed plane. Note the use of complex variables in Eq. (4)

$$x + iz = X + iZ + \frac{b^2}{4 \cdot (X + iZ)} \tag{4}$$

Using such mapping, the problem of satisfying boundary conditions is simplified. The differential equation that describes the problem (Laplace Equation) is the same in the new coordinates system, and the same kinds of singularities can be imposed. In fact, the solution of flow around an infinite circular cylinder is widely known, and the main goal of this approach is to bring a more complicated geometry to a this frame in which solution is easier.

To achieve a general solution, the problem may be decomposed in a non-circulatory solution and a circulatory one. The non-circulatory satisfies the normal velocity boundary condition, and the circulatory is necessary to satisfy Kutta condition. Source-sink pairs and vortices are employed in transformed plane to compose non-circulatory and circulatory solutions, respectively. Non-circulatory solution is imposed to satisfy the normal velocities at surfaces, using sources and sinks with opposite signs around the circle. Circulatory solution is used just to satisfy Kutta condition, without altering normal velocities. The vortices are distributed in order to maintain finite velocity at trailing edge. These solutions can be transformed back to original plane of flat plate: the source-sink pairs become doublets, and the vortices just remain vortices.

The conformal mapping maintains local angles between segments, and the tangential/normal directions correspond to tangential/radial directions in the circle. At the surface, these tangential and vertical velocities correspond in the way of Eqs. (5) and (6), in which *u'* is the perturbation velocity along *x*-axis and the θ angle corresponds to the geometric position in transformed plane measure in counterclockwise direction from trailing edge. q_θ and q_r are velocity components in tangential and radial directions, respectively.

$$|u'| = \frac{|q_\theta|}{|2\sin\theta|} \tag{5}$$

$$|w| = \frac{|q_r|}{|2\sin\theta|} \tag{6}$$

In order to satisfy vertical velocity condition (at the original plane), source-sink pairs at the circle surface are employed. Its total contribution to the tangential velocity at a position given by θ is shown in Eq. (7). In this equation, w_a is the vertical velocity at the flat plate surface. θ represents the angle of the position at which tangential velocity is calculated. Influence of all sources and sinks are considered in Eq. (7).

$$q_\theta(\theta,t) = \frac{2}{\pi}\int_0^\pi \frac{w_a \sin^2\phi\, d\phi}{(\cos\phi - \cos\theta)} \tag{7}$$

Relating this velocity with a potential and a pressure disturbance, it's possible to obtain non-circulatory part of the solution. After integrating the pressure difference in the airfoil, non-circulatory lift and moment are exhibited in Eqs. (8) and (9), respectively. In these equations, quantities $\dot{h}$, $\ddot{h}$, $\dot{\alpha}$ and $\ddot{\alpha}$ represent derivatives of vertical position and angle of attack, and the subscript *NC* indicates non-circulatory solution.

$$\mathrm{L}_{\mathrm{NC}} = \pi\rho b^2\left(\ddot{h} + U\dot{\alpha} - ba\ddot{\alpha}\right) \tag{8}$$

$$M_{yNC} = \pi\rho b^2\left[U\dot{h} + ba\ddot{h} + U^2\alpha - b^2\left(\frac{1}{8} + a^2\right)\ddot{\alpha}\right] \tag{9}$$

In order to satisfy Kutta condition, vortices are imposed without affecting radial velocity in the transformed plane. This is accomplished by positioning a pair of counter-rotating vortices, obeying images' method. In this way, one vortice is located at the wake and the other in the interior of the circle. To make tangential velocity zero at θ=0, the distributed vorticity must satisfy Eq. (10).

$$\frac{2}{\pi}\int_0^\pi \frac{w_a \sin^2\phi\, d\phi}{(\cos\phi - 1)} + \frac{1}{\pi b}\int_b^\infty \sqrt{\frac{\xi + b}{\xi - b}}\,\gamma_W(\xi,t)\, d\xi = 0 \tag{10}$$

In Eq. (10), ξ represents the non-dimensional position *x/b*, and ϕ has the same definition of angle θ. γ_W is vorticity located at wake.

