COMPARISONS BETWEEN GITT AND FVM SOLUTIONS FOR THERMALLY DEVELOPING FLOW IN A RECTANGULAR DUCT

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Abstract. *Both traditional discretization-based numerical methods and alternative hybrid analytical-numerical techniques have been successfully applied for solving convective heat transfer problems. Despite the number of studies dedicated to separately solving a given problem by one methodology or the other, there are very few studies dedicated to comparing the computational solution performance of these approaches. In this context, this paper presents a comparison of Finite Volumes Method (FVM) and Generalized Integral Transform Technique (GITT) solutions for a three dimensional steady-state convective heat transfer problem. The selected problem is that of thermally developing laminar flow within a rectangular cross-section duct. The flow is considered kinetically developed and a constant wall temperature condition is employed. Both solutions are computationally implemented using the Mathematica system and, in order to guarantee a fair comparison, both implementations employ the same numerical ordinary differential equation (ODE) integrator to handle the solution in the flow direction. The comparisons are made by observing the convergence behavior of the Nusselt number for different positions along the flow direction. The convergence of the velocity profile at different positions as well as the average velocity is also examined. The results provide an indication of cases and regions in which one methodology can give better results than the other.*

Keywords: *Integral Transform Technique, Finite Volumes Method, Thermally Developing Flow, Rectangular Duct*

1. NOMENCLATURE

x, y, z	Coordinates	**Greek Symbols**	
a, b	Cross-section dimensions	θ_m	Dimensionless mean stream temperature
u^*	Dimensionless velocity	$\overline{\theta}$	Transformed dimensionless temperature
$\overline{u}$	Average velocity	ξ, η, φ	Dimensionless coordinates
D_h	Hydraulic diameter	μ, λ, γ	Eigenvalues
K_0, K_1, K_2	Aspect ratios	Ψ, X, Y	Eigenfunctions
H	Dimensionless pressure	**Subscripts**	
T	Temperature	n, i, j, r, s, l, p	GITT indexes
A, B, F, G, Q, W	Coefficients	i, j	FVM indexes
N	Norm of eigenfunctions	$i_{\max}$, $i_{\max}$	Grid divisions for FVM
Nu	Nusselt number	$k_{\max}$	Total number of FVM cells
		$l_{\max}$	Truncation order for GITT

2. INTRODUCTION

For a long time, analytical techniques were the only available solution methods for diffusion and convection-diffusion problems, those of which could only be applied to a narrow class of mostly linear problems. Discrete methods also originated a long time ago and hybrid schemes, such as the Integral Transform approaches, are more recent methods that have been successfully applied to convection-diffusion problems.

The Generalized Integral Transform Technique (GITT) (Cotta, 1993) deals with expansions of the sought solution in terms of bases of infinite orthogonal eigenfunctions, maintaining the solution process always within a continuum domain. Unlike Classic Integral Transform (Mikhailov and Özişik, 1984), the transformation of the original problem results in a coupled system, making the method applicable to a virtually infinite numbers of problems. The resulting system is generally composed by a group of differential equations, that can be easily solved by well established numerical routines that allow precision control. Nevertheless, as the infinite series must be truncated so that any application can be made, a truncation error is introduced.

The Finite Volume Method (FVM) (Maliska, 1995) appears as widely used option to a variety of convection-diffusion problems, due to its conservative nature and ease of application. However, as any discrete method, approximations of integrals and derivatives in terms of nodal points on a computational domain are necessary, resulting in a solution error that decays with grid refinement.

Interesting applications of the GITT include a variety of convection-diffusion problems. For heat transfer in internal forced convection different investigations were carried out employing the GITT. Among the recent advancements for these type of problems, one should mention (Macêdo, Maneschy et al., 2000; Nacimento, Quaresma et al., 2002, 2006), which deals with non-Newtonian flows in circular shaped ducts, (Maia, Aparecido et al., 2006), which presents a solution for non-Newtonian flows in elliptical cross-section ducts, and Lima, Quaresma et al. (2007), which investigates the MHD flow and heat transfer within parallel-plates channels. For flow in ducts of arbitrary geometry, some particular solutions have been developed by Aparecido and Cotta (1990); Ding and Manglik (1996); Barbuto and Cotta (1997); Guerrero, Quaresma et al. (2000); nonetheless, a general methodology was described in (Sphaier and Cotta, 2000, 2002), being potentially promising for these types of geometries.

Both discrete and spectral approaches have been demonstrated to be effective methodologies to solve convective-diffusion problems, but there is lack of studies comparing them. Some previous investigations (Chalhub, Dias et al., 2008; Chalhub and Sphaier, 2009; de Queiroz, Nogueira et al., 2009; Nogueira and Sphaier, 2009) already compared the GITT with the FVM showing cases in which each methodology could be more adequately employed. In spite of the relevance of these studies, the analyzed problems presented dependence on only two spatial variables. This work is focused on comparing the computational performance of the FVM and the GITT for a three-dimensional problem.

