Proceedings of ENCIT 2010
13th Brazilian Congress of Thermal Sciences and Engineering
December 05-10, 2010, Uberlandia, MG, Brazil

BED-FORMS IN RIVERS: INSTABILITIES OF A GRANULAR BED SHEARED BY A FREE SURFACE TURBULENT FLOW

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Abstract. *The transport of granular matter by a fluid flow is frequently found in nature and in industry. When the shear stresses exerted by the fluid flow on a granular bed are bounded to some limits, a mobile granular layer known as bed-load takes place in which the grains stay in contact with the fixed part of the granular bed. Under these conditions, an initially flat granular bed may be unstable, generating ripples and dunes, such as those observed in deserts, in rivers, but also in pipelines conveying sand. In a recent article (Franklin, 2010a), the mechanisms of these instabilities were explained and a linear stability analysis was presented, showing, for the first time, a scaling between the fluid flow conditions and the length-scale of the initial bed-forms. This analysis didn't take into considerations free surface effects. In another recent article (Franklin, 2010b), a non-linear stability analysis in the same scope of Franklin (2010a) was presented, shedding light into the evolution of the bed-forms after their initial (linear) phase. Franklin (2010b) showed that, after the initial phase of the instability, some non-linear modes interact (resonate) with the initial granular bed undulations, and there is saturation of the bed-forms amplitude. This explains why in some cases the wavelengths predicted by the linear theory are in agreement with bed-forms measured in nature, clearly on the non-linear phase. The aim of this communication is to present a simplified analysis, based on Franklin (2010a) and Franklin (2010b), of the effects of the free surface in the determination of the size of bed-forms (ripples and dunes) observed in nature. Typical examples are ripples and dunes observed in rivers*

Keywords*: Secondary non-linear instabilities, granular bed, turbulent flow, free-surface*

1. INTRODUCTION

The transport of solid particles entrained by a fluid flow is frequently found in nature and in industry. It is present, for example, in the erosion of river banks, in the displacement of desert dunes and in hydrocarbon pipelines conveying sand. When shear stresses exerted by the fluid flow on the granular bed are able to move some grains, but are relatively small compared to the grains weight, the flow is not able to transport grains as a suspension. Instead, a mobile layer of grains known as bed-load takes place in which the grains stay in contact with the fixed part of the granular bed. The thickness of this mobile layer is a few grain diameters (Bagnold (1941), Raudkivi (1976)).

An initially flat granular bed may become unstable and give rise to bed-forms when submitted to a fluid flow. These forms, initially two-dimensional, may grow and generate patterns such as ripples or dunes. In nature, some examples affecting human activities are the aeolian and the aquatic dunes. The aquatic ripples and dunes observed on the bed of some rivers create a supplementary friction between the bed and the water, affecting the water depth and being related to flood problems. In cases where their size is comparable to the water depth, water flows can experiment strong depth variations, seriously affecting navigation (Engelund and Fredsoe, 1982). In industry, examples are mostly related to closed-conduit flows conveying grains, such as hydrocarbon pipelines conveying sand. In such cases, the bed-forms generate supplementary pressure loss, but also pressure and flow rate transients (Kuru et al., 1995; Franklin, 2008).

The stability of a granular bed is given by the balance between local grains erosion and deposition. If there is erosion at the crests of the granular bed, the amplitude of initial bed undulations decreases and the bed is stable. On the contrary, the bed is unstable. If there is neither erosion nor deposition at the crests, there is neutral stability. The regions of erosion and deposition can be found from the mass conservation of grains. The mass conservation implies that there is erosion in regions where the gradient of the flow rate of grains is positive and deposition where it is negative, so that the phase lag between the flow rate of grains and the bed-form is a stability criterion. If the maximum of the flow rate of grains is upstream a crest, there must be deposition at the crest and the bed is unstable, otherwise the bed is stable. To answer the stability question, the mechanisms creating a phase lag between the shape of the granular bed and the flow rate of grains need to be known.

The instabilities of a granular bed give rise to ripples and dunes, that are deformed and displaced by the fluid flow, so that they can be viewed as waves that develop is space and in time. In these cases, it is frequent to perform temporal stability analyses, the spatial and temporal analyses being related by the *Gaster Relation* (Schmid and Henningson, 2001; Drazin and Reid, 2004; Charru, 2007).

