HYBRID SOLUTION OF MULTIDIMENSIONAL PHASE CHANGE HEAT CONDUCTION PROBLEMS VIA INTEGRAL TRANSFORMS

Evaldiney Ribeiro Monteiro, monteiro@ufpa.br
João Nazareno Nonato Quaresma, quaresma@ufpa.br
School of Mining Engineering and the Environment, FEMMA, Campus de Marabá, UFPA
School of Chemical Engineering, Universidade Federal do Pará, FEQ/UFPA

Renato Machado Cotta, cotta@mecanica.coppe.ufrj.br
Laboratory of Transmission and Technology of Heat-LTTC
Mechanical Eng. Dept. - POLI & COPPE/UFRJ, Rio de Janeiro, RJ, Brasil

Abstract. *Heat conduction with phase change in more than one spatial dimension, also referered to in the literature as the Generalized Stefan problem, is addressed via the Generalized Integral Transform Thechnique (GITT) to yield a hybrid numerical-analytical solution to this class of nonlinear moving boundary formulations. The GITT is employed with a double transformation that results in a transformed ordinary differential system, which is numerically solved with an initial value problem solver with automatic errror control. A reordering scheme is adopted to rewrite the inversion formula as a single summation and select the most important modes in the eigenfunction expansion. A thorough convergence analysis is provided, to demonstrate the adequacy of the proposed methodology.*

Keywords*: Phase change, heat conduction, moving boundary problems, hybrid methods, integral transforms.*

1. INTRODUCTION

Diffusion problems with moving boundaries are present in different areas of science and engineering, including heat transfer problems with phase change such as in melting, solidification and ablation phenomena, generally coined as Stefan problems. Different authors studied this class of problems and obtained results by either analytical or numerical methods of solution for mostly one-dimensional formulations and only a few multidimensional situations.

One of the first applicable analytical studies in the multidimensional formulation of heat conduction with phase change that appears in the literature is due to Rasmussen (1977). Two approximate analytical approaches are also described in (Gupta and Kumar, 1986), who applied the integral method to a two-dimensional heat conduction problem with a moving boundary. Using the same methodology, Gupta and Banik (1990) and Kharab (2000) obtained additional results under a prescribed constant heat flux to the fixed surface boundary condition. Gammon and Howarth (1995) presented an analytical solution developed for the Stefan problem involving varying wall heat flux, for large Stefan number. More recently (Yigit, 2008) proposed an approximated analytical solution to the two-dimensional solidification problem employing a linear perturbation method, which is critically compared with a numerical solution. Such approximate analytical solutions, despite their simplicity, are quite limited in terms of accuracy and applicability as the problem formulation is further complicated. Several other works related to the generalized Stefan problem, but employing purely numerical methods, are presented by different authors. Numerical solutions with immobile contour using transformed system in curvilinear coordinates are presented in Furzeland (1977) and in Greydanos and Rasmussen (1989), which used an approximate numerical treatment of the solidification or melting of a solid so that the nonlinear effects could be included in the model. Gibou and Fedkiu (1995) developed a third order accurate scheme for the Stefan problem, based on the finite difference method. Gupta (2000) proposed a moving grid numerical scheme in conjunction with the method of lines to handle multidimensional solidification taking place over a temperature range. Some authors have also investigated the inverse Stefan problem, such as Ang et al. (1997) that studied the problem of melting ice by means of a two-dimensional regularization technique to determine the temperature at the boundary. Huang and Tsai (1998) presented the solution of the inverse moving boundary problem to determine the unknown transient temperature readings using the conjugate gradient method.

Therefore, it remains of interest to produce alternative approaches to the available time-consuming and rather approximate either analytical or purely numerical approaches, such as through hybrid numerical-analytical methodologies (Cotta, 1990; Cotta, 1993; Cotta & Mikhailov, 1997; Cotta, 1998; Cotta & Mikhailov, 2006). Diffusion problems with moving boundaries of a priori unknown position, such as in phase change problems, were for the first time handled through the Generalized Integral Transform Technique (GITT) by Diniz et al. (1990), in connection with one-dimensional ablation of thermal protection systems for space vehicles, then further extended in more recent contributions . In this context, the present work is aimed at developing a hybrid numerical-analytical solution for a two-dimensional Stefan problem, extending the ideas of the GITT. A double transformation is applied followed by the corresponding reordering scheme to account only for the most important modes in the inverse formula. A convergence analysis of the eigenfunction expansion solution is provided in order to demonstrate the consistency of the generated results.

2. ANALYSIS

The present analysis considers heat conduction with phase change as described in Fig. 1, using the energy and moving boundary heat balance equations, with thermally insulated fixed boundaries and phase change temperature at the moving boundary. Therefore, the mathematical formulation for this problem in dimensionless form is written as (Gupta & Kumar, 1986):

$$\frac{\partial\theta(X,Y,\tau)}{\partial\tau}=\frac{\partial^2\theta(X,Y,\tau)}{\partial X^2}+\frac{\partial^2\theta(X,Y,\tau)}{\partial Y^2},\quad 0<X<1,\quad 0<Y<S(X,\tau)\quad \tau>0 \tag{1}$$

$$\theta(X,Y,0)=F(X,Y),\quad 0\le X\le 1,\quad 0\le Y\le S(X,\tau) \tag{2}$$

$$\frac{\partial\theta(0,Y,\tau)}{\partial X}=0;\quad \frac{\partial\theta(1,Y,\tau)}{\partial X}=0,\quad 0\le Y\le S(X,\tau)\quad \tau>0 \tag{3,4}$$

$$\frac{\partial\theta(X,0,\tau)}{\partial Y}=0;\quad \theta(X,S(X,\tau),\tau)=0,\quad 0\le X\le 1\quad \tau>0 \tag{5,6}$$

