HYBRID SOLUTION THROUGH INTEGRAL TRANSFORM FOR FULL NAVIER-STOKES EQUATIONS IN CONCENTRIC ANNULAR DUCTS WITH ROTATION OF THE INNER CYLINDER

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Abstract. *A hybrid solution through the Generalized Integral Transform Technique (GITT) is obtained for the full Navier-Stokes equations in laminar flow of Newtonian fluids within annular ducts with rotation of the inner cylinder. The mathematical formulation is constructed based on the cylindrical coordinates system in the entrance region of the annular channel. Numerical results for the velocity field were produced for different values of the governing parameters, i.e., Reynolds numbers, radii ratios and rotation parameters. The results were confronted with previously reported ones, providing critical comparisons while illustrating the employed integral transform approach.*

Keywords: *Annular ducts, Navier-Stokes equations, Integral transform, Hydrodynamically developing flow.*

1. INTRODUCTION

The flow in concentric annular ducts is a classic problem in fluid mechanics. With the presence of rotation of the inner cylinder, several flow patterns can be found. Therefore, the accurate knowledge of hydrodynamic and thermal behavior is important in the design of equipments and the optimization of operations conditions.

In addition, the flow in annular regions can be found in different industrial applications, such as, heat exchangers and petroleum columns drilling. Studies involving the analysis of hydrodynamic entrance region and fully developed flow using boundary layer equations are very common in the literature, as well as the use of numerical methods for solving this type of problem (Coney and El-Shaarawi, 1974a; 197b; Kakaç and Yücel, 1974). However, few works using the full Navier-Stokes equations in the solution of concentric annular flow with rotation of the inner wall are found in the literature.

The flow in the entrance region of concentric annular duct with rotation of the inner wall was performed by Coney and El-Shaarawi (1974a), using the finite difference method to the solution of the boundary layers equations, considering different radii ratios and parameters of rotation intensity. The local Nusselt number and the bulk temperature through annular ducts were numerically obtained by Coney and El-Shaarawi (1974b) with objective of analyzing the thermal entrance region. A solution for the fully developed flow and the hydrodynamic developing flow in annular ducts using the full Navier-Stokes equations for three-dimensional laminar flow was proposed by Velusamy (1994). The hydrodynamic developing flow in concentric annular ducts using the full Navier-Stokes equations in terms of streamfunction formulation was successfully studied by Pereira (1995) and Pereira et al. (1998), within the governing parameters as: radii ratios and Reynolds numbers.

Along the years, a hybrid methodology based on eigenfunction expansions have developed and become an excellent alternative for the solution of annular flow problems, like those proposed by Viana et al. (2001) and Nascimento et al. (2002) that studied thermally developing flow of Herschel-Bulkley and Bingham fluids in concentric annular ducts, respectively.

In this context, one intends to use the Generalized Integral Transform Technique for handling the problem of developing laminar flow with rotation of the inner cylinder in order to obtain an error-controlled solution for benchmark purposes.

2. PROBLEM FORMULATION AND SOLUTION METHODOLOGY

In order to analyze hydrodynamically developing flow of a Newtonian fluid through concentric annular duct with rotation of the inner cylinder, as shown in Fig. 1, the following simplifying assumptions are considered: steady state and laminar flow with constant physical properties. The flow is modeled by the continuity (automatically satisfied) and the Navier-Stokes equations in cylindrical coordinates, which written in terms of streamfunction and primitive variables are given by:

$$E^4\psi = \frac{Re}{2(1-\gamma)}\left[\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\left(E^2\psi\right)}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial\left(E^2\psi\right)}{\partial r} - \frac{2}{r^2}\frac{\partial\psi}{\partial z}E^2\psi - 2\xi^2 v_\theta \frac{\partial v_\theta}{\partial z}\right] \tag{1.a}$$

$$\left[\frac{1}{r}\frac{\partial\psi}{\partial z}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right) - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial v_\theta}{\partial z}\right]\frac{Re}{2(1-\gamma)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{\partial^2 v_\theta}{\partial z^2} \tag{1.b}$$

The following boundary conditions are needed to solve the system of equations (1.a,b):

$$\psi = C_1;\ \ \frac{\partial\psi}{\partial r} = 0;\ \ v_\theta = 1,\quad r = \gamma \tag{1.c-f}$$

$$\psi = C_2;\ \ \frac{\partial\psi}{\partial r} = 0;\ v_\theta = 0,\ \ r = 1 \tag{1.g-j}$$

$$\psi = C_1 - \frac{\left(r^2-\gamma^2\right)}{2};\ \ \frac{\partial^2\psi}{\partial z^2} = 0;\ v_\theta = 0,\ z = 0 \tag{1.k-n}$$

$$\psi = \psi_\infty(r);\ \frac{\partial\psi}{\partial z} = 0;\ v_\theta = v_{\theta,\infty}(r),\ z \to \infty \tag{1.o-r}$$

The use of a streamfunction formulation in the solution of Navier-Stokes equations automatically satisfies the continuity equation, eliminates the pressure gradient and improves the computational performance. The streamfunction in cylindrical coordinates are related to the radial and axial velocity components, in the form

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial z},\qquad v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r} \tag{2}$$

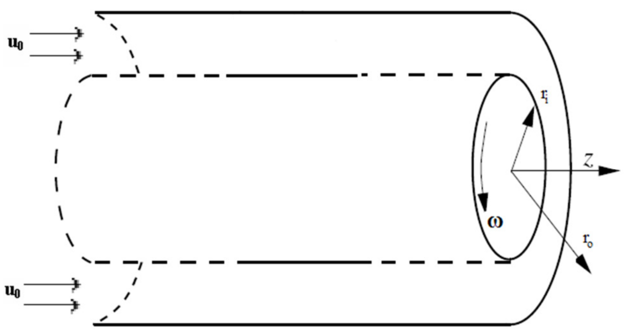

Figure 1. Geometry and coordinate system.