In a recent article (Franklin, 2010a), the mechanisms of this instability were explained and a linear stability analysis was presented, in the specific case of granular beds sheared by turbulent boundary-layers of liquids, without free-surface effects. It was seen that the basic mechanisms are three: the fluid flow perturbation by the shape of the bed, which is known to be the unstable mechanism (Jackson et al., 1975; Hunt et al., 1988; Weng et al., 1991), the relaxation effects related to the transport of grains and the gravity effects, which are the stable mechanisms (Valance and Langlois (2005) and Charru (2006) in the case of viscous flows, Franklin (2010a) in the case of turbulent flows). The linear stability analysis of Franklin (2010a) showed that the length-scale of the initial bed-forms varies with the fluid flow conditions.

Franklin (2010b) presented a nonlinear stability analysis in the same scope of Franklin (2010a). The approach used was the weakly nonlinear analysis (Landau and Lifchitz, 1994; Schmid and Henningson, 2001; Drazin and Reid, 2004; Charru, 2007), useful whenever a dominant mode can be proved to exist. This means that the modes resonating with this dominant one will grow much faster than the others, which can be neglected. The analysis is then made on a bounded number of modes. Franklin (2010b) showed that after the initial exponential growth (linear phase), the granular bed instabilities saturate, i.e., they attenuate their growth rate and maintain the same wavelength.

In both articles, Franklin (2010a) and Franklin (2010b), the results of the analysis were compared to some experimental data concerning ripples in closed-conduit flows (which is a case without free-surface effects). In particular, the dependence of the wavelength on the fluid flow conditions and the saturation of the bed-form amplitude were confirmed by experimental results. Nevertheless, the analysis of Franklin (2010a) and Franklin (2010b) are not directly applicable to cases were free surface effects are expected, such as the large wavelength bed-forms (dunes) in river flows.

This communication presents a simplified analysis, based on Franklin (2010a) and Franklin (2010b), of the instabilities on a granular bed in the presence of free surface effects. The main purpose is to understand the size of ripples and dunes observed in nature. Typical examples are the ripples and dunes observed in rivers.

The next two sections present a summary of the linear stability analysis of Franklin (2010a) and of the nonlinear analysis of Franklin (2010b), respectively. The following section discusses the free-surface effects the both analyses. A conclusion section follows.

2. LINEAR ANALISYS

Franklin (2010a) presented a linear stability analysis of the initial bed-forms on a granular bed sheared by a turbulent liquid flow, without free-surface effects. The stability analysis was based on four equations, and a brief description is given below. Please, refer to Franklin (2010a) for more details concerning the linear stability analysis. The four basic equations employed in the analysis describe the fluid flow perturbation by the shape of the bed (Eq. 1), the gravity effects (modeled in the previous equation), the transport of granular matter by a fluid flow (Eq. 2), the relaxation effects related to the transport of grains (Eq. 3), and the mass conservation of granular matter (Eq. 4).

For a hill with a height h, a surface rugosity y_0 and a length $2L$ between the half-heights, the perturbation of the longitudinal shear stress (dimensionless) caused by the fluid on the bed can be written as (Jackson and Hunt, 1975; Hunt et al., 1988; Weng et al., 1991)

$$\hat{\tau} = B_A\left(\frac{1}{\pi}\int\frac{\partial_x h}{x-\xi}d\xi + B_e\partial_x h\right) \qquad (1)$$

where ξ is an integration variable and $B_e = B - B_g/B_A$ (the term B_g/B_A was introduced in Franklin (2010a)). B_A and B come from the fluid flow perturbation and are considered as constants as they vary with the logarithm of L/y_0. B_g is a coefficient taking into account the weight of the grains (gravity effects) and the friction between them, and is also a constant. If the perturbation is supposed small compared to a basic flow, the fluid flow over the bed can be written as the basic flow, unperturbed, plus the flow perturbation. For the shear stress on the bed surface $\tau = \tau_0(1+\hat{\tau})$ where τ_0 is the shear stress caused by the basic flow on the bed. For a developed turbulent liquid flow over a granular bed, the basic flow is a rough turbulent boundary-layer, which near the bed has the well known logarithmic profile $u = u_*\kappa^{-1}\log\left(yy_0^{-1}\right)$ where κ is the Kármán constant, $u(y)$ is the unperturbed velocity profile and u_* is the friction velocity.