In addition, the heat balance at the nonlinear moving boundary is given by:

$$\frac{\partial S(X,\tau)}{\partial\tau}+Ste\left[\left(\frac{\partial S(X,\tau)}{\partial X}\right)^2+1\right]\frac{\partial\theta(X,S(X,\tau),\tau)}{\partial Y}=0,\quad 0<X<1,\quad \tau>0 \tag{7}$$

$$S(X,0)=G(X),\quad 0\le X\le 1 \tag{8}$$

$$S(0,\tau)=S_0(\tau);\quad S(1,\tau)=S_1(\tau),\quad \tau>0 \tag{9,10}$$

$$\frac{dS_0(\tau)}{d\tau}+Ste\frac{\partial\theta(0,S_0(\tau),\tau)}{\partial Y}=0;\quad \frac{dS_1(\tau)}{d\tau}+Ste\frac{\partial\theta(1,S_1(\tau),\tau)}{\partial Y}=0\quad \tau>0 \tag{11,12}$$

$$S_0(0)=G(0);\quad S_1(0)=G(1) \tag{13,14}$$

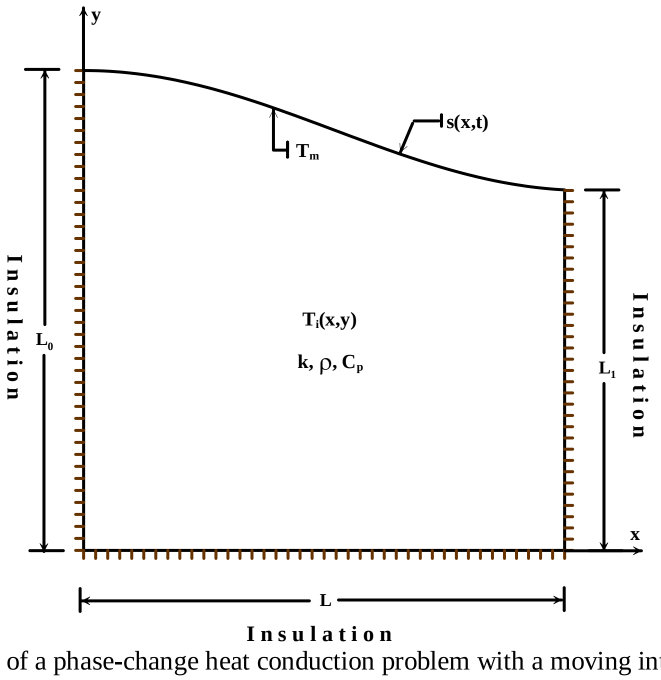

Figure 1. Description of a phase-change heat conduction problem with a moving interface as function of position and time.

The following dimensionless groups were employed in the above formulation:

$$\theta(X,Y,\tau)=\frac{T(x,y,t)-T_m}{\Delta T_c};\quad X=\frac{x}{L};\quad Y=\frac{y}{L};\quad \tau=\frac{\alpha_s t}{L^2};\quad S(X,\tau)=\frac{s(x,t)}{L};\quad Ste=\frac{C_p}{L_{fs}}\Delta T_c \tag{15-20}$$

where L is the domain length between the two insulated surfaces, α_s is the thermal diffusivity of the phase change material, C_p is the specific heat, L_{fs} is the phase change latent heat, Ste is the Stefan number and ΔT_c is a reference temperature difference.

The problem defined by Eqs. (7-10) presents non homogeneous boundary conditions for the moving boundary heat balance equation. Therefore, before applying the integral transformation process, it is advisable to filter the original problem in order to make the boundary conditions homogeneous, and thus speed up the convergence of the eigenfunction expansions. Here, a new and quite straightforward filter expression is proposed, given as:

$$S(X,\tau)=(1-X)S_0(\tau)+XS_1(\tau)+H(X,\tau) \tag{21}$$

where $S_0(\tau)$ and $S_1(\tau)$ are, respectively, the positions of the moving boundary at the borders X=0 and 1. The resulting homogeneous problem for the filtered surface position, $H(X,\tau)$, may then be integral transformed in the X direction, to yield an ODE system for the transformed boundary positions, to be numerically solved together with the transformed temperatures, as a function of time. After application of the filter, the nonlinear moving boundary problem is given by:

$$\frac{\partial H(X,\tau)}{\partial \tau}+(1-X)\frac{dS_0(\tau)}{d\tau}+X\frac{dS_1(\tau)}{d\tau}+Ste\left[\left(S_1(\tau)-S_0(\tau)+\frac{\partial H(X,\tau)}{\partial X}\right)^2+1\right]\frac{\partial \theta(X,S(X,\tau),\tau)}{\partial Y}=0, \tag{22}$$
$$0<X<1,\quad \tau>0$$

$$H(X,0)=G(X)-(1-X)S_0(0)-XS_1(0),\quad 0\le X\le 1 \tag{23}$$

$$H(0,\tau)=0;\quad H(1,\tau)=0,\quad \tau>0 \tag{24,25}$$

An analytical solution for the problem defined by Eqs. (1) to (4) and Eqs. (22) to (25) is unlikely to be obtainable, in light of the nonlinear nature of the moving boundary equation formulation. A hybrid analytical-numerical solution of the problem is then here developed by extending the ideas of the generalized integral transform technique (GITT) as described in (Cotta, 1990; Cotta, 1993; Cotta & Mikhailov, 1997; Cotta, 1998; Cotta & Mikhailov, 2006). First, appropriate auxiliary eigenvalue problems are selected to offer a basis for the eigenfunction expansion. Therefore, the following τ and X-dependent eigenvalue problem is proposed for the transformation in the Y direction, and the subsequent simpler eigenvalue problem for the transformation in the X direction:

$$\frac{\partial^2 \psi_i(X,Y,\tau)}{\partial Y^2}+\mu_i^2(X,\tau)\psi_i(X,Y,\tau)=0,\quad 0<X<1,\quad 0<Y<S(X,\tau)\quad \tau>0 \tag{26}$$

$$\frac{\partial \psi_i(X,0,\tau)}{\partial Y}=0;\quad \psi_i(X,S(X,\tau),\tau)=0,\quad 0<X<1\quad \tau>0 \tag{27,28}$$

$$\frac{d^2\Gamma_k(X)}{dX^2}+\lambda_k^2\Gamma_k(X)=0,\quad 0<X<1 \tag{29}$$

$$\frac{d\Gamma_k(0)}{dX}=0;\quad \frac{d\Gamma_k(1)}{dX}=0 \tag{30,31}$$

Equations (20) to (22) and (23) to (25) can be analytically solved to yield, respectively, the related eigenfunctions and eigenvalues as:

$$\psi_i(X,Y,\tau)=Cos\left[\beta_i\frac{Y}{S(X,\tau)}\right];\quad \beta_i=(2i-1)\frac{\pi}{2};\quad \mu_i(X,\tau)=\frac{\beta_i}{S(X,\tau)};\quad i=1,2,3... \tag{32-34}$$

$$\Gamma_k(X)=Cos[\lambda_k X];\quad \lambda_k=(k-1)\pi;\quad k=1,2,3... \tag{35,36}$$