The second half of Eq. (9) is designated $2 \cdot Q$. The pressure difference in a point is expressed as function of distributed vorticity in Eq. (11).

$$p_U - p_L = \left(p_U - p_L\right)_{NC} - 2\rho UQ \left\{ \cot\theta + \left[\frac{1-\cos\theta}{\sin\theta}\right] \frac{\int_b^\infty \frac{\xi}{\sqrt{\xi^2 - b^2}} \gamma_W(\xi,t)\, d\xi}{\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}} \gamma_W(\xi,t)\, d\xi} \right\} \tag{11}$$

In Eq. (11), the subscript stands for "non-circulatory". The problem is the ratio between the unknown integrals. At this point, if the movements are supposed harmonic, and the wake is supposed to oscillate in a wavelike manner, it makes possible to identify this ratio like in Eq. (12).

$$C(k) = \frac{\int_b^\infty \frac{\xi}{\sqrt{\xi^2 - b^2}} \gamma_W(\xi,t)\, d\xi}{\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}} \gamma_W(\xi,t)\, d\xi} = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \tag{12}$$

In Eq. (12), H stands for a Henkel Function, with subscript for its order and superscript for its kind. *k* is the reduced frequency defined as $k = \omega b / U$

Finally, the complete expressions of lift and moment are obtained as in Eqs. (13) and (14).

$$\mathrm{L} = \pi\rho b^2 \left(\ddot{h} + U\dot{\alpha} - ba\ddot{\alpha}\right) + 2\pi\rho UbC(k)\left[\dot{h} + U\alpha + b\left(\frac{1}{2} - a\right)\dot{\alpha}\right] \tag{13}$$

$$\begin{aligned} M_y = {} & \pi\rho b^2 \left[ba\ddot{h} - Ub\left(\frac{1}{2} - a\right)\dot{\alpha} - b^2\left(\frac{1}{8} + a^2\right)\ddot{\alpha}\right] + \\ & 2\pi\rho Ub^2\left(a + \frac{1}{2}\right)C(k)\left[\dot{h} + U\alpha + b\left(\frac{1}{2} - a\right)\dot{\alpha}\right] \end{aligned} \tag{14}$$

To analyze in frequencies, it's necessary to decompose the arbitrary movement in harmonic forms. Periodic movements, like the ones which will be analyzed in the examples, are decomposed in series of coefficients calculated according to Eq. (15)

$$c_n = \frac{1}{T}\int_0^T f(\tau) e^{-in\omega_0\tau}\, d\tau \tag{15}$$

In Eq. (15), the $f(\tau)$ function may be the vertical position of the plate or the angle of attack, and ω_0 is the fundamental frequency, the lowest one. The composition is an infinite series, just as in Eq. (16), which will be truncated to perform the analysis.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \tag{16}$$

Composing the harmonic motions, a stationary solution is obtained. It corresponds to one kind solution analyzed in the present work. The other one refers the analysis by indicial movements.

It's possible to derive an expression for the indicial lift and then integrate it by a convolution product between the movement and the indicial response it generates.

In small disturbances theory, the indicial lift created by a unitary angle of attack variation is given by the Eq. (17), after Wagner (Bisplinghoff et al, 1955).

$$L = 2\pi\rho U b w \phi(s) \tag{17}$$

In Eq. (17), $\phi(s)$ is the Wagner function at non-dimensional time s defined as $s=tU/b$.

Thus, a lift variation at time t_o originated by an airfoil's abrupt motion can be obtained through the differentiation of the Eq. (17) that results in:

$$dL = 2\pi\rho U b \phi(s - Ut_0 / b) dw_{b/2}(Ut_0 / b) \tag{18}$$

Equation (18) clearly illustrates that the increasing of the lift occurs exactly by the velocity's variation at the three-quarter chord point and this is governed through Wagner function along the time. Integrating the expression in $\sigma = Ut_0/b$, it is possible to calculate the circulatory lift as a function of s (and consequently as a function of time t), as in Eq. (19). The term $w_{b/2}$ stands for vertical velocity at three-quarter chord point.

$$L = 2\pi\rho U b \int_{-\infty}^{s} \phi(s-\sigma) \frac{dw_{b/2}(\sigma)}{d\sigma} d\sigma \tag{19}$$