The following dimensionless groups were used in Eqs. (1):

$$r = \frac{r^*}{r_o};\ z = \frac{z^*}{r_o};\ v_z = \frac{v_z^*}{u_o};\ v_r = \frac{v_r^*}{u_o};\ v_\theta = \frac{v_\theta^*}{\omega r_i};\ \ \gamma = \frac{r_i}{r_o} \tag{3.a-f}$$

$$p = \frac{p^*}{\rho u_o^2};\ Re = \frac{\rho D_h u_o}{\mu};\ D_h = 2r_o(1-\gamma)\,;\quad \xi = \frac{\omega r_i}{u_o} = \frac{1}{Re}\sqrt{\frac{2Ta(1+\gamma)}{(1-\gamma)}};\ Ta = \frac{2(\omega r_i)^2(r_o - r_i)^3}{\nu^2(r_o + r_i)} \tag{3.g-k}$$

In the solution of Eqs. (1), in order to improve the computational performance, a filter is proposed, which preserves the original characteristics of the problem and becomes homogeneous the boundary conditions, therefore:

$$\psi(r,z) = \psi_\infty(r) + \phi(r,z);\quad v_\theta(r,z) = v_{\theta,\infty}(r) + v_{\theta,F}(r,z);\quad v_{\theta,\infty}(r) = \frac{\gamma}{\left(1-\gamma^2\right)}\left(\frac{1}{r} - r\right) \tag{4.a-c}$$

$$\psi_\infty(r) = C_2 - \frac{2}{\beta}\left[r_m^2\left(r^2\ln r - \gamma^2\ln r\right) - \frac{\left(r^2-\gamma^2\right)}{4}\left(2r_m^2 - 2 + r^2 + \gamma^2\right)\right];\ \ r_m = \left\{\frac{\left(1-\gamma^2\right)}{2\ln(1/\gamma)}\right\}^{1/2}\ ;\ \ \beta = 1 + \gamma^2 - 2r_m^2 \tag{4.d-f}$$

The following system of equations is obtained for the filtered potentials:

$$E^4\phi = \frac{Re}{2(1-\gamma)}\left[\frac{1}{r}\frac{\partial\phi}{\partial z}\left[\frac{\partial(E^2\phi)}{\partial r}+\frac{d(E^2\psi_\infty)}{dr}\right]-\frac{1}{r}\frac{\partial\phi}{\partial r}\frac{\partial(E^2\phi)}{\partial z}-\frac{1}{r}\frac{d\psi_\infty}{dr}\frac{\partial(E^2\phi)}{\partial z}-\frac{2}{r}\frac{\partial\phi}{\partial z}(E^2\phi+E^2\psi_\infty)-2\xi^2(v_{\theta,F}+v_{\theta,\infty})\frac{\partial v_{\theta,F}}{\partial z}\right] \tag{5.a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_{\theta,F}}{\partial r}\right)-\frac{v_{\theta,F}}{r^2}+\frac{\partial^2 v_{\theta,F}}{\partial z^2}=\frac{Re}{2(1-\gamma)}\left[\frac{1}{r}\frac{\partial\phi}{\partial z}\left(\frac{\partial v_{\theta,F}}{\partial r}+\frac{v_{\theta,F}}{r}+\frac{dv_{\theta,\infty}}{dr}+\frac{v_{\theta,\infty}}{r}\right)-\frac{1}{r}\left(\frac{\partial\phi}{\partial r}+\frac{d\psi_\infty}{dr}\right)\frac{\partial v_{\theta,F}}{\partial z}\right] \tag{5.b}$$

with the following boundary conditions:

$$\phi(\gamma,z)=0;\quad \frac{\partial\phi(\gamma,z)}{\partial r}=0;\ v_{\theta,F}(\gamma,z)=0 \tag{5.c-e}$$

$$\phi(1,z)=0;\quad \frac{\partial\phi(1,z)}{\partial r}=0;\ v_{\theta,F}(1,z)=0 \tag{5.f-h}$$

$$\phi(r,0)=C_1-\frac{(r^2-\gamma^2)}{2}-\psi_\infty;\quad \frac{\partial^2\phi}{\partial z^2}=0;\ v_{\theta,F}(r,0)=0 \tag{5.i-k}$$

$$\phi(r,\infty)=0;\quad \frac{\partial\phi(r,\infty)}{\partial z}=0;\ v_{\theta,F}(r,\infty)=0 \tag{5.l-n}$$

where, $C_1=0$ and $C_2=-(1-\gamma^2)/2$.