It can be shown that the eigenfunctions $\psi_i(X,Y,\tau)$ and $\Gamma_k(X)$ enjoy the following orthogonality properties with their corresponding normalization integrals, $N_i(X,\tau)$ and M_k

$$\int_0^{S(X,\tau)} \psi_i(X,Y,\tau)\psi_j(X,Y,\tau)dY = \begin{cases} 0, & i \neq j \\ L_i(X,\tau) & i = j \end{cases}; \quad L_i(X,\tau) = L_i^* S(X,\tau); \quad L_i^* = \frac{1}{2} \tag{37-39}$$

$$\int_0^1 \Gamma_k(X)\Gamma_l(X)dX = \begin{cases} 0, & k \neq l \\ M_k & i = j \end{cases}; \quad M_k = \begin{cases} 1, & k = 1 \\ \frac{1}{2}, & k > 1 \end{cases} \tag{40,41}$$

Similarly, the eigenvalue problem for the transformation of the moving boundary heat balance equation, with its corresponding analytical solution, is given by:

$$\frac{d^2\Omega_m(X)}{dX^2} + \alpha_m^2 \Omega_m(X) = 0, \quad 0 < X < 1 \tag{42}$$

$$\Omega_m(0) = 0; \quad \Omega_m(1) = 0 \tag{43,44}$$

$$\Omega_m(X) = Sin(\alpha_m X); \quad \alpha_m = m\pi; \quad m = 1,2,3... \tag{45,46}$$

$$\int_0^1 \Omega_m(X)\Omega_n(X)dX = \begin{cases} 0, & m \neq n \\ N_m & m = n \end{cases}; \quad N_m = \frac{1}{2} \tag{47,48}$$

The above orthogonality properties allow the definition of the integral transform pairs for the temperature and the filtered moving boundary position:

$$\tilde{\tilde{\theta}}_{ik}(\tau) = \int_0^1 \int_0^{S(X,\tau)} \frac{\psi_i(X,Y,\tau)\Gamma_k(X)}{L_i(X,\tau)M_k} \theta(X,Y,\tau)dYdX, \qquad \text{transform} \tag{49}$$

$$\theta(X,Y,\tau) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \psi_i(X,Y,\tau)\Gamma_k(X)\tilde{\tilde{\theta}}_{ik}(\tau), \qquad \text{inverse} \tag{50}$$

$$\bar{H}_m(\tau) = \frac{1}{N_m}\int_0^1 \Omega_m(X)H(X,\tau)dX, \qquad \text{transform} \tag{51}$$

$$H(X,\tau) = \sum_{m=1}^{\infty} \Omega_m(X)\bar{H}_m(\tau), \qquad \text{inverse} \tag{52}$$

To obtain the transformed ordinary differential system for the transformed potentials $\tilde{\tilde{\theta}}_{ik}(\tau)$ and $\bar{H}_m(\tau)$, the partial differential equation (1) is multiplied by $\psi_i(X,Y,\tau)\Gamma_k(X)$, integrated over the domain $[0, S(X,\tau)]$ in the *Y* direction and the domain $[0,1]$ in the *X* direction, and the inverse formula, Eq. (50), is employed in place of the temperature $\theta(X,Y,\tau)$. The partial differential equation (22) is multiplied by $\Omega_m(X)/N_m$, integrated over the domain $[0,1]$ in the *X* direction, and the inverse formula, Eq. (52), is employed in place of the moving boundary position $H(X,\tau)$, resulting in the following system:

$$\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} A_{ijkl}(\tau)\frac{d\tilde{\tilde{\theta}}_{jl}(\tau)}{d\tau} + B_{ik}(\tau)\frac{dS_0(\tau)}{d\tau} + C_{ik}(\tau)\frac{dS_1(\tau)}{d\tau} + \sum_{m=1}^{\infty} D_{ikm}(\tau)\frac{d\bar{H}_m(\tau)}{d\tau} = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} E_{ijkl}(\tau)\tilde{\tilde{\theta}}_{jl}(\tau) \tag{53}$$

$$\sum_{n=1}^{\infty} F_{mn}(\tau)\frac{d\bar{H}_n(\tau)}{d\tau} + G_m(\tau)\frac{dS_0(\tau)}{d\tau} + F_m(\tau)\frac{dS_1(\tau)}{d\tau} = Ste\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} I_{ikm}(\tau)\tilde{\tilde{\theta}}_{ik}(\tau) \tag{54}$$

$$\frac{dS_0(\tau)}{d\tau} = \frac{Ste}{S_0(\tau)}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \beta_i Sin(\beta_i)\tilde{\tilde{\theta}}_{ik}(\tau), \qquad \frac{dS_1(\tau)}{d\tau} = \frac{Ste}{S_1(\tau)}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \beta_i Sin(\beta_i)Cos(\lambda_k)\tilde{\tilde{\theta}}_{ik}(\tau) \tag{55,56}$$

The same operations can be performed over the initial conditions given by Eqs. (2) and (23), to furnish

$$\tilde{\tilde{\theta}}_{ik}(0) = a_i b_k, \quad \bar{H}_m(0) = a_m \tag{57,58}$$

where,

$$A_{ijkl}(\tau)=\delta_{ij}\left\{\begin{array}{l}S_0^2(\tau)A1_{kl}+S_1^2(\tau)A2_{kl}+2S_0(\tau)S_1(\tau)A3_{kl}+\\ \sum_{m=1}^{\infty}\left[2S_0(\tau)A4_{klm}+2S_1(\tau)A5_{klm}+\sum_{n=1}^{\infty}A6_{klmn}\bar{H}_n(\tau)\right]\bar{H}_m(\tau)\end{array}\right\} \tag{59}$$

$$A1_{kl}=\int_0^1(1-X)^2\frac{\Gamma_k(X)\Gamma_l(X)}{M_k}dX\,,\qquad A2_{kl}=\int_0^1 X^2\frac{\Gamma_k(X)\Gamma_l(X)}{M_k}dX \tag{60,61}$$