3. PROBLEM FORMULATION

The studied problem is that of heat transfer in steady incompressible laminar flow within a rectangular duct. The flow is considered kinetically (or hydrodynamically) developed, but thermally developing.

The problem for the velocity field is given by the following dimensionless equation

$$\frac{\partial^2 u^*}{\partial \xi^2} + K_0^2 \frac{\partial^2 u^*}{\partial \eta^2} = H, \qquad \text{in} \qquad 0 \le \eta \le 1, \quad 0 \le \xi \le 1 \tag{1}$$

and the the following boundary conditions:

$$u^*(1,\eta) = 0, \qquad \left(\frac{\partial u^*}{\partial \xi}\right)_{\xi=0} = 0, \qquad \text{in} \qquad 0 \le \eta \le 1, \tag{2}$$

$$u^*(\xi,1) = 0, \qquad \left(\frac{\partial u^*}{\partial \eta}\right)_{\eta=0} = 0 \qquad \text{in} \qquad 0 \le \xi \le 1, \tag{3}$$

where the value of H is calculated to ensure that u^* is normalized with the cross-section average velocity, which can be obtained by the following relation:

$$\overline{u} = \int_0^1 \int_0^1 u(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \, = \, 1 \tag{4}$$

The dimensionless energy equation, neglecting the effects of heating due to viscous dissipation, assuming constant properties and high Peclet numbers, along with the associated boundary conditions are shown bellow:

$$u^* \frac{\partial \theta}{\partial \varphi} = K_1^2 \frac{\partial^2 \theta}{\partial \xi^2} + K_2^2 \frac{\partial^2 \theta}{\partial \eta^2} \qquad \text{in} \qquad 0 \le \eta \le 1, \quad 0 \le \xi \le 1, \quad \varphi \ge 0 \tag{5}$$

$$\theta(1,\eta,\varphi) = 0, \qquad \left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=0} = 0, \qquad \text{in} \qquad 0 \le \eta \le 1, \quad \varphi \ge 0 \tag{6}$$

$$\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0} = 0 \qquad \theta(\xi,1,\varphi) = 0, \qquad \text{in} \qquad 0 \le \xi \le 1, \quad \varphi \ge 0 \tag{7}$$

$$\theta(\xi,\eta,0) = 1, \qquad \text{in} \qquad 0 \le \eta \le 1, \quad 0 \le \xi \le 1 \tag{8}$$

The dimensionless variables and parameters for this problem are defined by:

$$\xi = \frac{x}{a}, \qquad \eta = \frac{y}{b}, \qquad \varphi = \frac{\alpha\, z}{D_h^2\, \overline{u}}, \qquad \theta = \frac{T - T_{\text{wall}}}{T_{\text{in}} - T_{\text{wall}}} \tag{9}$$

$$K_0 = \frac{a}{b}, \qquad K_1 = \frac{D_h}{a}, \qquad K_2 = \frac{D_h}{b}, \tag{10}$$

where a and b are the dimensions of the duct walls in the x and y directions, respectively, and D_h is the hydraulic diameter.

With the adopted dimensionless variables, the Nusselt number based on the hydraulic diameter is given by:

$$\text{Nu}(\varphi) = \frac{h\, D_h}{k} = \left(\frac{-K_1 K_2}{K_1 + K_2}\right) \frac{K_0^{-1} \int_0^1 \left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=1} \mathrm{d}\eta \, + \, K_0 \int_0^1 \left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1} \mathrm{d}\xi}{\theta_m} \tag{11}$$

where the value of h is averaged over the perimeter of the cross section such that $h = h(\varphi)$.

The dimensionless mean stream temperature, necessary for calculating the Nusselt number, is obtained from:

$$\theta_m(\varphi) = \int_0^1 \int_0^1 u^* \theta \,\mathrm{d}\xi \,\mathrm{d}\eta \tag{12}$$