The flow rate of grains in equilibrium with the fluid flow is known as “saturated flow rate of grains”. From Bagnold (1941)

$$q_{sat}/Q_{sat} \sim (1+\hat{\tau})^{3/2} \qquad (2)$$

where q_{sat} is the saturated volumetric flow rate of grains by unit of width and Q_{sat} is the saturated volumetric flow rate of grains by unit of width over a flat surface (basic state). If the fluid flow over the bed changes, the flow rate of grains will lag some distance (or time) with respect to the fluid flow (relaxation effect). Charru et al. (2004) proposed for the local volumetric flow rate of grains, by unit of width

$$\partial_x q = \frac{q_{sat}-q}{L_{sat}} \qquad (3)$$

where $L_{sat} \sim L_d = d\left(u_* U_S^{-1}\right)$ is a distance called “saturation length”, d is the mean grain diameter and U_s is the typical settling velocity of a grain. Finally, the two-dimensional mass conservation of grains is

$$\partial_t h + \partial_x q = 0 \tag{4}$$

Given the small character of perturbations, the solutions of the preceding equations are plane waves. The bed height h and the flow rate of grains q can be decomposed in their normal modes

$$h(x,t) = H\, e^{\sigma t - i\omega t + ikx} \tag{5}$$

$$q(x,t)\Big/Q_{sat} = 1 + Q\, e^{\sigma t - i\omega t + ikx} \tag{6}$$

where σ is the growth rate, ω is the frequency and k is the longitudinal wave-number. Inserting Eqs (5) and (6) into Eqs. (1), (2), (3) and (4), the growth rate σ and the phase velocity $c=\omega/k$ of instabilities can be obtained from the non-trivial solution

$$\sigma = \frac{3Q_{sat}k^2\left(B_e - B_A|k|L_{sat}\right)}{2\left(1+\left(kL_{sat}\right)^2\right)} \tag{7}$$

$$c = \frac{3Q_{sat}|k|\left(B_A + B_e|k|L_{sat}\right)}{2\left(1+\left(kL_{sat}\right)^2\right)} \tag{8}$$

The most unstable (or amplified) mode is the one for which instabilities grow faster, corresponding to $\partial\sigma\big/\partial k = 0$ .From Eq. (17), we obtain

$$\lambda_{\max} \approx \frac{3B_A}{2B_e}L_{sat} \tag{9}$$

$$\sigma_{\max} \approx \frac{2B^3}{9B_A^2}\left(B_A - 2\right)Q_{sat}\frac{1}{L_{sat}^{\ \ 2}} \tag{10}$$

$$c_{\max} \approx \frac{B}{B_A}Q_{sat}\frac{1}{L_{sat}} \tag{11}$$

where the subscript *max* is related to the most unstable mode.

The stability analysis showed the existence of long-wave instability, with the fluid flow conditions, the relaxation effects and the gravity effects playing an important role. The saturation length-scale L_{sat}, related to the relaxation effects, was seen to be the major responsible for the stabilization of small waves, also playing a role in the growth rate, that varies as $\sigma_{\max} \sim L_{sat}^{-2}$. On the other hand, gravity was seen to play a smaller role in the stabilization of small waves, but to strongly affect the growth rate. Changes in the fluid flow were seen to cause variations in the growth rate proportional to the shear velocity: $\sigma_{\max} \sim u_*$.

The most relevant result of the analysis of Franklin (2010a) concerns the dependence of the wavelength of the most unstable mode on the fluid flow conditions. The linear analysis showed that wavelength scales with the fluid flow as $\lambda_{\max} \sim u_*$. Figure 1, extracted from Franklin (2010a), shows the dimensionless growth rate σt_d and the dimensionless phase velocity c/U_S of the initial perturbations versus the dimensionless wave-number kL_{sat}, where $t_d=d/U_S$ is the typical settling time. In this figure, the continuous curves correspond to a baseline, where $d=1mm$ and $B_g=0$ (no gravity effect), and the dashed curves correspond to variations in the shear velocity u_* (fluid flow effects): dashed curves corresponding to values of u_* equal to half of the values used in the continuous curves. Because U_S and d are the same in both cases, the saturation length varies as $L_{sat} \sim u_*$, which means that in the dashed curve L_{sat} is half of the value for the continuous curve. For the most unstable mode, Fig. 2 shows that a decrease in u_* by a factor *2* implies a decrease in the wavelength λ_{max} and in the growth rate σ_{max} by a factor *2*, and a decrease in the phase velocity c_{max} by a factor *4*.