$$A3_{kl}=\int_0^1 X(1-X)\frac{\Gamma_k(X)\Gamma_l(X)}{M_k}dX\,,\quad A4_{klm}=\int_0^1(1-X)\frac{\Gamma_k(X)\Gamma_l(X)}{M_k}\Omega_m(X)dX \tag{62,63}$$

$$A5_{klm}=\int_0^1 X\frac{\Gamma_k(X)\Gamma_l(X)}{M_k}\Omega_m(X)dX\,,\quad A6_{klmn}=\int_0^1\frac{\Gamma_k(X)\Gamma_l(X)}{M_k}\Omega_m(X)\Omega_n(X)dX \tag{64,65}$$

$$B_{ik}=-\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left\{B1_{ij}\left[S_0(\tau)B2_{kl}+S_1(\tau)B3_{kl}+\sum_{m=1}^{\infty}B4_{klm}\bar{H}_m(\tau)\right]\right\}\tilde{\bar{\theta}}_{jl}(\tau) \tag{66}$$

$$B1_{ij}=\int_0^1\eta\frac{\psi_i(\eta)}{L_i^*}\frac{d\psi_j(\eta)}{d\eta}d\eta\,,\qquad B2_{kl}=A1_{kl}\,,\quad B3_{kl}=A3_{kl}\,,\quad B4_{klm}=A4_{klm} \tag{67-68}$$

$$C_{ik}=-\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left\{C1_{ij}\left[S_0(\tau)C2_{kl}+S_1(\tau)C3_{kl}+\sum_{m=1}^{\infty}C4_{klm}\bar{H}_m(\tau)\right]\right\}\tilde{\bar{\theta}}_{jl}(\tau) \tag{69}$$

$$C1_{ij}=B1_{ij}\,,\qquad C2_{kl}=A3_{kl}\,,\quad C3_{kl}=A2_{kl}\,,\quad C4_{klm}=A5_{klm} \tag{70-73}$$

$$D_{ikm}=-\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left\{D1_{ij}\left[S_0(\tau)D2_{klm}+S_1(\tau)D3_{klm}+\sum_{n=1}^{\infty}D4_{klmn}\bar{H}_n(\tau)\right]\right\}\tilde{\bar{\theta}}_{jl}(\tau) \tag{74}$$

$$D1_{ij}=C1_{ij}\,,\qquad D2_{klm}=B4_{klm}\,,\quad D3_{klm}=C4_{klm}\,,\quad D4_{klmn}=A6_{klmn} \tag{75-78}$$

$$E_{ijkl}(\tau)=E1_{ij}E1t_{klmn}(\tau)+E8_{ij}E2t_{klmn}(\tau)-2E12_{ij}E3t_{klmn}(\tau)+E19_{ij}E4t_{klmn}(\tau)+E26_{ij}E27_{kl} \tag{79}$$

$$E1t_{klmn}(\tau)=2\left[S_1(\tau)-S_0(\tau)\right]^2E2_{kl}+\sum_{m=1}^{\infty}\left\{\begin{array}{l}4\left[S_1(\tau)-S_0(\tau)\right]E3_{klm}-S_0(\tau)E4_{klm}-\\ S_1(\tau)E5_{klm}+\sum_{n=1}^{\infty}\left[2E6_{klmn}-E7_{klmn}\right]\bar{H}_n(\tau)\end{array}\right\}\bar{H}_m(\tau) \tag{80}$$

$$E2t_{klmn}(\tau)=\left[S_1(\tau)-S_0(\tau)\right]^2E9_{kl}+\sum_{m=1}^{\infty}\left\{2\left[S_1(\tau)-S_0(\tau)\right]E10_{klm}+\sum_{n=1}^{\infty}E11_{klmn}\bar{H}_n(\tau)\right\}\bar{H}_m(\tau) \tag{81}$$

$$\begin{aligned}E3t_{klmn}(\tau)&=S_0(\tau)\left[S_1(\tau)-S_0(\tau)\right]E13_{kl}+S_1(\tau)\left[S_1(\tau)-S_0(\tau)\right]E14_{kl}+\\ &\sum_{m=1}^{\infty}\left\{2\left[S_1(\tau)-S_0(\tau)\right]E15_{klm}+S_0(\tau)E16_{klm}+S_1(\tau)E17_{klm}+\sum_{n=1}^{\infty}E18_{klmn}\bar{H}_n(\tau)\right\}\bar{H}_m(\tau)\end{aligned} \tag{82}$$

$$\begin{aligned}E3t_{klmn}(\tau)&=S_0(\tau)\left[S_1(\tau)-S_0(\tau)\right]E13_{kl}+S_1(\tau)\left[S_1(\tau)-S_0(\tau)\right]E14_{kl}+\\ &\sum_{m=1}^{\infty}\left\{2\left[S_1(\tau)-S_0(\tau)\right]E15_{klm}+S_0(\tau)E16_{klm}+S_1(\tau)E17_{klm}+\sum_{n=1}^{\infty}E18_{klmn}\bar{H}_n(\tau)\right\}\bar{H}_m(\tau)\end{aligned} \tag{83}$$

$$\begin{aligned}E4t_{klmn}(\tau)&=S_0^2(\tau)E20_{kl}+S_1^2(\tau)E21_{kl}+2S_0(\tau)S_1(\tau)E22_{kl}+\\ &\sum_{m=1}^{\infty}\left\{2\left[S_0(\tau)E23_{klm}+S_1(\tau)E24_{klm}\right]+\sum_{n=1}^{\infty}E25_{klmn}\bar{H}_n(\tau)\right\}\bar{H}_m(\tau)\end{aligned} \tag{84}$$

$$E1_{ij}=D1_{ij}\,,\qquad E2_{kl}=\delta_{kl}\,,\quad E3_{klm}=\int_0^1\frac{\Gamma_k(X)\Gamma_l(X)}{M_k}\frac{d\Omega_m(X)}{dX}dX\,,\quad E4_{klmn}=-\alpha_m^2A4_{klm} \tag{85-88}$$

$$E5_{klmn} = -\alpha_m^2 A5_{klm}, \quad E6_{klmn} = \int_0^1 \frac{\Gamma_k(X)\Gamma_l(X)}{M_k} \frac{d\Omega_m(X)}{dX} \frac{d\Omega_n(X)}{dX} dX, \quad E7_{klmn} = -\alpha_m^2 A6_{klmn} \tag{89-91}$$