4. INTEGRAL TRANSFORM SOLUTION

Since the velocity profile is fully-developed, a fully analytical solution using expansions of the velocity solution in terms of a basis of infinite orthogonal eigenfunctions can be employed, resulting in the following profile:

$$u^* = \frac{H\xi^2}{2} - \frac{H}{2} + \sum_{n=1}^{\infty} A_n \Psi_n(\xi) \cosh\left(\frac{\mu_n \eta}{K_0}\right) \tag{13}$$

where the eigenfunctions (Ψ_n) and eigenvalues (μ_n) are given by:

$$\Psi_n(\xi) = \cos(\mu_n \xi) \tag{14}$$

$$\mu_n = \pi\,(n - 1/2) \qquad \text{where} \qquad n = 1, 2, 3, \ldots \tag{15}$$

and the constants A_n and the norms N_n are given by:

$$A_n = -\frac{\int_0^1 \left(\frac{H\xi^2}{2} - \frac{H}{2}\right)\cos(\mu_n \xi)\,\mathrm{d}\xi}{N_n \cosh\left(\frac{\mu_n}{K_0}\right)}; \qquad N_n = \int_0^1 \Psi_n^2 \,\mathrm{d}\xi = \frac{1}{2} \tag{16}$$

The integral transform solution of the thermal problem is accomplished by employing the Generalized Integral Transform Technique (Cotta, 1993). The solution process begins with the definition of the transformation pair:

$$\text{Transform} \quad \Rightarrow \quad \overline{\theta}_{ir}(\varphi) = \int_0^1 \int_0^1 \theta(\xi, \eta, \varphi)\, X_r(\xi)\, Y_i(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{17}$$

$$\text{Inversion} \quad \Rightarrow \quad \theta(\xi, \eta, \varphi) = \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \frac{\overline{\theta}_{ir}(\varphi)\, X_r(\xi)\, Y_i(\eta)}{N_r\, N_i} \tag{18}$$

where X_r's and Y_i's are the orthogonal solutions of Sturm-Liouville eigenvalue problems that, for this problem, are selected to be:

$$X_r''(\xi) + \gamma_r^2\, X_r(\xi) = 0, \qquad \text{in} \qquad 0 \le \xi \le 1, \tag{19}$$

$$X_r'(0) = 0, \qquad X_r(1) = 0, \tag{20}$$

and

$$Y_i''(\eta) + \lambda_i^2\, Y_i(\eta) = 0, \qquad \text{in} \qquad 0 \le \eta \le 1, \tag{21}$$

$$Y_i'(0) = 0, \qquad Y_i(1) = 0, \tag{22}$$

These eigenvalue problems lead to infinite nontrivial solutions as shown bellow:

$$X_r(\xi) = \cos(\gamma_r\, \xi), \qquad \gamma_r = \pi\,(r - 1/2)\,, \qquad \text{for} \qquad r = 1, 2, 3, \ldots \tag{23}$$

$$Y_i(\eta) = \cos(\lambda_i\, \eta), \qquad \lambda_i = \pi\,(i - 1/2)\,, \qquad \text{for} \qquad i = 1, 2, 3, \ldots \tag{24}$$

The norms of the eigenfunctions $X_r(\xi)$ and $Y_i(\eta)$ are given by:

$$N_r = \int_0^1 X_r^2(\xi)\,\mathrm{d}\xi = \frac{1}{2}; \qquad N_i = \int_0^1 Y_i^2(\eta)\,\mathrm{d}\eta = \frac{1}{2} \tag{25}$$

The problem is transformed using the transformation operator defined in eq. (17). Almost all the terms of the equation can be directly transformed and for the non-transformable terms the inversion formula (18) is applied.

$$\sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \left(B_{jsir}\, \overline{\theta}_{ir}'(\varphi) \right) - F_{js}\, \overline{\theta}_{js}(\varphi) = 0 \tag{26}$$

The transformed equation consists of a system of coupled first order ordinary differential equations and a set of boundary conditions specified only at the channel entrance ($\varphi = 0$), which are obtained by the transformation of equation (8).

$$\overline{\theta}_{ir}(0) = \int_0^1 \int_0^1 X_r \, Y_i \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{27}$$

The coefficients in the transformed system are given by:

$$B_{jsir} = \int_0^1 \int_0^1 \frac{u^* \, Y_i \, X_r \, Y_j \, X_s}{N_r \, N_i} \, \mathrm{d}\xi \, \mathrm{d}\eta; \qquad F_{js} = -K_1^2 \, \gamma_s^2 - K_2^2 \, \lambda_j^2 \tag{28}$$

Once system (26)-(28) is solved, the dimensionless temperature $\theta(\xi, \eta, \varphi)$ can be readily calculated from the inversion formula (18). The solution of this coupled ODE system can be easily accomplished computationally; however, it is necessary that the infinite summation representation be truncated to a finite order. Since a double summation representation is present, the truncation of the infinite representation must be preceded by a reordering scheme in order to ensure an efficient convergence behavior. The reordering scheme consists on mapping combinations of i and r pairs into a single index, l, and similarly to j and s, as described below:

$$l \longleftrightarrow (i, r) \qquad \text{and} \qquad p \longleftrightarrow (j, s) \tag{29}$$

where the reordering associations are chosen so that the larger terms (in magnitude) are summed first.