Finally, adding the non-circulatory lift to Eq. (19) and calculating an analogous expression for the moment M, Eq. (20) and Eq. (21) give the total lift and moment as a function of s.

$$\begin{aligned} L &= \pi\rho b^2 [\ddot{h} + U\dot{\alpha} - ab\ddot{\alpha}] \\ &- 2\pi\rho U b [w_{b/2}(0)\phi(s) + \int_0^s \phi(s-\sigma) \frac{dw_{b/2}(\sigma)}{d\sigma} d\sigma] \end{aligned} \tag{20}$$

$$\begin{aligned} M_y &= \pi\rho b^2 [ab\ddot{h} - Ub(0.5-a)\dot{\alpha} - b^2(0.125 + a^2)\ddot{\alpha}] \\ &- 2\pi\rho U b^2 (a+0.5)[w_{b/2}(0)\phi(s) + \int_0^s \phi(s-\sigma) \frac{dw_{b/2}(\sigma)}{d\sigma} d\sigma] \end{aligned} \tag{21}$$

3. RESULTS

The methods described above were used for solving three sinusoidal motions: vertical and pitching movements and a superposition of them.

In the examples, a flat plate of semi-chord b=1.5m is submitted to a flow in undisturbed velocity U=100m/s.

The first case is that in which the plate executes only a sinusoidal heaving motion of amplitude of 0.3 meters and a frequency of 10 Hz, just like $h(t) = 0.3 \cdot \sin(20\pi \cdot t)$. The reduced frequency is k=0.94. Thus, the moment when the airfoil would reach a zero vertical velocity should be t=0.025 seconds so that the lift coefficient would be zero. However, it is possible to see in Fig. 2 that, beforehand, the lift coefficient of the flat plate cl is zero for t=0.015 s, which means an advance in time by 0.01. This is confirmed by the fact that lift coefficient is not maximum at t=0 and the respective period time, and it could me maximum at theses times because the profile's velocity is maximum at them. The maximum lift occurs 0.01s before the time predicted.

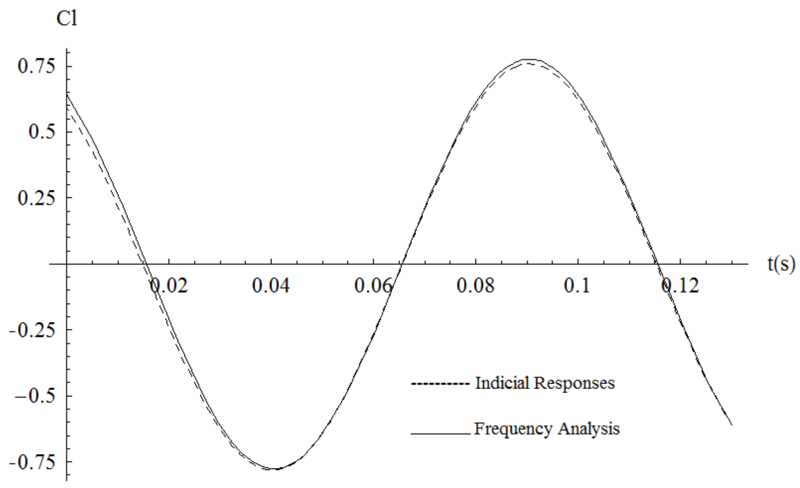

Figure 2. Lift coefficient – heaving motion.

Considering the moment coefficient *cm* in Fig. 3 (relative to the center at mid-chord of flat plate), it's remarkable the existence of a positive moment even for a negative lift (about t=0.022 s). This suggests that the instantaneous center of pressure has moved downstream through the origin of the system, since the moment is about the center of the airfoil.