$$E8_{ij} = \int_0^1 \eta^2 \frac{\psi_i(\eta)}{L_i^*} \frac{d^2\psi_j(\eta)}{d\eta^2} d\eta, \quad E9_{kl} = \delta_{kl}, \quad E10_{klm} = E3_{klm}, \quad E11_{klmn} = E6_{klmn} \tag{72-75}$$

$$E12_{ij} = E1_{ij}, \quad E13_{kl} = \int_0^1 (1-X) \frac{\Gamma_k(X)}{M_k} \frac{d\Gamma_l(X)}{dX} dX, \quad E14_{kl} = \int_0^1 X \frac{\Gamma_k(X)}{M_k} \frac{d\Gamma_l(X)}{dX} dX \tag{92-94}$$

$$E15_{klm} = \int_0^1 \frac{\Gamma_k(X)}{M_k} \frac{d\Gamma_l(X)}{dX} \Omega_m(X) dX, \quad E16_{klm} = \int_0^1 (1-X) \frac{\Gamma_k(X)}{M_k} \frac{d\Gamma_l(X)}{dX} \frac{d\Omega_m(X)}{dX} dX \tag{95,96}$$

$$E17_{klm} = \int_0^1 X \frac{\Gamma_k(X)}{M_k} \frac{d\Gamma_l(X)}{dX} \frac{d\Omega_m(X)}{dX} dX, \quad E18_{klmn} = \int_0^1 \frac{\Gamma_k(X)}{M_k} \frac{d\Gamma_l(X)}{dX} \frac{d\Omega_m(X)}{dX} \Omega_n(X) dX \tag{97,98}$$

$$E19_{ij} = \delta_{ij}, \quad E20_{kl} = -\lambda_l^2 A1_{kl}, \quad E21_{kl} = -\lambda_l^2 A2_{kl}, \quad E22_{kl} = -\lambda_l^2 A3_{kl}, \quad E23_{klm} = -\lambda_l^2 A4_{klm} \tag{99-103}$$

$$E24_{klm} = -\lambda_l^2 A5_{klm}, \quad E25_{klmn} = -\lambda_l^2 A6_{klmn}, \quad E26_{ij} = -\delta_{ij}\beta_j^2, \quad E27_{kl} = \delta_{kl} \tag{104-107}$$

$$F_{mn}(\tau) = S_0(\tau)F1_{mn} + S_1(\tau)F2_{mn} + \sum_{p=1}^{\infty} F3_{mnp} \bar{H}_p(\tau) \tag{108}$$

$$G_m(\tau) = S_0(\tau)G1_m + S_1(\tau)G2_m + \sum_{n=1}^{\infty} G3_{mn} \bar{H}_n(\tau) \tag{109}$$

$$H_m(\tau) = S_0(\tau)H1_m + S_1(\tau)H2_m + \sum_{n=1}^{\infty} H3_{mn} \bar{H}_n(\tau) \tag{110}$$

$$I_{ikm}(\tau) = I1_i \left\{ \left[1 + \left(S_1(\tau) - S_0(\tau)\right)^2\right] I2_{mk} + \sum_{n=1}^{\infty} \left[2\left(S_1(\tau) - S_0(\tau)\right) I3_{mkn} + \sum_{p=1}^{\infty} I4_{mknp} \bar{H}_p(\tau) \right] \bar{H}_n(\tau) \right\} \tag{111}$$

$$F1_{mn} = \int_0^1 (1-X) \frac{\Omega_m(X)\Omega_n(X)}{N_m} dX, \quad F2_{mn} = \int_0^1 X \frac{\Omega_m(X)\Omega_n(X)}{N_m} dX \tag{112,113}$$

$$F3_{mnp} = \int_0^1 \frac{\Omega_m(X)\Omega_n(X)}{N_m} \Omega_p(X) dX, \quad G1_m = \int_0^1 (1-X)^2 \frac{\Omega_m(X)}{N_m} dX \tag{114,115}$$

$$G2_m = \int_0^1 X(1-X) \frac{\Omega_m(X)}{N_m} dX, \quad G3_{mn} = F1_{mn}, \quad H2_m = \int_0^1 X^2 \frac{\Omega_m(X)}{N_m} dX \tag{116-118}$$

$$H3_{mn} = F2_{mn}, \quad I1_i = \beta_i (-1)^{i+1}, \quad I2_{mk} = \int_0^1 \frac{\Omega_m(X)}{N_m} \Gamma_k(X) dX \tag{119-121}$$

$$I3_{mkn} = \int_0^1 \frac{\Omega_m(X)}{N_m} \Gamma_k(X) \frac{d\Omega_n(X)}{dX} dX, \quad I4_{mknp} = \int_0^1 \frac{\Omega_m(X)}{N_m} \Gamma_k(X) \frac{d\Omega_n(X)}{dX} \frac{d\Omega_p(X)}{dX} dX \tag{122,123}$$

$$a_i = \int_0^1 (1-\eta) \frac{\psi_i(\eta)}{L_i^*} d\eta, \quad b_k = \int_0^1 \frac{\Gamma_k(X)}{M_k} dX, \quad a_m = \int_0^1 \frac{\Omega_m(X)}{N_m} H(X,0) dX \tag{124-126}$$

3. RESULTS AND DISCUSSION

Numerical results for the moving boundary position and temperature field were obtained from a code developed in the FORTRAN 90 programming language. The code was implemented on a PENTIUM DUAL CORE 1.73 GHz microcomputer, and the system given by Eqs. (53) to (58) was handled through the subroutine DIVPAG from the IMSL Library (1991). Before numerically solving the transformed system, it is most appropriate to implement a reordering scheme (Correa et al., 1997; Cotta & Mikhailov, 1997) so as to pre-select the most relevant contributing modes of the expansion to the final converged solution. In this sense, the double summations in Eqs. (53-56) are rewritten as single sums, as well as the inverse formula Eq.(50), and computational costs are markedly reduced, since only the most relevant components of the expansions are being in fact numerically evaluated. The hybrid solution was then computed using up to forty five terms (N = 45) in the reordered eigenfunction expansions, and results were obtained for two different initial condition functions of the moving boundary as presented in (Gupta and Kumar, 1986). These two cases were here named Cases A and B, corresponding to the initial positions $S(X,0)=0.5-0.2\text{Cos}(\pi X/2)$ and $S(X,0)=0.5-0.2\text{Cos}(\pi X)$.