The next step is the choice of the appropriate eigenvalue problem to the solution of the original problem given by Eqs. (5). The following eigenvalue problems with their respective eigenvalues, eigenfunctions and orthogonality properties are given as:

- For the component $\phi(r,z)$:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)^2\Omega_i(r)=\lambda_i^4\Omega_i(r) \tag{6.a}$$

$$\Omega_i(\gamma)=0;\qquad \frac{d\Omega_i(\gamma)}{dr}=0;\qquad \Omega_i(1)=0;\qquad \frac{d\Omega_i(1)}{dr}=0 \tag{6.b-e}$$

$$\underset{\sim}{\mathbf{P}}=\begin{bmatrix} J_0(\lambda_i\gamma) & Y_0(\lambda_i\gamma) & \dfrac{I_0(\lambda_i\gamma)}{I_0(\lambda_i)} & 1 \\ J_0(\lambda_i) & Y_0(\lambda_i) & 1 & \dfrac{K_0(\lambda_i)}{K_0(\lambda_i\gamma)} \\ -J_1(\lambda_i\gamma) & -Y_1(\lambda_i\gamma) & \dfrac{I_1(\lambda_i\gamma)}{I_0(\lambda_i)} & -\dfrac{K_1(\lambda_i\gamma)}{K_0(\lambda_i\gamma)} \\ -J_1(\lambda_i) & -Y_1(\lambda_i) & \dfrac{I_1(\lambda_i)}{I_0(\lambda_i)} & -\dfrac{K_1(\lambda_i)}{K_0(\lambda_i\gamma)} \end{bmatrix};\quad \sum_{j=1}^{4}P_{jk}A_{ji}=0,\ \ k=1,2,3,4;\quad Det(\underset{\sim}{\mathbf{P}})=0 \tag{6.f-h}$$

$$\Omega_i(r)=A_{1i}J_0(\lambda_i r)+A_{2i}Y_0(\lambda_i r)+A_{3i}\frac{I_0(\lambda_i r)}{I_0(\lambda_i)}+A_{4i}\frac{K_0(\lambda_i r)}{K_0(\lambda_i\gamma)} \tag{6.i}$$

$$\int_\gamma^1 r\Omega_i(r)\Omega_j(r)dr=\begin{cases}0, i\neq j\\ M_i, i=j\end{cases};\quad M_i=\int_\gamma^1 r\Omega_i^2(r)dr=[J_0(\lambda_i)+A_{2i}Y_0(\lambda_i)]^2-\gamma^2[J_0(\lambda_i\gamma)+A_{2i}Y_0(\lambda_i\gamma)]^2 \tag{6.j,k}$$

- For the component $v_{\theta,F}(r,z)$:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dX_i(r)}{dr}\right)-\left(\frac{1}{r^2}-\mu_i^2\right)X_i(r)=0 \tag{7.a}$$

$$X_i(\gamma)=0;\qquad X_i(1)=0 \tag{7.b,c}$$

$$J_1(\mu_i)Y_1(\mu_i\gamma) - J_1(\mu_i\gamma)Y_1(\mu_i) = 0, \qquad i = 1,2,3...; \quad X_i(r) = \frac{J_i(\mu_i r)}{J_i(\mu_i)} - \frac{Y_i(\mu_i r)}{Y_i(\mu_i)} \tag{7.d,e}$$

$$\int_\gamma^1 rX_i(r)X_j(r)dr = \begin{cases} 0, i \neq j \\ N_i, i = j \end{cases}; \quad N_i = \int_0^1 rX_i^2(r)dr = \frac{2}{\pi^2}\frac{[J_1^2(\mu_i\gamma) - J_1^2(\mu_i)]}{\mu_i^2 Y_1^2(\mu_i)J_1^2(\mu_i\gamma)J_1^2(\mu_i)} \tag{7.f,g}$$

The eigenvalues problems allow the definition of the following integral transform pairs:

- For the component $\phi(r,z)$:

$$\bar{\phi}_i(z) = \int_\gamma^1 r\tilde{\Omega}_i(r)\phi(r,z)dr, \quad \text{transform;} \quad \phi(r,z) = \sum_{i=1}^{\infty}\tilde{\Omega}_i(r)\bar{\phi}_i(z), \quad \text{inverse} \tag{8.a,b}$$

- For the component $v_{\theta,F}(r,z)$:

$$\bar{v}_{\theta,i}(z) = \int_\gamma^1 r\tilde{X}_i(r)v_{\theta,F}(r,z)dr, \quad \text{transform;} \quad v_{\theta,F}(r,z) = \sum_{i=1}^{\infty}\tilde{X}_i(r)\bar{v}_{\theta,i}(z), \quad \text{inverse} \tag{8.c,d}$$

where, $\tilde{\Omega}_i(r) = \Omega_i(r)/\sqrt{M_i}$ and $\tilde{X}_i(r) = X_i(r)/\sqrt{N_i}$ are the normalized eigenfunctions.