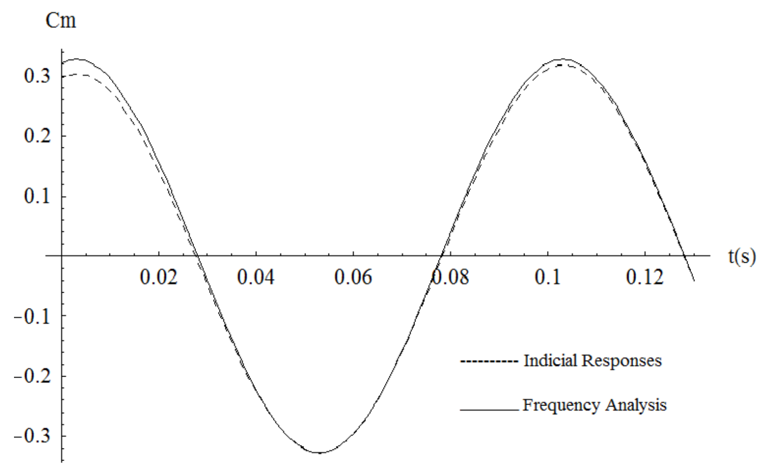

Figure 3. Moment coefficient about the center of the airfoil – heaving motion.

In respect to both graphics it is noticeable that the non-dimensional coefficients oscillate in the same frequency of the heaving motion.

In the second example, flat plate is exposed to the same freestream velocity of 100m/s along *x*-direction. The motion is just a sinusoidal pitching one about an axis passing through the mid-chord of the flat plate. Frequency is one third of the one used in heaving motion described previously. So, reduced frequency associated is k=0.31. Amplitude is of 10 degrees.

The general effect of pitching is a variation in angle of attack accompanied by increments or decrements in lift and moment caused by the rotation itself. Such effects can be observed in lift curve (Fig. 4), which presents a positive sustentation at zero angle of attack. At t=0.3s, for example, the angle of attack is null, but the lift is already positive. The zero lift occurred at a moment just before. It's seen that the presence of a wake advances expected behavior, so.

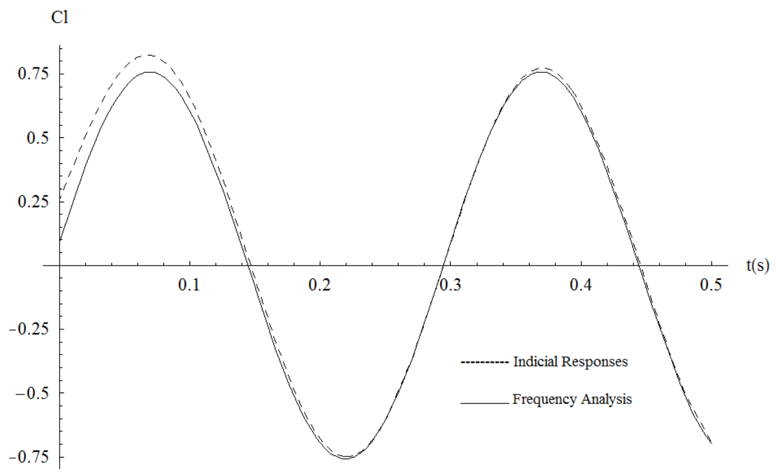

Figure 4. Lift coefficient – pitching movement.

Moment coefficient shows a negative value at null angle of attack, while the lift is positive, how illustrated in Fig. 5. This shows that this lift increment at this moment cannot be located at quarter chord, but at a point located downstream from mid-chord.

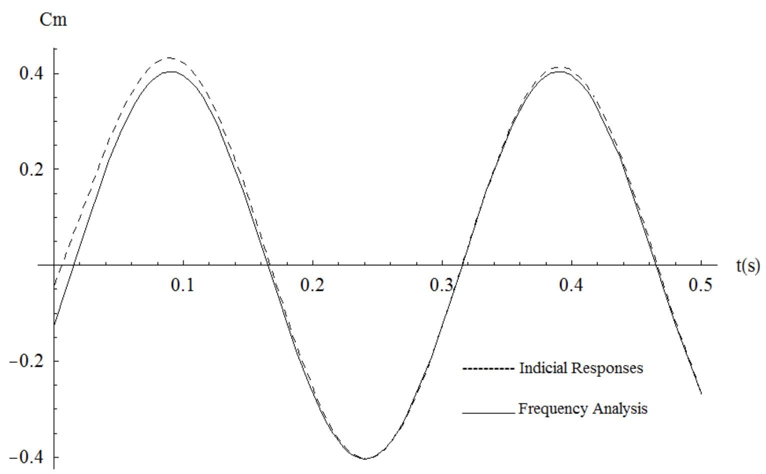

Figure 5. Moment coefficient about the center of the airfoil – pitching movement.