Figures 2.a, b illustrate the convergence behavior of the present results for the moving boundary position evolution, at the two insulated borders, X=0 and 1, for increasing values of the truncation order, from N=5 up to 45. In both cases convergence to the graph scale is achieved at around N=35 terms, as we observe from the merging of the results for N=35, 30 and 45. Also, convergence appears to be slightly slower towards the steady-state solution.

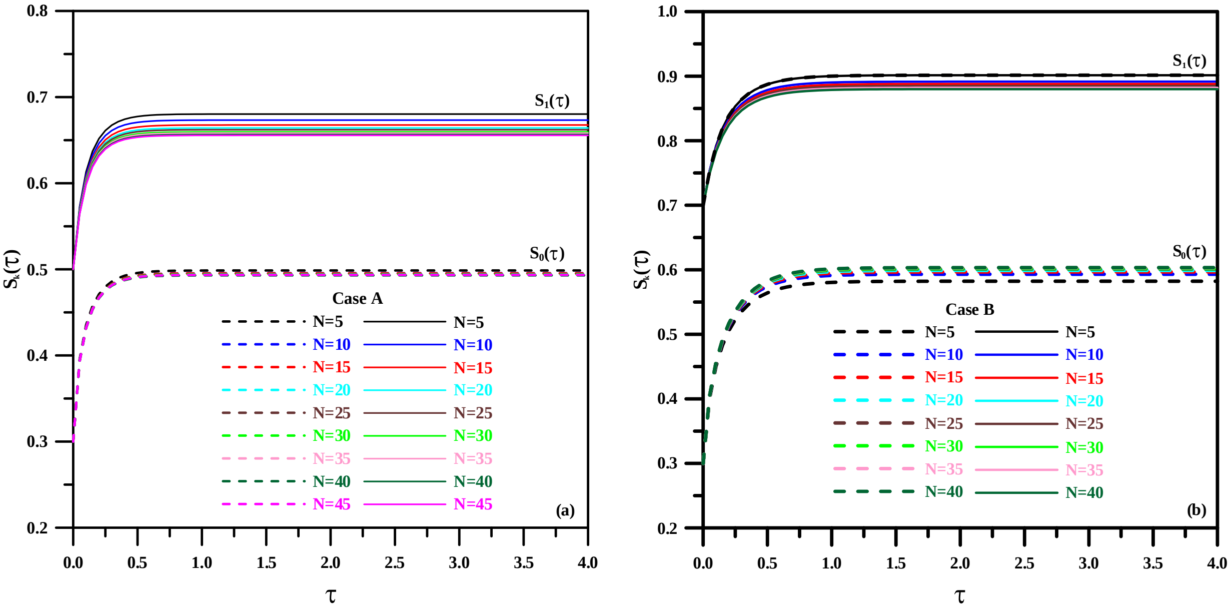

Figure 2. Convergence behavior of the moving boundary evolution at X = 0 and X = 1: a) Case A. b) Case B.

Figures 3.a,b illustrate both the behavior of the moving boundary profile and the convergence of this variable for truncation orders from N=5 up to 40. Again, we may observe the achievement of a steady-state situation for a dimensionless time between the two last sets of curves, at τ=0.4 to 4.0, and the requirement of a slightly higher number of terms as the solution progresses to larger times.

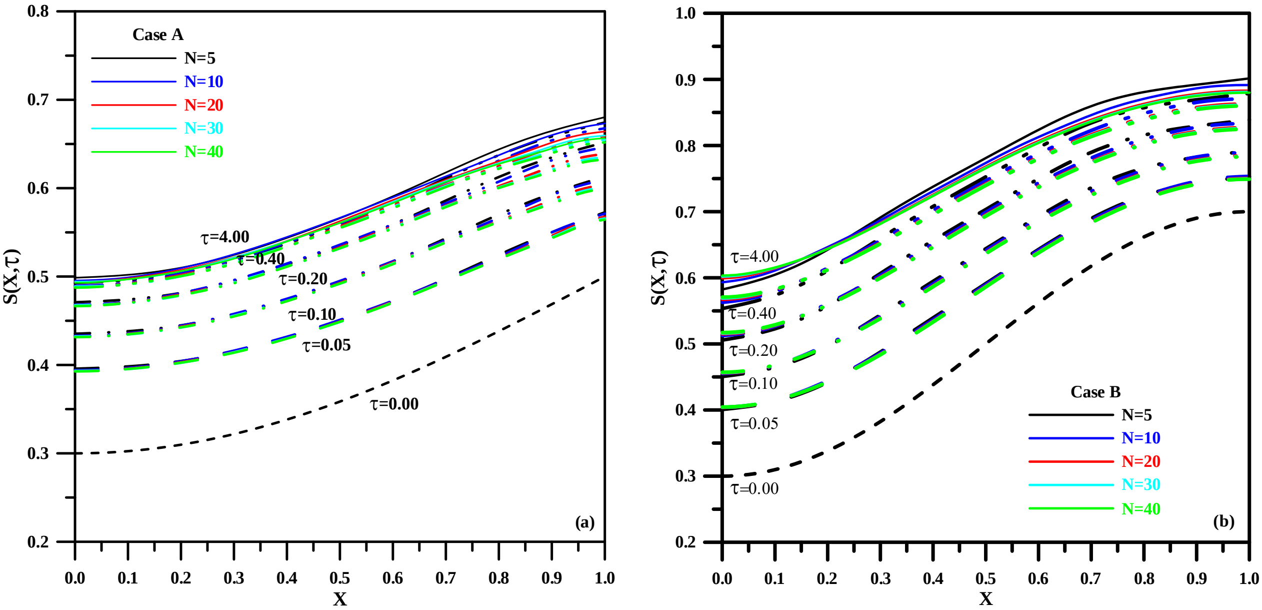

Figure 3. Convergence behavior of the moving boundary profile at different times: a) Case A. b) Case B.

Figures 4.a,b show the time evolution of the dimensionless temperature at the two bordering positions, X=0 and 1, at the bottom plane in Fig.1, Y=0, for cases A and B, respectively. As we can see, all four temperature evolutions show an excellent convergence behavior, with full agreement to the graph scale even at truncation orders as low as N=10. In both cases A and B, we observe that the temperatures at the borders X=0 (dashed lines) experience a faster transient, as expected, since the initially smaller width of the medium at this position provides larger temperature gradients and higher heat transfer rates. Also, case A has overall the smaller initial widths of the heat exchanging medium, thus resulting in shorter transients than case B. In this case B we also have a larger difference between the initial positions of the boundary at X=0 and 1, which results in the more marked difference between the evolution of the temperatures at these positions.