With the reordering procedure, the transformed problem becomes:

$$\sum_{l=1}^{\infty} \overline{\theta}_l' \, B_{lp} - \overline{\theta}_p \, F_p = 0; \qquad \overline{\theta}_l(0) = G_l(\xi, \eta) \tag{30}$$

where:

$$G_l(\xi, \eta) = \int_0^1 \int_0^1 X_r(\xi) \, Y_i(\eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \tag{31}$$

Alternatively, the reordered system can be written in matrix form:

$$\boldsymbol{B} \, \overline{\boldsymbol{\theta}}' - \boldsymbol{F} \, \overline{\boldsymbol{\theta}} = 0; \qquad \overline{\boldsymbol{\theta}}(0) = \boldsymbol{G} \tag{32}$$

where $\boldsymbol{B}$ represents the B matrix, $\boldsymbol{G}$ represents the G constant matrix and $\boldsymbol{F}$ represents the diagonal matrix containing the vector F_p, in other words:

$$F_{lp} = F_p \, \delta_{lp} \tag{33}$$

Using the definition of inverse matrix we arrive at a simplified explicit equation, as shown bellow.

$$\overline{\boldsymbol{\theta}}' = \boldsymbol{B}^{-1} \, \boldsymbol{F} \, \overline{\boldsymbol{\theta}} \tag{34}$$

This system is then numerically solved using the *Mathematica* routine **NDSolve** and the temperature is calculated using the inversion formula (18).

Finally, the Nusselt number is computed from the expression:

$$\mathrm{Nu}(\varphi) = \frac{-1}{\theta_m} \left(\frac{K_1 K_2}{K_1 + K_2} \right) \left(K_0^{-1} \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \overline{\theta}_{ir}(\varphi) \, Q_{ir} + K_0 \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \overline{\theta}_{ir}(\varphi) \, W_{ir} \right) \tag{35}$$

$$Q_{ir} = X_r'(1) \int_0^1 Y_i \, \mathrm{d}\eta; \qquad W_{ir} = Y'(1) \int_0^1 X_r \, \mathrm{d}\xi \tag{36}$$

Using the same reordering technique explained previously, the Nusselt expression can be simplified to include a single summation, resulting in:

$$\mathrm{Nu}(\varphi) = \frac{-1}{\theta_m} \left(\frac{K_1 K_2}{K_1 + K_2} \right) \left(K_0^{-1} \sum_{l=1}^{\infty} \overline{\theta}_l(\varphi) \, Q_l + K_0 \sum_{l=1}^{\infty} \overline{\theta}_l(\varphi) \, W_l \right) \tag{37}$$

5. FINITE VOLUMES METHOD

The solution by the FVM is accomplished by integrating the equation within a finite volume of height $\Delta\eta = 1/j_{\max}$ and length $\Delta\xi = 1/i_{\max}$ and employing second-order approximations for the integrations and derivations, which leads to the following discretized system using standard FVM notations for momentum and energy equations respectively:

$$\frac{u_E^* - 2u_P^* + u_W^*}{\Delta\xi^2} + K_0^2 \frac{u_N^* - 2u_P^* + u_S^*}{\Delta\eta^2} = H \tag{38}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{\theta_E - 2\theta_P + \theta_W}{\Delta\xi^2} + K_2^2 \frac{\theta_N - 2\theta_P + \theta_S}{\Delta\eta^2} \tag{39}$$

This equation is valid for internal volumes. For peripheral volumes, the application of the boundary conditions lead to:

- For $1 < i < i_{\max}$ and $j = 1$:

$$\frac{u_E^* - 2u_P^* + u_W^*}{\Delta\xi^2} + K_0^2 \frac{u_N^* - u_P^*}{\Delta\eta^2} = H \tag{40}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{\theta_E - 2\theta_P + \theta_W}{\Delta\xi^2} + K_2^2 \frac{\theta_N - \theta_P}{\Delta\eta^2} \tag{41}$$

- For $1 < i < i_{\max}$ and $j = j_{\max}$:

$$\frac{u_E^* - 2u_P^* + u_W^*}{\Delta\xi^2} + K_0^2 \frac{-3u_P^* + u_S^*}{\Delta\eta^2} = H \tag{42}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{\theta_E - 2\theta_P + \theta_W}{\Delta\xi^2} + K_2^2 \frac{-3\theta_P + \theta_S}{\Delta\eta^2} \tag{43}$$