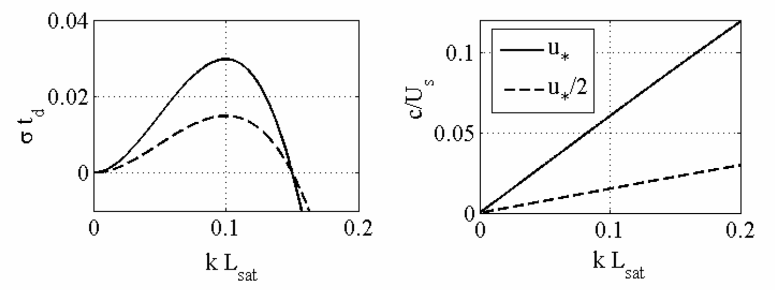

Figure 2. Dimensionless growth rate σt_d and dimensionless phase velocity c/U_S of the initial instabilities versus the dimensionless wave-number kL_{sat}, in the case of fluid flow variation. Figure extracted from Franklin (2010a).

Different from previous stability analysis for turbulent regime, it was proposed in Franklin (2010a) that the initial wavelength varies with the flow conditions of the carrier liquid. This could explain, for the first time, some previous experimental results.

3. NON-LINEAR ANALYSIS

Franklin (2010b) presented a nonlinear stability analysis in the same scope of Franklin (2010a). The approach used was the weakly nonlinear analysis (Landau and Lifchitz, 1994; Schmid and Henningson, 2001; Drazin and Reid, 2004; Charru, 2007), useful whenever a dominant mode can be proved to exist. This means that the modes resonating with this dominant one will grow much faster than the others, which can be neglected. The analysis is then made on a bounded number of modes.

In the case of a granular bed sheared by a turbulent liquid flow, Franklin (2010b) wrote the normal modes as

$$h(x,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} A_n(t)e^{inkx} \tag{12}$$

where $A_{-n} = ae^{-i\phi} = A_n^*$

Franklin (2010b) then expanded Eq. 2 in a Taylor series until $O(\varepsilon^2)$ and replaced the first term in the RHS of Eq. 1, the convolution product, by a linear term in h. As noted by Andreotti et al., 2002, this convolution product is a non-local term that varies with the shape of the bed. It can then be replaced by a bed dimensionless shape h/L, where L is a characteristic length of the bed-form). Combining the expansion and the replacement of the convolution product with Eqs. 1 to 4, we arrive in the following equation

$$\partial_t h + B_1(h)^2 + B_2(\partial_x h)^2 + B_3 h\partial_x h + B_4 h + B_5\partial_x h + B_6 = 0 \tag{13}$$

where B_1 to B_5 involve Q_{sat}, L_{sat}, L and constants, so that B_1 to B_5 are only functions of u_* and d, and they may be treated as constants in an analysis of a given granular bed submitted to a given fluid flow. B_6 is a constant, obtained from $c \sim h^{-1}$ (Franklin, 2010a). Normalizing the problem by its characteristic length (k^{-1}), and inserting the normal modes of the form of Eq. (12) in Eq. (13) gives

$$\begin{aligned} &\frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\frac{dA_n}{dt} + A_nB_4 + iB_5nA_n\right]e^{inx} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[A_n^2B_1 + B_2(inA_n)^2\right]e^{2inx} + \\ &+\frac{1}{2}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left[B_3A_piqA_q\right]e^{i(q+p)x} + B_6 = 0 \end{aligned} \tag{14}$$

By inspecting Eq. (14), we can see that it is the third term in the equation that can resonate with the linear part (in the first term). This resonance will only occur if $q + p = n$. In this case, the third term in eq. (14) can be written as

$$-\frac{iB_3}{2}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\left[pA_{p+n}A_p^*\right]e^{inx} \tag{15}$$

and, keeping in Eq. (14) only the terms that resonate with the linear part, we find

$$\frac{dA_n}{dt} = \sigma_n A_n + iB_3 \sum_{p=-\infty}^{\infty} \left[pA_{p+n} A_p^* \right] \tag{16}$$

where $\sigma_n = -(B_4 + inB_5)$

From Eq. (16), it can be seen that the non-linearities are in the third term. If this term is neglected, we find that the solution is stable for $\sigma_n<0$ and unstable for $\sigma_n>0$. Once the initial (linear) instability takes place, the perturbations grow in an exponential way and, after a time-scale equal to σ_n^{-1}, they can no-longer be analyzed by a linear approach. In order to better understand the behavior of the nonlinear part of Eq. (16), we can analyze the first two modes