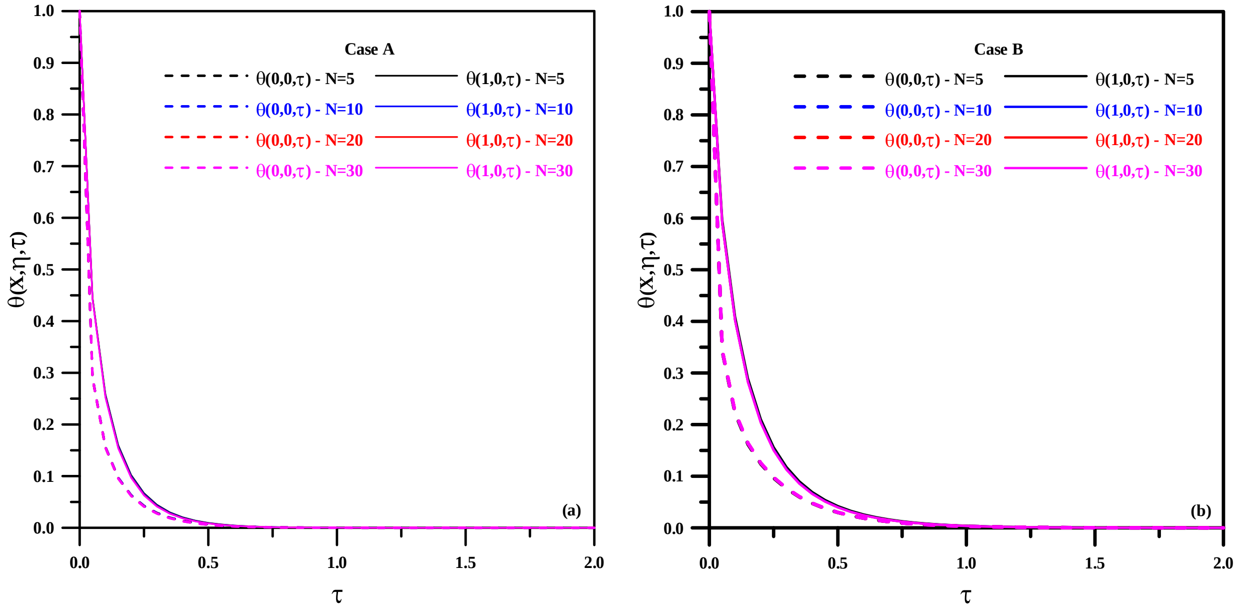

Figure 4. Convergence behavior of the temperature evolution at the borders (X=0 and X=1) of the base (Y=0): a) Case A. b) Case B.

Figures 5.a,b present the temperature distributions along the horizontal direction, X, again at the base of the medium (Y=0) for different values of time, considering the two cases of initial position of the moving boundary, respectively. One may observe that the temperature distributions depart from a uniform initial profile, then experiencing within the initial transient stage a marked variation, evolving towards again an uniform profile for large times. The temperatures convergence behavior is here also illustrated and noticed to be indeed excellent throughout the domain.

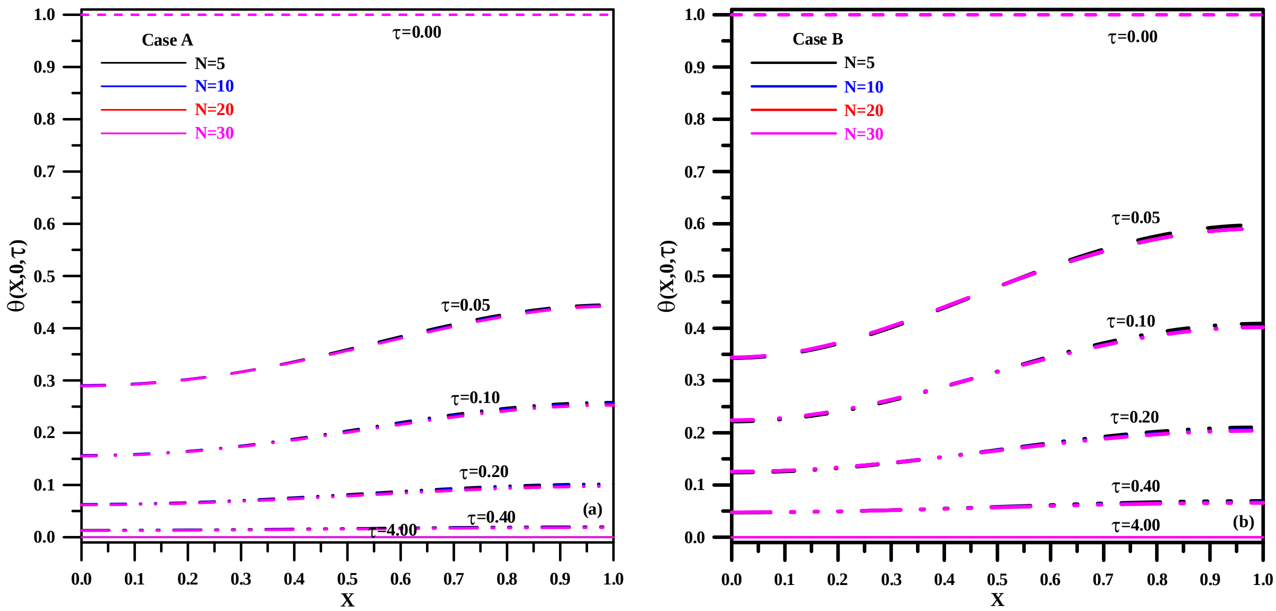

Figure 5. Convergence behavior of the temperature profiles at the base (Y=0) for different times: a) Case A. b) Case B.

Figures 6.a,b illustrate the temperature profiles in terms of the normalized vertical direction $\eta = Y/S(X,\tau)$, along the insulated border X=0 for different times. One may observe the progressive cooling of the medium, from top (η=1) to bottom (η=0), departing from an initial linear condition. Again, the medium with overall larger widths presents a longer transient period, noticeable even at this boundary where the initial positions are the same for the two cases A and B.