- For $i = 1$ and $1 < j < j_{\max}$:

$$\frac{u_E^* - u_P^*}{\Delta\xi^2} + K_0^2 \frac{u_N^* - 2u_P^* + u_S^*}{\Delta\eta^2} = H \tag{44}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{\theta_E - \theta_P}{\Delta\xi^2} + K_2^2 \frac{\theta_N - 2\theta_P + \theta_S}{\Delta\eta^2} \tag{45}$$

- For $i = i_{\max}$ and $1 < j < j_{\max}$:

$$\frac{3u_P^* + u_W^*}{\Delta\xi^2} + K_0^2 \frac{u_N^* - 2u_P^* + u_S^*}{\Delta\eta^2} = H \tag{46}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{-3\theta_P + \theta_W}{\Delta\xi^2} + K_2^2 \frac{\theta_N - 2\theta_P + \theta_S}{\Delta\eta^2} \tag{47}$$

- For $i = 1$ and $j = 1$:

$$\frac{u_E^* - u_P^*}{\Delta\xi^2} + K_0^2 \frac{u_N^* - u_P^*}{\Delta\eta^2} = H \tag{48}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{\theta_E - \theta_P}{\Delta\xi^2} + K_2^2 \frac{\theta_N - \theta_P}{\Delta\eta^2} \tag{49}$$

- For $i = i_{\max}$ and $j = 1$:

$$\frac{-3u_P^* + u_W^*}{\Delta\xi^2} + K_0^2 \frac{u_N^* - u_P^*}{\Delta\eta^2} = H \tag{50}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{-3\theta_P + \theta_W}{\Delta\xi^2} + K_2^2 \frac{\theta_N - \theta_P}{\Delta\eta^2} \tag{51}$$

- For $i = 1$ and $j = j_{\max}$:

$$\frac{u_E^* - u_P^*}{\Delta\xi^2} + K_0^2 \frac{-3u_P^* + u_S^*}{\Delta\eta^2} = H \tag{52}$$

$$u_P^* \frac{\partial\theta_P}{\partial\varphi} = K_1^2 \frac{\theta_E - \theta_P}{\Delta\xi^2} + K_2^2 \frac{-3\theta_P + \theta_S}{\Delta\eta^2} \tag{53}$$

- For $i = i_{\max}$ and $j = j_{\max}$:

$$\frac{-3\,u_P^* + u_W^*}{\Delta\xi^2} + K_0^2\,\frac{-3\,u_P^* + u_S^*}{\Delta\eta^2} = H \tag{54}$$

$$u_P^*\,\frac{\partial\theta_P}{\partial\varphi} = K_1^2\,\frac{-3\,\theta_P + \theta_W}{\Delta\xi^2} + K_2^2\,\frac{-3\,\theta_P + \theta_S}{\Delta\eta^2} \tag{55}$$

The boundary condition for energy equation in φ direction is given by:

$$\theta_P(0) = 0 \tag{56}$$

This initial value system of ordinary differential equations was also solved numerically by **NDSolve** routine from *Mathematica* software. The Nusselt number is obtained from equation (37) computing the derivative and integral numerically.

6. RESULTS AND DISCUSSION

After presenting the solution of the problem using the considered two different analyzed methodologies, computational implementations were developed, and numerical results are now presented for comparisons. The mesh used to solve the energy equation by the FVM is the same used to solve the momentum equation and, for a fair comparison, the number of terms used in the GITT to solve the velocity profile is the same as used for the temperature profile. Table 1 presents the velocity convergence for particular positions using the GITT. As can be seen, the convergence becomes more difficult in the areas near the the duct walls and it becomes better for low aspect ratios. Some special cases only need five terms for obtaining six converged digits. Another interesting aspect to note is that the average velocity presents an excellent convergence behavior.

Table 1. Velocity convergence for some particular positions for GITT.