The next step is the integral transformation process of the partial differential equations for the components $v_{\theta,F}(r,z)$ and $\phi(r,z)$. For this purpose, Eq. (5.a) is multiplied by $r\,\tilde{\Omega}_i(r)$, Eq. (5.b) by $r\,\tilde{X}_i(r)$, and integrated over the domains [γ,1] and [0,1], respectively, in the r-direction; the inverse formulae, Eqs. (8.b) and (8.d), are employed in place of the velocity distribution $v_{\theta,F}(r,z)$ and $\phi(r,z)$, resulting in the following transformed ordinary differential system:

$$\frac{d^4\bar{\phi}_i}{dz^4} = -\lambda_i^4\bar{\phi}_i + \sum_{j=1}^{\infty}\left\{\left(4A_{ij}\bar{\phi}_j + 2B_{ij}\bar{\phi}_j''\right) + \frac{Re}{2(1-\gamma)}\left[C_{ij}\bar{\phi}_j - D_{ij}\bar{\phi}_j''' - 2\xi^2E_{ij}\bar{v}_j' + \left(F_{ijk}\bar{\phi}_j'\bar{\phi}_k + G_{ijk}\bar{\phi}_j'\bar{\phi}_k'' - H_{ijk}\bar{\phi}_j'''\bar{\phi}_k - 2\xi^2I_{ijk}\bar{v}_j'\bar{v}_k\right)\right]\right\} \tag{9.a}$$

$$\frac{d^2\bar{v}_i}{dz^2} = \mu_i^2\bar{v}_i + \frac{Re}{2(1-\gamma)}\sum_{j=1}^{\infty}\left[\left(J_{ij}\bar{\phi}_j' - K_{ij}\bar{v}_j'\right) + \sum_{k=1}^{\infty}\left(L_{ijk}\bar{v}_j\bar{\phi}_k' - M_{ijk}\bar{v}_j'\bar{\phi}_k\right)\right] \tag{9.b}$$

The same integral transform process is made in the boundary conditions given by Eqs. (5.i) to (5.n), resulting:

$$\bar{\phi}_i(0) = \bar{f}_i; \quad \bar{\phi}_i''(0) = 0; \quad \bar{v}_i(0) = 0; \quad \bar{\phi}_i(\infty) = 0; \quad \phi_i'(\infty) = 0; \quad \bar{v}_i(\infty) = 0 \tag{9.c-h}$$

where the various coefficients are given by:

$$A_{ij} = \int_\gamma^1 \tilde{\Omega}_i\left(\dddot{\tilde{\Omega}}_j - \frac{\ddot{\tilde{\Omega}}_j}{r} + \frac{\dot{\tilde{\Omega}}_j}{r^2}\right)dr; \qquad B_{ij} = \int_\gamma^1 \tilde{\Omega}_i\left(\dot{\tilde{\Omega}}_j - r\ddot{\tilde{\Omega}}_j\right)dr \tag{10.a,b}$$

$$C_{ij} = \int_\gamma^1 \tilde{\Omega}_i\left[\tilde{\Omega}_i\left(\dddot{\psi}_\infty - \frac{3}{r}\ddot{\psi}_\infty + \frac{3}{r^2}\dot{\psi}_\infty\right) + \frac{1}{r}\dot{\tilde{\Omega}}_j\dot{\psi}_\infty - \dot{\tilde{\Omega}}_j\dot{\psi}_\infty\right]dr; \quad D_{ij} = \int_\gamma^1 \tilde{\Omega}_i\tilde{\Omega}_j\dot{\psi}_\infty dr; \; E_{ij} = \int_\gamma^1 r\tilde{\Omega}_i\tilde{X}_j v_{\theta,\infty}dr \tag{10.d-e}$$

$$F_{ij} = \int_\gamma^1 \tilde{\Omega}_i\left[\tilde{\Omega}_j\left(\dddot{\tilde{\Omega}}_k - \frac{3}{r}\ddot{\tilde{\Omega}}_k + \frac{3}{r^2}\dot{\tilde{\Omega}}_k\right) - \left(\ddot{\tilde{\Omega}}_j - \frac{1}{r}\dot{\tilde{\Omega}}_j\right)\dot{\tilde{\Omega}}_k\right] \tag{10.f}$$

$$G_{ijk} = \int_\gamma^1 \tilde{\Omega}_i\tilde{\Omega}_j\left(\dot{\tilde{\Omega}}_k - \frac{2}{r}\dot{\tilde{\Omega}}_k\right)dr; \quad H_{ij} = \int_\gamma^1 \tilde{\Omega}_i\tilde{\Omega}_j\dot{\tilde{\Omega}}_k dr; \quad I_{ij} = \int_\gamma^1 r\tilde{\Omega}_i\tilde{X}_j\tilde{X}_k dr; \; J_{ij} = \int_\gamma^1 \tilde{X}_i\tilde{\Omega}_j\left(\dot{v}_{\theta,\infty} + \frac{v_{\theta,\infty}}{r}\right)dr \tag{10.g-j}$$

$$K_{ij}=\int_{\gamma}^{1}\tilde{X}_i\tilde{X}_j\psi_\infty dr;\;\; L_{ijk}=\int_{\gamma}^{1}\tilde{X}_i\left(\dot{\tilde{X}}_j+\frac{\tilde{X}_j}{r}\right)\tilde{\Omega}_k dr;\;\; M_{ijk}=\int_{\gamma}^{1}\tilde{X}_i\tilde{X}_j\dot{\tilde{\Omega}}_k dr;\;\; \bar{f}_i=\int_{\gamma}^{1}r\tilde{\Omega}_i\left[C_1-\frac{\left(r^2-\gamma^2\right)}{2}-\psi_\infty\right]dr \quad (10.k\text{-}n)$$