Another noticeable fact is the decrement of maximum lift coefficient. While the maximum angle of attack achieved would lead to a 1.1 lift coefficient in static situation, the maximum here achieved is about 0.77, only. The wake is related to this reduction.

A general note is that frequency of moment and lift oscillations is the same of the movement, like the heaving motion. It's expected from the model of oscillating vortices.

The third motion is just a superposition of the last ones so that the airfoil pitches and moves vertically in the coordinate system. Initial phase is the same, null. Figures 6 and 7 illustrate lift and moment coefficients obtained.

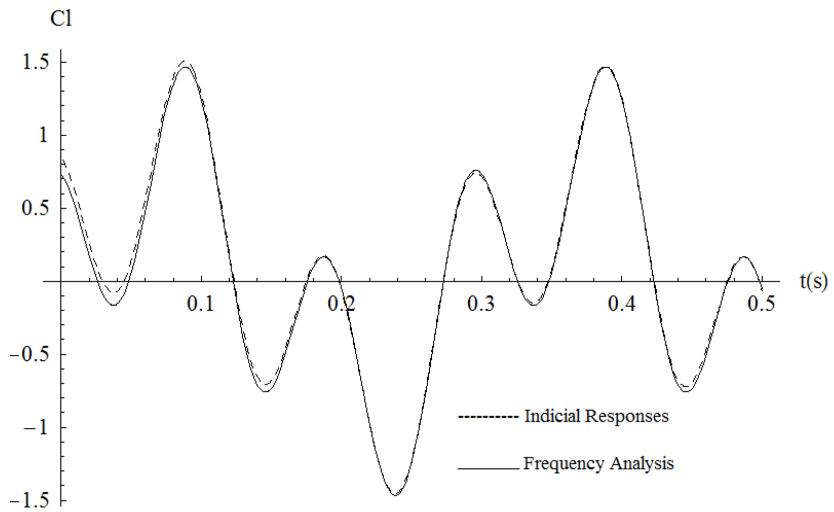

Figure 6. Lift coefficient – superposition of the motions.

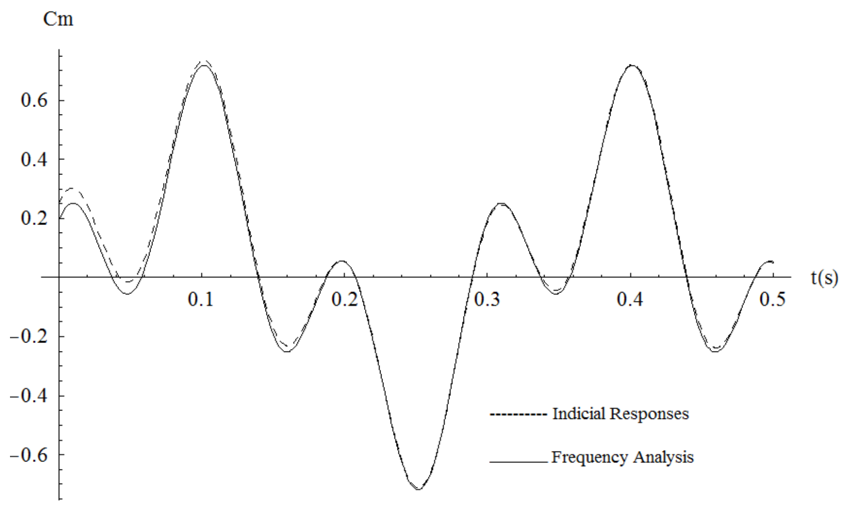

Figure 7. Moment coefficient about the center of the airfoil – superposition of the motions.

As a superposition of the previous motions, this solution must be the sum of the others since they both satisfy Laplace equation for incompressible flow. This was verified by the sum of them as expected.