$$\frac{dA_1}{dt} = \sigma_1 A_1 - B_3 i A_2 A_1^* + O\left(A_1^5\right) \tag{17}$$

$$\frac{dA_2}{dt} = \sigma_2 A_2 - B_3 i A_1^2 + O\left(A_1^4\right) \tag{18}$$

For $n >1$, on the onset of the instability, the temporal derivatives vary with $\sigma_1 A_n << |\sigma_n| A_n$ (due to the dominant effect of the fundamental). The time derivatives may then be neglected for $n >1$

$$A_2 = \frac{B_3 i}{\sigma_2} A_1^2 + O\left(A_1^4\right) \tag{19}$$

Inserting Eq. (19) into Eq. (17), we can find an equation for the fundamental similar to the Landau Equation (Landau and Lifchitz, 1994; Schmid and Henningson, 2001; Drazin and Reid, 2004; Charru, 2007)

$$\frac{dA_1}{dt} = \sigma_1 A_1 - \kappa_L A_1 |A_1|^2 + O\left(A_1^5\right) \tag{20}$$

where $\kappa_L = -B_3^2 / \sigma_2 > 0$. This corresponds to a supercritical bifurcation (Glendinning, 1999; Charru, 2007): the nonlinear term resonating with the linear one will saturate the instability, so that, after the initial exponential growth, the instability attenuate, reaching a finite value for the amplitude and maintaining the same wavelength. A bifurcation diagram can be drawn in order to visualize the saturation of the fundamental mode A_1 as a function of a control parameter (Glendinning, 1999).

Figure 2 shows the bifurcation diagram for the fundamental mode (dimensionless amplitude modulus $|A_1|$ versus the dimensionless linear growth rate σ_1) for a fixed value of the Landau constant ($\kappa_L=1$). The continuous curves correspond to stable states (attractors) and the dashed curve corresponds to the unstable states. From Fig.2, we can see that this is a supercritical bifurcation, the diagram corresponding to the well known supercritical pitchfork bifurcation. So, after the initial exponential growth (linear phase), the granular bed instabilities saturate, with their amplitude following the bifurcation diagram of Fig. 2, but keeping the same wavelength.

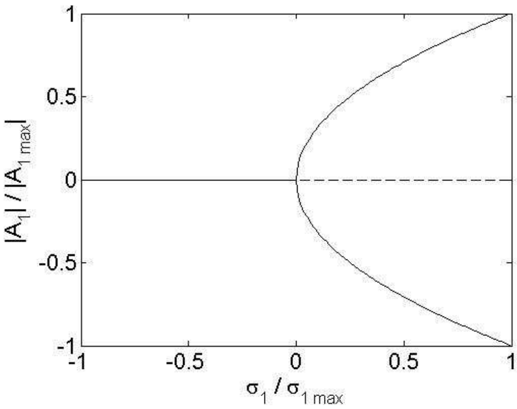

Figure 2. Bifurcation diagram for the fundamental mode (amplitude modulus $|A_1|$ versus the linear growth rate σ_1, normalized by their maximum values, respectively). The continuous curves correspond to the stable states (attractors) and the dashed curve corresponds to the unstable states. This diagram corresponds to the well known pitchfork bifurcation. Figure extracted from Franklin (2010b).

4. FREE-SURFACE EFFECTS

In the two previous sections, the effects related to the presence of a free surface were not considered. However, in many cases of practical and environmental interest, the presence of a free surface must be taken into account. One example is the presence of dunes in rivers whose depth is of the same order of the dune wavelength. In this case, free-surface effects affect the fluid flow near the bed. It is known from measurements in the field that in such cases there are at least two characteristic wavelengths, one scaling with the grains diameter and with an inner layer close to the bed (but not with the flow depth, as they are too small), and another scaling with the fluid flow depth.

In the last decades, many stability analyses have been done in order to understand the wavelength and the celerity of dunes in rivers (Kennedy, 1963; Reynolds, 1964; Engelund, 1970; Richards, 1980; Engelund and Fredsoe, 1982; Coleman and Fenton, 2000). The great majority of these works found that the wavelength of the initial instabilities scales with the liquid depth, i.e., there would be a primary instability affected by free-surface effects. This primary instability should be strong enough to allow the growth of bed-forms whose length scales with the liquid depth.