	$l_{\max}$	$\xi = 0;\ \eta = 0.99$	$\xi = 0.99;\ \eta = 0.99$	$\xi = 0.99;\ \eta = 0$	$\xi = 0;\ \eta = 0$	U_{avg}
$K_0 = 1$	5	0.0462064	0.00403787	0.0476841	2.09626	1.00002
	10	0.0478488	0.00289441	0.0476841	2.09626	1.00000
	20	0.0476993	0.00244196	0.0476841	2.09626	1.00000
	30	0.0476874	0.00234024	0.0476841	2.09626	1.00000
	40	0.0476851	0.00230695	0.0476841	2.09626	1.00000
	50	0.0476844	0.00229439	0.0476841	2.09626	1.00000
	60	0.0476842	0.00228938	0.0476841	2.09626	1.00000
	70	0.0476841	0.00228739	0.0476841	2.09626	1.00000
	80	0.0476841	0.00228663	0.0476841	2.09626	1.00000
$K_0 = 1/2$	5	0.0630276	0.00345703	0.040452	1.99180	1.00001
	10	0.0638741	0.00289054	0.040452	1.99180	1.00000
	20	0.0638053	0.00270850	0.040452	1.99180	1.00000
	30	0.0638011	0.00267899	0.040452	1.99180	1.00000
	40	0.0638005	0.00267197	0.040452	1.99180	1.00000
	50	0.0638004	0.00267004	0.040452	1.99180	1.00000
	60	0.0638003	0.00266947	0.040452	1.99180	1.00000
	70	0.0638003	0.00266931	0.040452	1.99180	1.00000
	80	0.0638003	0.00266926	0.040452	1.99180	1.00000
$K_0 = 1/4$	5	0.102491	0.00424850	0.035325	1.77368	1.00000
	10	0.102978	0.00394732	0.035325	1.77368	1.00000
	20	0.102947	0.00388253	0.035325	1.77368	1.00000
	30	0.102946	0.00387706	0.035325	1.77368	1.00000
	40	0.102946	0.00387637	0.035325	1.77368	1.00000
	50	0.102946	0.00387627	0.035325	1.77368	1.00000
	60	0.102946	0.00387625	0.035325	1.77368	1.00000
	70	0.102946	0.00387625	0.035325	1.77368	1.00000
	80	0.102946	0.00387625	0.035325	1.77368	1.00000

Table 2 shows the FVM velocity convergence at the same positions as table 1, where $k_{\max}$ is the total number of cells, given by $i_{\max}$ times $j_{\max}$. The results show that this technique has the same tendency as GITT in which the convergence is better far from the physical boundaries, especially from corners.

Table 2. Velocity convergence for some particular positions for FVM.

	k_{max}	$\xi = 0;\ \eta = 0.99$	$\xi = 0.99;\ \eta = 0.99$	$\xi = 0.99;\ \eta = 0$	$\xi = 0;\ \eta = 0$	U_{avg}
$K_0 = 1$	25	0.0487418	0.00149349	0.0487418	2.22222	1.01220
	100	0.0482089	0.00180884	0.0482089	2.12771	1.00158
	400	0.0480810	0.00212309	0.0480810	2.10412	1.00047
	625	0.0480658	0.00222419	0.0480658	2.10129	1.00018
	2500	0.0480455	0.00253815	0.0480455	2.09751	1.00001
	10000	0.0476862	0.00226995	0.0476862	2.09657	1.00001
$K_0 = 1/2$	25	0.0643124	0.00166496	0.0405975	2.04704	0.999311
	100	0.0645948	0.00205599	0.0406528	2.00264	1.00041
	400	0.0646514	0.00244309	0.0406663	1.99373	1.00006
	625	0.0646578	0.00256747	0.0406679	1.99287	0.999870
	2500	0.0646663	0.00295358	0.0406699	1.99194	0.999990
	10000	0.0638025	0.00264893	0.0404521	1.99180	1.00005
$K_0 = 1/4$	25	0.0957254	0.00213787	0.0354688	1.83914	1.00317
	100	0.103320	0.00279521	0.0354935	1.78963	1.00122
	400	0.105195	0.00342974	0.0355008	1.77755	1.00060
	625	0.105400	0.00363276	0.0355017	1.77613	0.999911
	2500	0.105669	0.00426221	0.0355027	1.77427	1.00002
	10000	0.102943	0.00384313	0.0353250	1.77382	1.00003

Table 3. Local Nusselt number for GITT.

		φ					
	l_{max}	10^{-3}	$10^{-2.5}$	10^{-2}	$10^{-1.5}$	10^{-1}	10^{0}
$K_0 = 1/4$	20	12.8440	8.30897	5.59702	4.81074	4.57258	4.45182
	30	12.7311	8.14621	5.59476	4.80942	4.57267	4.45296
	40	12.4780	7.59011	5.56874	4.80099	4.56530	4.25015
	50	12.3044	7.58249	5.56731	4.80069	4.56522	4.44703
	60	12.2556	7.58118	5.56693	4.80058	4.56518	4.44588
	70	11.3671	7.49885	5.55824	4.79751	4.56251	4.44410
	80	11.3509	7.49803	5.55804	4.79746	4.56250	4.44337
$K_0 = 1/2$	20	11.9250	7.03468	4.74042	3.73709	3.43697	3.39720
	30	11.5315	6.92651	4.73617	3.73561	3.43644	3.39678
	40	10.9281	6.70854	4.72650	3.73252	3.43433	3.40553
	50	10.3026	6.67719	4.72082	3.73099	3.43332	3.40134
	60	10.2317	6.67489	4.72013	3.73082	3.43326	3.39371
	70	9.90737	6.66729	4.71708	3.73004	3.43276	3.49277
	80	9.88674	6.66585	4.71671	3.72994	3.43272	3.39319
$K_0 = 1$	20	11.3155	6.64788	4.37650	3.28482	2.98606	2.98138
	30	10.7680	6.38035	4.36613	3.28141	2.98499	2.98041
	40	10.1572	6.30958	4.35946	3.27959	2.98415	2.97887
	50	9.68994	6.29783	4.35535	3.27856	2.98366	2.97877
	60	9.44564	6.29093	4.35284	3.27793	2.98337	2.96737
	70	9.40274	6.28813	4.35208	3.27773	2.98328	2.97811
	80	9.35783	6.28482	4.35128	3.27753	2.98319	2.97495