An important parameter used in the engineering is the Fanning friction factor, and it can be calculated by the following expression:

$$f=\frac{16\left(1-\gamma\right)^2}{Re\beta}+\frac{4\left(1-\gamma\right)}{\left(1+\gamma\right)}\left(\left.\frac{\partial^2\phi}{\partial r^2}\right|_{r=1}-\left.\frac{\partial^2\phi}{\partial r^2}\right|_{r=\gamma}\right);\quad \frac{\partial^2\phi}{\partial r^2}=\sum_{i=1}^{\infty}\frac{d^2\tilde{\Omega}_i\left(r\right)}{dr^2}\bar{\phi}_i\left(z\right) \quad (11.a,b)$$

For computational purposes, it is necessary to truncate the infinite expansions in a sufficiently large number of terms so as to achieve the user prescribed relative error target for obtaining the original potentials, in this case the streamfunction and the tangential velocity component values, where NF and NT are here the order of truncation of the infinite series, respectively. Also, in order to solve the transformed ODE system, efficient numerical algorithms for boundary value problems are to be employed, such as the subroutine DBVPFD from the IMSL Library (1991), which offers an automatic adaptive scheme for local error control of the numerical results for the transformed potentials. It is then necessary to rewrite the transformed ODE system as a first order one, by introducing the following dependent variables:

$$\bar{\phi}_i=W_i;\quad \frac{d\bar{\phi}_i}{dz}=\frac{dW_i}{dz}=W_{i+NF};\quad \frac{d\bar{\phi}_i}{dz}=\frac{dW_i}{dz}=W_{i+NF};\quad \frac{d^2\bar{\phi}_i}{dz^2}=\frac{d}{dz}\left(\frac{d\bar{\phi}_i}{dz}\right)=\frac{dW_{i+NF}}{dz}=W_{i+2NF} \quad (12.a\text{-}d)$$

$$\frac{d^3\bar{\phi}_i}{dz^3}=\frac{d}{dz}\left(\frac{d^2\bar{\phi}_i}{dz^2}\right)=\frac{dW_{i+2NF}}{dz}=W_{i+3NF};\quad \frac{d^4\bar{\phi}_i}{dz^4}=\frac{d}{dz}\left(\frac{d^3\bar{\phi}_i}{dz^3}\right)=\frac{dW_{i+3NF}}{dz},\qquad i=1,2,3,...,NF \quad (12.e,f)$$

$$W_{i+4NF}=\bar{v}_i;\quad W_{i+4NF+NT}=\frac{d\bar{v}_i}{dz}=\frac{dW_{i+4NF}}{dz};\quad \frac{dW_{i+4NF+NT}}{dz}=\frac{d^2\bar{v}_i}{dz^2};\qquad i=1,2,3,...,NT \quad (12.g\text{-}i)$$

Therefore, by making use of Eqs. (12), the transformed system can be rewritten as:

$$\frac{dW_{i+3NF}}{dz}=-\lambda_i^4W_i+\sum_{j=1}^{NF}\Bigg\{\left(4A_{ij}W_j+2B_{ij}W_{j+2NF}\right)+\frac{Re}{2\left(1-\gamma\right)}\Big[C_{ij}W_{j+NF}-D_{ij}W_{j+3NF}-2\xi^2E_{ij}W_{j+4NF+NT}$$
$$+\sum_{k=1}^{NF}\left(F_{ijk}W_{j+NF}W_k+G_{ijk}W_{j+NF}W_{k+2NF}-H_{ijk}W_{j+3NF}W_k-2\xi^2I_{ijk}W_{j+4NF+NT}W_{k+4NF}\right)\Big]\Bigg\} \quad (13.a)$$

$$\frac{dW_{i+4NF+NT}}{dz}=\mu_i^2W_{i+4NF}+\frac{Re}{2\left(1-\gamma\right)}\left\{\sum_{j=1}^{NF}\left(J_{ij}W_{j+NF}\right)-\sum_{j=1}^{NT}\left(K_{ij}W_{j+4NF+NT}\right)+\sum_{j=1}^{NT}\sum_{k=1}^{NF}\left(L_{ijk}W_{j+4NF}W_{k+NF}-M_{ijk}W_{j+4NF+NT}W_k\right)\right\} \quad (13.b)$$

with the boundary conditions:

$$W_i(0)=\bar{f}_i;\quad W_{i+2NF}(0)=0;\quad W_{i+4NF}\left(0\right)=0 \quad (13.c\text{-}e)$$

$$W_i(\infty)=0;\quad W_{i+NF}(\infty)=0;\quad W_{i+4NF}\left(\infty\right)=0 \quad (13.f\text{-}h)$$

$$W_i(0)=\bar{f}_i;\quad W_{i+2NF}(0)=0;\quad W_{i+4NF}\left(0\right)=0 \quad (13.i\text{-}k)$$

As the domain of the problem in the longitudinal direction is $z\in[0,\infty)$, a domain transformation should be accomplished, i.e. , $\eta\in[0,1]$. Therefore the following algebraic expression must be used in the system of equations (13):

$$\eta=1-e^{-cz};\quad \frac{d\eta}{dz}=c\left(1-\eta\right) \quad (14.a,b)$$

where c is a parameter of scale contraction.