Richards (1980) performed a linear stability analysis that showed the presence of two unstable modes: one scaling with the liquid depth (dune mode) and the other scaling with the grains diameter, but independent of the free-surface (ripple mode). He proposed that both modes would emerge as a primary instability.

Recently, Fourrière et al. (2010) presented a linear stability analysis for bed-forms in rivers, in turbulent regime. In their analysis, they found that the ripple mode (without free-surface effects) has a growth rate many times greater than the dune mode (with free surface effects). Also, they found a wavelength range where the growth rate is strongly negative, corresponding to wavelengths of the order of the liquid depth. They called this range “resonance region”. Any bed-form in this wavelength range has a strong negative growth rate.

Fourrière et al. (2010) showed that the time scale for the growth of dunes as a primary linear instability is greater than that for the appearance of dunes by the coalescence of ripples (which have a much faster growth rate and evolution). As a consequence, dunes observed in rivers are the product of a secondary instability resulting from the coalescence of ripples. These dunes grow until reaching a length-scale in the resonance range, where their growth is stopped by free surface effects. In summary, they found that ripples appear as a primary linear instability that eventually saturates (but they do not prove the saturation), and that dunes appear as a secondary instability resulting from the coalescence of ripples (unstable mechanism) and the free-surface effects in the resonance range (stable mechanism).

This section is devoted to the understanding of the bed-forms observed in fluid flows with a free-surface, such as the flows in open channels and rivers. We know from previous works that in such cases there are at least two characteristic wavelengths, one scaling with the grains diameter and with an inner layer close to the bed (ripple mode), and another scaling with the fluid flow depth (dune mode). The ripple mode isn’t affected by free-surface effects, so that the stability analyses made in sections 1 and 2 are valid for this case: we can predict the initial instabilities and saturation with the presented models. The dune mode, as proposed by the recent work of Fourrière et al. (2010), is considered here as a secondary instability resulting from the competition of ripples coalescence and free-surface effects.

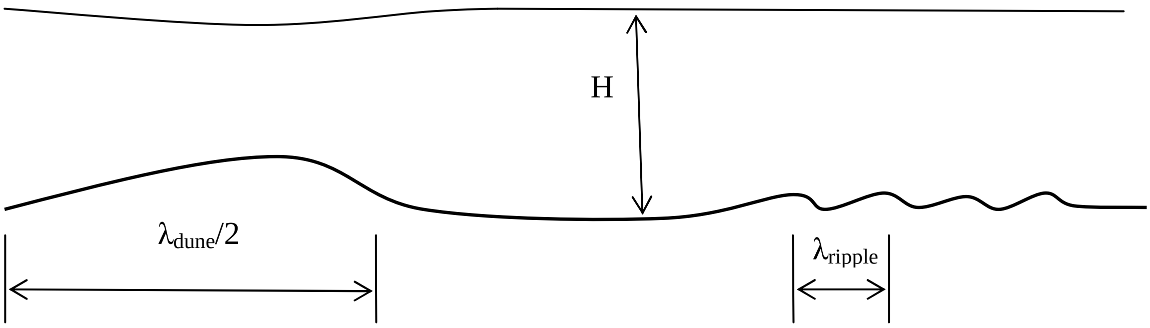

Figure 3. Definition of the ripple mode and of the dune mode: the ripple wavelength is much smaller than the liquid depth, and the dune wavelength is comparable to the liquid depth.

With these considerations, the ripple mode is well established by sections 1 and 2. We need just to find the resonance region to find the equilibrium length of the dune mode, i.e., we need to combine the free surface effects with the models presented here. In free-surface flows, one valid reasoning is that, as dunes grow by ripples coalescence, their length become comparable to the liquid depth. In the beginning, this affects the free-surface in a subcritical regime and excites gravity waves in the free surface in anti-phase with the dunes (180^0 out-of-phase). If they keep growing, they would eventually reach a supercritical regime, for which the excited surface waves are in phase with the dunes (0^0 out-of-phase). But the dunes have their growth rate stopped in a region around the transition from subcritical to supercritical: in this region, the excited surface waves are out-of-phase by nearly 90^0 lagging the dunes, resulting in erosion in the crests. This region dictates the characteristic length of dunes.