The local and mean Nusselt number based on the hydraulic diameter for the thermal entry region calculated by the Integral Transform Technique is shown in tables 3 and 4. By observing the data in this table, one can easily notice that the results have a tendency to converge quicker at positions farther away from the channel entrance. In addition, reducing the aspect ratio K_0 causes the convergence rate to become slightly worse. The same observation can be done about tables 5 and 6, which shows similar Nusselt results at the same positions presented for the Finite Volumes Method. The poor convergence near the duct entrance can be linked to the high gradients in this region.

Simulations by the GITT using the fully converged velocity profile (with at least 6 digits) were developed to calculate Nusselt number. This approach resulted in the exact same output presented in tables 3 and 4. This occurs due to the fact

Table 4. Mean Nusselt number for GITT.

		φ					
	$l_{\max}$	10^{-3}	$10^{-2.5}$	10^{-2}	$10^{-1.5}$	10^{-1}	10^{0}
$K_0 = 1/4$	20	14.9580	11.6796	8.11173	6.01356	5.08591	4.52762
	30	15.2696	11.6359	8.07010	5.99933	5.08119	4.52660
	40	17.0124	11.7121	7.98895	5.96563	5.06528	4.61436
	50	17.0133	11.6712	7.97470	5.96068	5.06363	4.51848
	60	16.9767	11.6521	7.96820	5.95849	5.06289	4.51082
	70	17.4065	11.5235	7.91501	5.93858	5.05470	4.51530
	80	17.3735	11.5112	7.91083	5.93719	5.05424	4.51519
$K_0 = 1/2$	20	15.0530	10.8215	7.18675	5.05496	4.00961	3.45988
	30	15.3172	10.7218	7.14206	5.03904	4.00404	3.45896
	40	16.0091	10.6243	7.08004	5.01584	3.99512	3.45604
	50	16.1629	10.5221	7.04134	5.00173	3.98989	3.45350
	60	16.0879	10.4888	7.02990	4.99789	3.98862	3.45464
	70	15.9840	10.4142	7.00266	4.98831	3.98520	3.43678
	80	15.9329	10.3955	6.99625	4.98616	3.98448	3.45375
$K_0 = 1$	20	14.4892	10.3164	6.78050	4.64768	3.55811	3.03914
	30	15.2421	10.2186	6.70978	4.62129	3.54839	3.03663
	40	15.5057	10.1038	6.66376	4.60444	3.54234	3.03520
	50	15.5014	10.0155	6.63148	4.59291	3.53828	3.03456
	60	15.3802	9.94787	6.60700	4.58436	3.53532	3.03409
	70	15.3061	9.91998	6.59719	4.58101	3.53418	3.03380
	80	15.2019	9.88181	6.58402	4.57659	3.53271	3.03359

Table 5. Local Nusselt number for FVM.

		φ					
	$k_{\max}$	10^{-3}	$10^{-2.5}$	10^{-2}	$10^{-1.5}$	10^{-1}	10^{0}
$K_0 = 1/4$	25	10.3925	7.29505	5.56481	4.69828	4.44549	4.35031
	100	10.9709	7.58127	5.51930	4.76039	4.52930	4.41402
	400	10.6992	7.45385	5.54022	4.78535	4.55180	4.43342
	625	10.6822	7.45524	5.54264	4.78842	4.55457	4.43593
	2500	10.6853	7.45718	5.54601	4.79256	4.55830	4.43936
	10000	10.6847	7.45781	5.54687	4.79360	4.55924	4.44020
$K_0 = 1/2$	25	10.7105	6.94796	4.728	3.6792200	3.38212	3.33964
	100	9.96081	6.63006	4.70047	3.71436	3.41684	3.37625
	400	9.74144	6.63927	4.70784	3.72470	3.42774	3.38638
	625	9.73811	6.63960	4.70886	3.72602	3.42916	3.38835
	2500	9.73526	6.64025	4.71028	3.72781	3.43110	3.39157
	10000	9.73475	6.64045	4.71064	3.72826	3.43158	3.39209
$K_0 = 1$	25	10.9719	6.50712	4.28184	3.22028	2.90188	2.89361
	100	9.24292	6.25534	4.33128	3.26036	2.96023	2.95440
	400	9.28795	6.26214	4.34270	3.27218	2.97681	2.97149
	625	9.28711	6.26334	4.34417	3.27368	2.97889	2.97363
	2500	9.28665	6.26506	4.34618	3.27573	2.98171	2.97653
	10000	9.28665	6.26552	4.34669	3.27623	2.98241	2.97725

that the velocity profile converges quickly with only a few terms at almost all points in the physical domain, except at critical positions, as displayed on table 1.