3. RESULTS AND DISCUSSION

Numerical results for the velocity field and for the product of the Fanning friction factor-Reynolds number were obtained from a code developed in the FORTRAN 90 programming language. The code was implemented on a PENTIUM IV Core 2 Duo E7400 2.8 GHz microcomputer, and the system given by Eqs. (13) was handled through the subroutine DBVPFD from the IMSL Library (1991). A relative error target of 10^{-5} was employed throughout the computations, for varying the values of the Reynolds numbers, Re, radii ratios, γ, and rotation parameters, ξ.

Table 1 shows the convergence behavior of the axial velocity component in the entrance region of the annular duct, evaluated at axial positions, z=0.54 and z=2.7 for different truncation orders N=NF=NT. The Reynolds number adopted is equal to 300 and the aspect ratio 0.1 at five radial positions. In this case, the rotation parameter ξ corresponds to the annular flow without rotation of the inner cylinder, i.e., ξ=0. It was found that the results shown excellent agreement with those of Pereira (1995). Comparing with the results obtained by Kakaç and Yücel (1974) there was a discrepancy between the results, since in this case, the authors used the formulation of boundary layer in the solution of the problem.

Table 1. Convergence analysis of the axial velocity component for the case without rotation of the inner cylinder.

$v_z(r,z)$										
γ=0.1; Re=300; ξ=0										
	z=0.54					z=2.7				
N	**r=0.145**	**r=0.325**	**r=0.550**	**r=0.775**	**r=0.955**	**r=0.145**	**r=0.325**	**r=0.550**	**r=0.775**	**r=0.955**
7	0.9055	1.093	1.090	1.186	0.5441	0.5027	1.281	1.383	1.121	0.2896
11	0.6411	1.097	1.106	1.193	0.5052	0.4953	1.274	1.389	1.121	0.2884
15	0.5917	1.108	1.109	1.202	0.4988	0.4939	1.272	1.391	1.121	0.2881
19	0.5829	1.110	1.110	1.204	0.4970	0.4936	1.272	1.391	1.121	0.2881
23	0.5799	1.112	1.110	1.205	0.4962	0.4935	1.272	1.391	1.121	0.2881
27	0.5787	1.112	1.110	1.206	0.4959	0.4935	1.272	1.391	1.121	0.2881
31	0.5782	1.112	1.110	1.206	0.4958	0.4935	1.272	1.391	1.121	0.2882
35	0.5780	1.112	1.110	1.206	0.4958	0.4935	1.271	1.391	1.121	0.2882
39	**0.5779**	**1.112**	**1.110**	**1.206**	**0.4958**	**0.4935**	**1.271**	**1.391**	**1.121**	**0.2882**
a	0.5791	1.112	1.110	1.205	0.4960	0.4935	1.272	1.391	1.121	0.2881
b	0.5800	1.202	1.209	1.177	0.4060	0.499	1.309	1.415	1.094	0.2780

a - Pereira (1995); b - Kakaç and Yücel (1974).

From Figure 2, it is observed that the results of GITT adhere completely with those obtained with the software Comsol Multiphysics™ (2006) that employs the finite element method. The flattened behavior at the duct centerline can be attributed to the influence of higher velocity gradients caused by the development of the boundary layer near the duct walls. When the flow advances for z→∞, the velocity profile tends to a parabolic distribution.

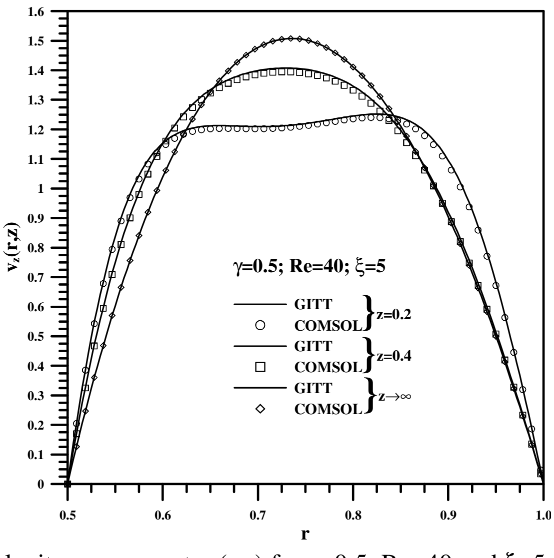

Figure 2. Developing of axial velocity component $v_z(r,z)$ for γ=0.5, Re=40 and ξ=5 at different axial positions.

Figures 3 and 4 present the development of the axial and tangential velocity components, respectively, at different axial positions. The Reynolds numbers employed were 40, 300 and 2000 with radii ratios 0.1 and 0.5, and the values of the rotation parameter ξ corresponding to1, 5 and 10.

Analyzing the influence of the rotation parameter, it was verified that as such parameter increases, the axial velocity component begins to lose its parabolic configuration and present a displacement of velocity profile near the inner cylinder for aspect ratios 0.1 and 0.5, maintaining fixed the Reynolds number. This is due to the fact that there is an increase of pressure drop, therefore causing a greater acceleration of fluid particles that are close to the inner cylinder that is rotating. However, when the aspect ratio becomes equal to 0.9, the velocity profile almost loses influence of rotation, and has a symmetry tendency in the velocity profile. This is explained by the fact that the geometry approaches to a parallel-plates channel when the radii ratio tends to 1. It is observed that as the radii ratio increases, for the same Reynolds number, the tangential velocity component reaches more quickly the fully developed flow region at the axial positions analyzed, where maximum values are verified at positions near the rotating inner cylinder, until to become close to value of the outer cylinder that is at rest. Regarding the effect of the rotation parameter, can be emphasized that the influence of the tangential velocity component becomes practically negligible.