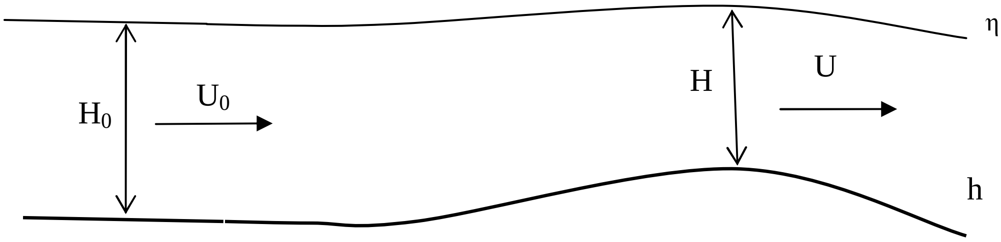

Figure 4. Definition of the free-surface η, the dune height h, the liquid depth H and the fluid velocity U. The subscript *0* corresponds to the fluid flow over a flat bed.

This reasoning can be incorporated mathematically in the model presented in sections 1 and 2 in a simple way, in order to give us the scales of length and celerity of the dune mode. One simple approach to understand the effects that the free-surface may cause on the fluid flow is to approximate the fluid flow far from the bed (above the logarithmic layer) as a potential flow. With this approximation we can estimate the behavior of the free-surface by applying the mass conservation and the Bernoulli equation to a potential liquid flow. Equations (21) and (22) apply the mass conservation and Bernoulli equation, respectively, for the two-dimensional flow depicted in Fig. 4. In these equations, the subscript *0* corresponds to the fluid flow over a flat bed.

$$U\left(H_0 - h + \eta\right) = U_o H_0 \tag{21}$$

$$\frac{U^2}{2} + g\eta = \frac{U_0^2}{2} \tag{22}$$

Considering small aspect ratio dunes (in nature we find usually an aspect ratio between *0.01* and *0.1*), an expansion in power series may be built with the small parameter $\varepsilon = h/H_0$, where U_A, U_B, η_A and η_A are *O(1).*

$$U = U_A + \varepsilon U_B + O(\varepsilon^2) \tag{23}$$

$$\eta = \eta_A + \varepsilon\eta_B + O(\varepsilon^2) \tag{24}$$

Inserting the expansions (Eqs. 23 and 24) in Eqs. (21) and (22), and separating the terms by their orders, we find

$$O(1):\begin{cases} U_A + U_A \dfrac{\eta_A}{H_0} = U_0 \\ \dfrac{1}{2}U_A^2 + g\eta_A = \dfrac{1}{2}U_0^2 \end{cases} \tag{25}$$

where $U_A = U_0$ and $\eta_A = 0$ can be a solution (and it is physically expected). In $O(\varepsilon)$

$$O(\varepsilon):\begin{cases} -U_A + U_A \dfrac{\eta_B}{H_0} + U_B + \dfrac{U_B}{H_0}\eta_A = 0 \\ \qquad U_A U_B + g\eta_B = 0 \end{cases} \tag{26}$$

The solution of the system given by Eq. (26) is $\eta_B = H_0\left(Fr^2/\left(Fr^2-1\right)\right)$ and $U_B = -U_0\left(1/\left(Fr^2-1\right)\right)$, where $Fr = U_0\big/\sqrt{gH_0}$ is the Froude number: the ratio between the mean flow velocity and the celerity of surface gravity waves. From Eqs. (23) and (24), the solution can now be written:

$$U = U_0\left[1 - \frac{h}{H_0}\left(\frac{1}{Fr^2-1}\right)\right] + O(\varepsilon^2) \tag{27}$$

$$\eta = h\left(\frac{Fr^2}{Fr^2-1}\right) + O(\varepsilon^2) \tag{28}$$

Equations (27) and (28) are sufficient for the discussion of the wavelength of the dunes proposed here. However, a simplified transport model together with the mass conservation (Eq. 4) may give us some information about the celerity of the dunes. Supposing, that the shear stress on the bed is proportional to the square of the mean velocity, and approximating the bed-load flow rate by Eq. (2), we may write the bed-load flow rate as $q \approx CU^3$, where C is a constant. In this case, Eq. (2) gives

$$\frac{\partial h}{\partial t} + C_0\frac{\partial h}{\partial x} = 0 \tag{29}$$

where the term C_0 corresponds to

$$C_0 = -\frac{3CU^2U_0}{H_0}\frac{1}{Fr^2-1} \tag{30}$$

Equation (29) corresponds to a wave propagation equation for the bed-form, with the celerity C_0. We analyze now the effects on the free-surface and on the granular-bed based on Eqs. (27), (28) and (30).