Finally, for validation purposes, table 7 displays a comparison for the Nusselt number calculated in this paper with the work done by Chandrupatla and Sastri (1977) for square duct case ($K_0 = 1$). As one can observe, that the results are in perfect accordance with the literature.

Table 6. Mean Nusselt number for FVM.

	$k_{\max}$	φ: 10^{-3}	$10^{-2.5}$	10^{-2}	$10^{-1.5}$	10^{-1}	10^{0}
$K_0 = 1/4$	25	15.0686	10.4644	7.55066	5.79149	4.92546	4.41662
	100	15.9606	11.1075	7.77664	5.86941	5.00876	4.48408
	400	16.2592	11.0527	7.74751	5.87642	5.02731	4.50357
	625	16.1886	11.0323	7.74242	5.87677	5.02943	4.50604
	2500	16.0904	11.0018	7.73480	5.87700	5.03219	4.50925
	10000	16.0635	10.9935	7.73271	5.87701	5.03287	4.51024
$K_0 = 1/2$	25	14.2808	10.1873	7.01440	4.96364	3.94205	3.40131
	100	15.3338	10.1952	6.91885	4.95129	3.96253	3.43639
	400	14.9752	10.0613	6.88086	4.94549	3.96816	3.44741
	625	14.9204	10.0436	6.87580	4.94470	3.96887	3.44885
	2500	14.8437	10.0190	6.86887	4.94361	3.96984	3.45082
	10000	14.8238	10.0127	6.86710	4.94332	3.97007	3.45132
$K_0 = 1$	25	13.8929	10.0105	6.59659	4.54151	3.47293	2.95173
	100	14.7371	9.70437	6.51203	4.54171	3.50767	3.00983
	400	14.3604	9.59124	6.48341	4.54052	3.51730	3.02616
	625	14.3124	9.57647	6.47971	4.54036	3.51851	3.02821
	2500	14.2463	9.55634	6.47469	4.54013	3.52015	3.03098
	10000	14.2295	9.55122	6.47342	4.54007	3.52056	3.03166

Table 7. Comparison of the results in the thermal entry region calculated with the work done by Chandrupatla and Sastri (1977)†.

φ	$\mathrm{Nu}^{\dagger}$	Nu	$\mathrm{Nu}_m^{\dagger}$	Nu_m
0.1000	2.976	2.983	3.514	3.531
0.0500	3.074	3.077	4.024	4.052
0.0400	3.157	3.158	4.253	4.287
0.0250	3.432	3.431	4.841	4.895
0.0200	3.611	3.608	5.173	5.240
0.0125	4.084	4.080	5.989	6.096
0.0100	4.357	4.350	6.435	6.568
0.0075	4.755	4.745	7.068	7.246
0.0050	5.412	5.397	8.084	8.349

7. CONCLUSIONS

This paper provided a comparison between two solutions strategies for calculating Nusselt number in thermally developing Newtonian laminar flow within a rectangular duct: the Generalized Integral Transform Technique and Finite Volumes Method (FVM). In order to perform a fair comparison of the two methodologies, the original partial differential equation (PDE) formulation was converted to an ordinary differential equation (ODE) system, either through Integral Transformation (GITT) or spatial discretization (FVM) within the domain cross-section. Both solution strategies were computationally implemented in the *Mathematica* platform, and the resulting ODE system, for both approaches, was solved using the ODE solver NDSolve. Simulation results were then conducted to compare the solution behavior of both methodologies. The results showed that, in general, for the velocity profile, the FVM requires a very refined mesh to achieve the same convergence obtained with the integral transformation approach with a relatively small truncation order. For the Nusselt number, the FVM needs an even more refined mesh and the GITT also requires more terms for obtaining satisfactory convergence rates. For both methods, the results were worse in regions near the physical boundaries due to the higher temperature gradients that occur in these regions. As suggestions for future works, one could compare these techniques for solving problems in other geometries, or even attempt do develop combined solution schemes using both methodologies, thereby joining the advantages associated with each methodology.

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10. RESPONSIBILITY NOTICE