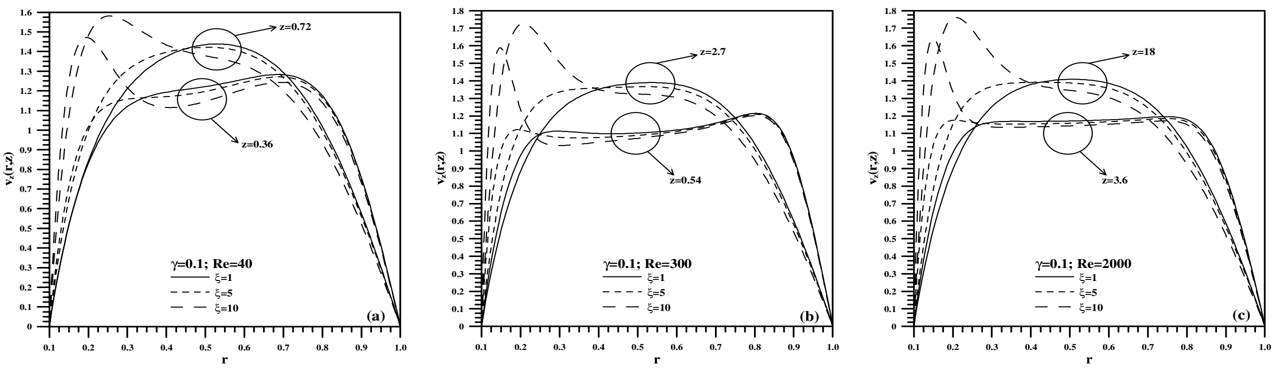

Figure 3. Developing of the axial velocity component $v_z(r,z)$ at various axial positions for γ=0.1: (a) Re=40; (b) Re=300 and (c) Re=2000.

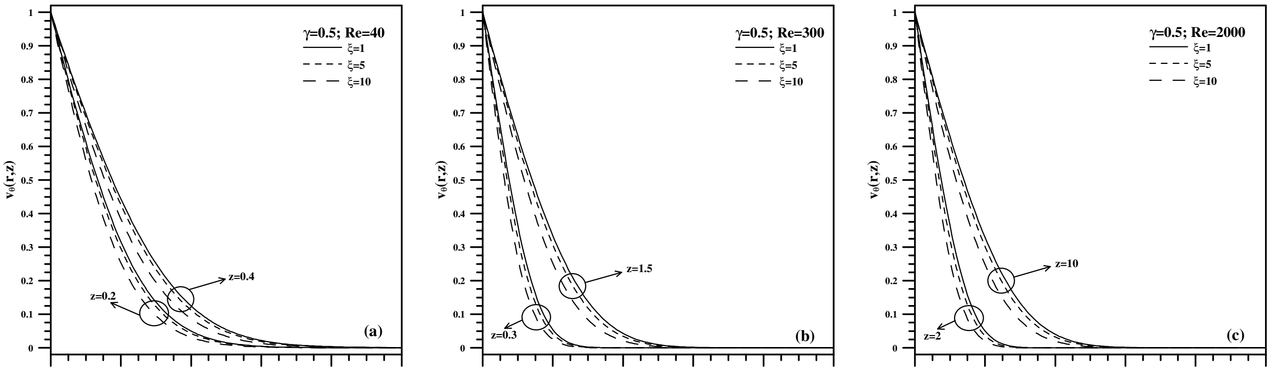

Figure 4. Developing of the tangential velocity component $v_\theta(r,z)$ at various axial positions for γ=0.5: (a) Re=40; (b) Re=300 and (c) Re=2000.

Figures 5 and 6 show the behavior of the product fRe along the duct lenght, for radii ratios equal to 0.1 and 0.9, and Reynolds number 40, 300 and 2000. The rotation parameter (ξ) corresponds to 1, 5 and 10. It was verified for the same Reynolds number, the value of the product fRe decreases as the radii ratio increases. When the aspect ratio is equal to 0.9, the value of the product fRe approaches the limiting case of the parallel-plates, where the product fRe tends to 24. The influence of the rotation parameter is practically negligible for the analyzed cases.