- Case I: subcritical flow

In this case, $0<Fr<1$, which implies that $\eta<0$ and $U>U_0$. This means that the free surface above the bed-form is a trough, and that the mean velocity is accelerated in the crest region. The surface gravity waves are 180^0 out-of-phase with respect to the bed-forms. The maximum of the fluid velocity is around the crest, with an upstream out-of-phase component that this simplified model is not capable to obtain (so that the fluid flow is an unstable mechanism). Also, in this case $C_0>0$, so that the bed-forms have a downstream celerity. From these characteristics, we can conclude that those forms are dunes (Engelund and Fredsoe , 1982).

- Case II: supercritical flow

In this case, $Fr>1$, which implies that $\eta>0$ and $U<U_0$. This means that the free surface above the bed-form is a bump, and that the mean velocity is decelerated in the bed-form crest region. The surface gravity waves are 0^0 out-of-phase with respect to the bed-forms The minimum of the fluid velocity is around the crest, with an upstream out-of-phase component that this simplified model is not capable to obtain. Also, in this case $C_0<0$, so that the bed-forms have an upstream celerity. From those characteristics, we can conclude that those forms are the usually called anti-dunes (Engelund and Fredsoe , 1982).

- Case III: transition

In this case, $Fr \approx 1$. In this region, the excited surface waves are out-of-phase by nearly 90^0 lagging the dunes. This means that the maximum flow velocities are shifted downstream the bed-form crests, implying erosion on the crests and stability (the fluid flow is a stable mechanism in this case). This condition bounds the length-scale of dunes by means of the surface waves and bed-form interaction, and for this reason it is called a resonance region.

It is considered here that dunes are a secondary instability, i.e., they are generated from the coalescence of ripples. In this case, as the dunes grow up the local values of Fr increase and tend to $Fr \approx 1$. When $Fr \approx 1$, the fluid flow becomes a stable mechanism, limiting then the size of dunes.

From this, we can conclude that the wavelength λ_{dune} and the celerity of dunes C_{dune}, in the presence of a free-surface, have the following scales

$$\lambda_{dune} \sim H_0 \tag{31}$$

$$C_{dune} \sim -\frac{3CU_0^3}{H_0}\frac{1}{Fr^2-1} \tag{32}$$

5. CONCLUSIONS

The transport of granular matter by a fluid flow is frequently found in nature and in industry. When the shear stresses exerted by the fluid flow on a granular bed are bounded to some limits, a mobile granular layer known as bed-load takes place in which the grains stay in contact with the fixed part of the granular bed. Under these conditions, an initially flat granular bed may be unstable, generating ripples and dunes, such as those observed in deserts, but also in pipelines conveying sand.

It is known from observations in rivers and in open-channels that two different kinds of bed-forms may coexist. Those forms have different wavelengths, one scaling with the grains diameter and with an inner layer close to the bed (but not with the flow), called ripple, and another scaling with the fluid flow depth, called dune. Many previous works were made in an attempt to understand the generation and evolution of these forms, but until now there is not a real consensus about it.

This paper presented a complete model capable to predict the formation and the evolution of sand patterns observed in rivers and in open-channels. Based on previous works (Franklin, 2010a; Franklin, 2010b; Fourrière et al., 2010), it was argued here that ripples are a primary linear instability, which saturate with the same wavelength predicted by the linear analysis, and that dunes are a secondary instability, formed by the coalescence of saturated ripples.

The ripples formation and evolution were already presented by Franklin (2010a) and Franklin (2010b) and a brief description was made here. To obtain the length-scale of dunes, a simplified model was proposed (potential approximation far from the granular bed surface), in which the coalescence of saturated ripples is the unstable mechanism, and the fluid flow, influenced by free-surface effects, is the stable mechanism. It was then shown that near the subcritical – supercritical transition, $Fr \approx 1$, surface gravity waves are out-of-phase in such a manner that the maximum of the fluid velocity occurs downstream the crest, so that the flow is a stable mechanism when $Fr \approx 1$.

Finally, it was proposed here the scales for the wavelength and celerity of ripples (the same from Franklin (2010a) and Franklin (2010b)) and for the wavelength and celerity dunes.

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