Figures 7.a,b illustrate the same type of temperature distributions and evolutions, but for the opposing boundary at X=1. Since the vertical lengths of both cases are larger at this other border, the two graphs show longer transients in comparison to the corresponding previous figures.

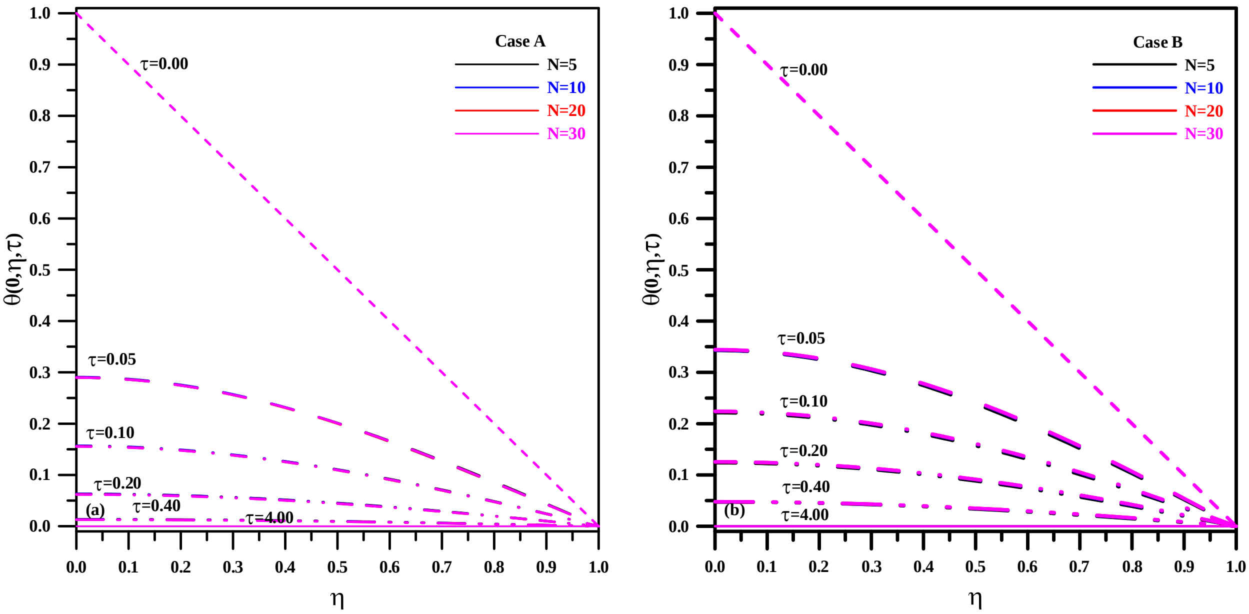

Figure 6. Convergence behavior of the temperature profiles at the border (X=0) along normalized vertical direction $\eta=Y/S(X,\tau)$: a) Case A. b) Case B.

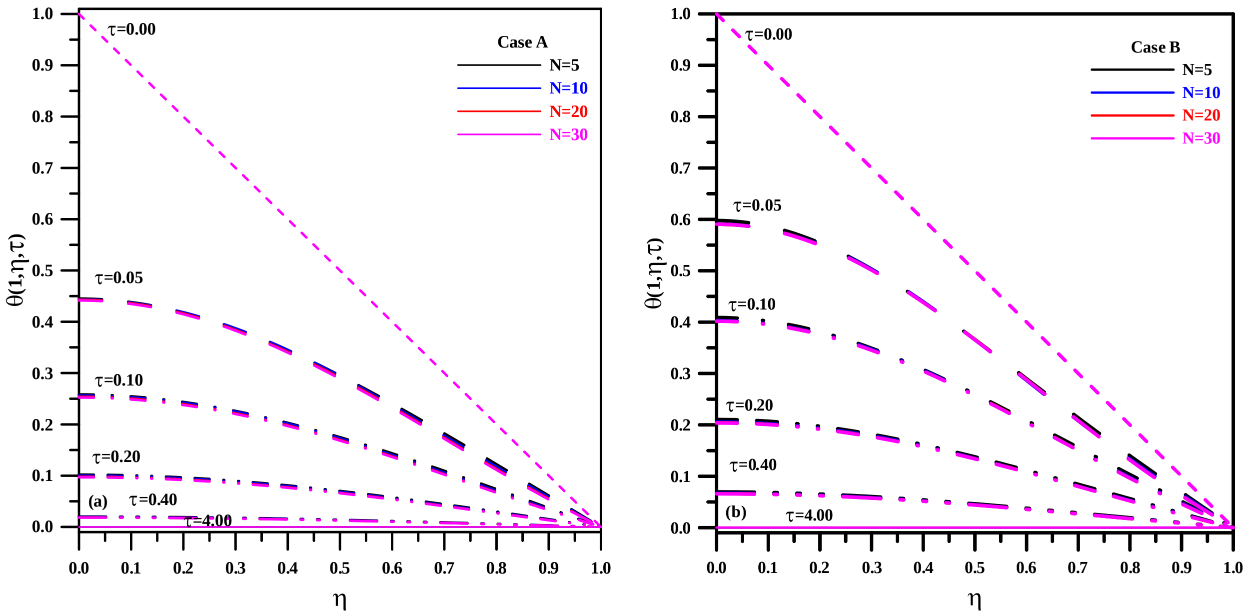

Figure 7. Convergence behavior of the temperature profiles at the border (X=1) along normalized vertical direction $\eta=Y/S(X,\tau)$: a) Case A. b) Case B.

4. CONCLUSIONS

A hybrid numerical-analytical treatment of the two-dimensional Stefan problem is advanced, by extending the ideas in the Generalized Integral Transform Technique (GITT). A double integral transformation of the temperature is proposed based on an eigenvalue problem that incorporates the information on the moving boundary. A reordering scheme is organized to reduce the double summations into single ones, thus significantly reducing computational costs in the numerical solution of the transformed differential system. The moving boundary position itself is expanded in eigenfunctions, preceded by application of a filtering solution that extracts the information at the two borders, thus yielding a homogeneous eigenvalue problem for the boundary heat balance equation. An extensive convergence analysis is undertaken, for both the moving boundary position and the temperature eigenfunction expansions, demonstrating the adequacy and robustness of the proposed hybrid solution path.

5. ACKNOWLEDGEMENTS

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