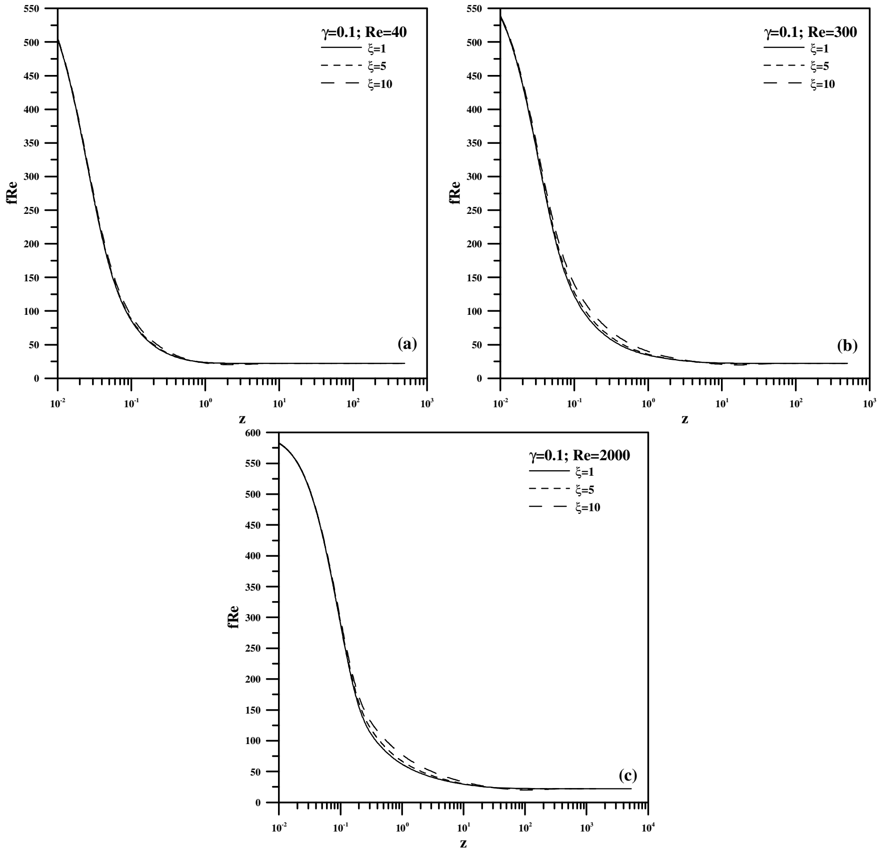

Figure 5. Product fRe along the duct length for γ=0.1: (a) Re=40; (b) Re=300 and (c) Re=2000.

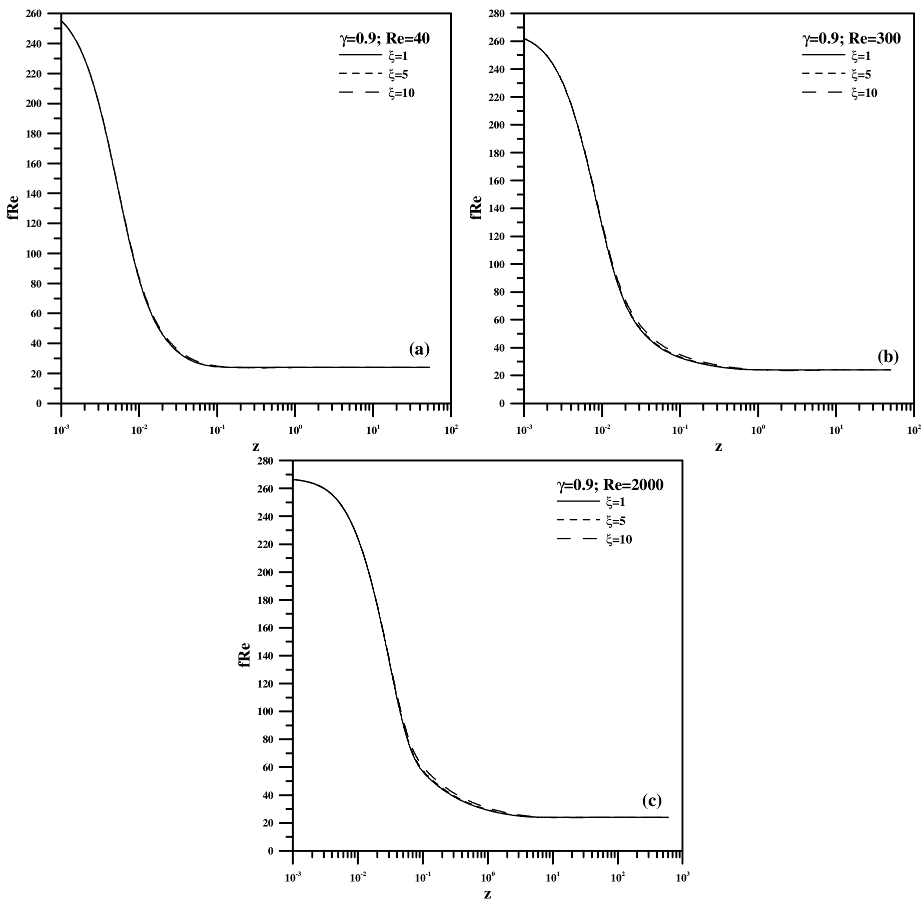

Figure 6. Product fRe along the duct length for γ=0.9: (a) Re=40; (b) Re=300 and (c) Re=2000.

4. CONCLUSIONS

The Generalized Integral Transform Technique (GITT) has been demonstrated in the hybrid numerical–analytical solution of laminar flow problems within annular ducts with rotation of the inner cylinder. The influence of parameters such as radii ratio, rotation parameter and Reynolds number in the development of velocity components along the duct, as well as, for the product fRe were analyzed. An excellent agreement of the present work with results of the Comsol Multiphysics™ (2006) shows the efficiency of integral transform approach in handling this type of problem.

5. REFERENCES

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6. RESPONSIBILITY NOTICE

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