In order to analyze the differences between transient regime and stationary solution in respect to all motions studied, it is necessary to see the effects of the wake, which develops with time in transient. These effects may be described in terms of evolution of Wagner function. This function is used in convolution of Eq. (19), with each infinitesimal movement influencing the subsequent ones.

In stationary case, at the beginning of the cycle, the influence of the entire previous cycle is decaying to zero as Wagner function associated to these movements approaches unit. Thus, the last movements of previous cycle still influencing constitute the difference between stationary and transient cases. In transient, there is no remaining influence of last movements of previous cycle.

The non-dimensional time required for the Wagner function to achieve 0.9 is approximately 12.8. This time corresponds to a situation in which the last movements of previous cycle have almost completed its influence, and the influence of this cycle approaches zero. This can be identified with 0.2s in heaving motion analysis. A plot of the difference between the two solutions can be seen in Fig. 8. This shows a rapid decrease in amplitude of the difference, such that at 0.2s the amplitude of the variation is almost constant, in accordance with expectations.

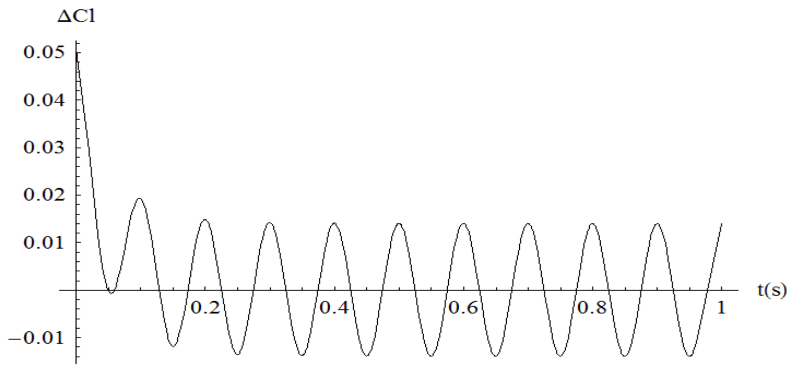

Figure 8. Lift coefficient difference between both methods – heaving motion

It is important to note that the difference occurs only in circulatory solution. The non-circulatory component is exactly the same in indicial and frequency analysis. Because of this component, the lift is not zero when the plane in pitching passes through zero angle of attack, even for the first moments in indicial analysis. It does not make sense to compare the transient in a relative way, like a percentage error, since the aerodynamic coefficients may pass through zero and any ratio would not be define in some points.

Physically, indicial analysis deals with a wake under development. From the beginning, there are vortices at wake, but these start its propagation with free stream velocity from that moment. When they fill the wake near the airfoil leading edge, a stationary situation is achieved.

But even for long time, the solutions do not coincide perfectly. Indeed, there is an oscillation of the difference. This is due to numerical approximations of the functions employed to construct the graphics. Wagner function, for example, was taken from such an exponential approximation.

4. CONCLUSION

The frequency of 10Hz taken as example shows the occurrence of unsteady phenomena, leading to reduction in lift coefficients and phase difference between aerodynamic coefficients and the movements. The continuous shedding of a vortex wake was responsible by influences over the airfoil, and the progressive appearance of this wake could be analyzed by comparing two solutions, one based on indicial responses and another based on harmonic motions.

Indicial responses may take into account the progressive development of a wake, while the frequency analysis assumes a developed wake. The differences can be related to Wagner function evolution, allowing estimations of transient time length.

Expected linearity was observed by comparing a conjugate movement of pitching and heaving motion with the individual ones, in terms of the aerodynamic coefficients. Another interesting observation is about the phase at each moment: if a particular position is compared with its static analogous, phase differences have been seen due to circulatory and non-circulatory influences. Indeed, the maximum and minimum aerodynamic coefficients don't occur at the moments expected from common sense. The circulatory part of solution is dependent of Theodorsen function, which varies according to reduced frequency.

Solutions by indicial responses and harmonic motions showed high concordance. The existence of an admittance function for indicial responses, however, makes this approach more easily applicable to a random movement. The fundamental solution by singularities and considering harmonic motion lies in the basis of Wagner Function, however, and is necessary as a first step to more general approaches like indicial one.

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