SIMULATION OF HYPERBOLIC FLOWS THROUGH UNSATURATED POROUS MEDIA EMPLOYING AN APPROXIMATE RIEMANN SOLVER

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1. INTRODUCTION

This work describes the filling up of a rigid porous medium employing an alternative procedure to treat the nonlinear hyperbolic associated problem. Hyperbolic systems adequately describe flows through unsaturated porous media; however they may not admit a regular solution, but a generalized one, involving shock waves. Despite its limitation to one-dimensional cases, Glimm's scheme is a reliable method to treat discontinuous problems, preserving the shock identity and presenting mathematically ensured accuracy. Nevertheless, its implementation requires the complete solution of the associated Riemann problem which, besides its inherent difficulty, increases computational costs. In this work Glimm's method is implemented by employing an approximate Riemann solver, circumventing the requirement of an exact solution of the associated Riemann problem (Saldanha da Gama and Martins-Costa, 2008).

A time evolution through Glimm's scheme is performed by solving the associated Riemann problem between each two consecutive time steps. This work compares results obtained using the classical exact solutions of the Riemann problem with those obtained employing an approximate Riemann solver (Saldanha da Gama and Martins-Costa, 2008), built by connecting any two given states by a discontinuity. This Riemann solver is approximates the Riemann problem solution by piecewise constant functions satisfying the jump conditions, but not necessarily the entropy condition, giving rise to a simple and efficient procedure for simulating hyperbolic systems; circumventing the requirement of a complete solution for the associated Riemann problem.

The mechanical model employs a mixture theory approach instead of the widely used way of handling with transport phenomena in porous media – the volume averaging technique, in which concentration and velocity components are described as volumetric averages in order that the momentum transport may be described in a classical continuum mechanics context. A comparison among different models for transport in porous media employing a volume averaging approach is found in Alazmi and Vafai (2000).

The mixture theory, convenient for modeling multicomponent systems, describes the flow by considering three overlapping continuous constituents, representing the porous matrix (solid constituent), the fluid (liquid constituent) and an inert gas included to account for the compressibility of the mixture as a whole (Martins-Costa and Saldanha da Gama, 2005), leading to an apparent thermomechanical independence among the constituents. In order to provide dynamical interaction, an additional term, absent in a Continuum Mechanics description – playing the role of a momentum source – is required to account for the mechanical coupling among the constituents in the balance equations.

The problem mathematical description consists of a nonlinear hyperbolic system of two partial differential equations representing the mass and momentum conservation for the liquid constituent, since the solid constituent is supposed rigid and at rest and the gas constituent is assumed inert. This system is approximated by Glimm's scheme, implemented using two distinct approaches to deal with the associated Riemann problem.

Comparison of the standard procedure – namely Glimm's scheme being implemented by employing the exact solution of the associated Riemann problem for each two consecutive steps – with the results obtained through the approximate Riemann solver for the non-saturated flow of a fluid through a rigid porous matrix allows observing the good performance of the approximate Riemann solver.

2. MECHANICAL MODEL

The flow through an unsaturated porous medium is described by using a mixture theory viewpoint in which a chemically non reacting continuous mixture of a rigid solid constituent at rest, a liquid constituent – from now on denoted as fluid constituent and an inert gas, playing the role of the third constituent, is considered. Consequently, to model an isothermal flow, it suffices to solve mass and momentum balance equations for the fluid constituent. The balance equations are given by (Atkin and Craine, 1976; Rajagopal and Tao, 1995)

$$\begin{aligned} &\frac{\partial \rho_F}{\partial t} + \nabla \cdot (\rho_F \mathbf{v}_F) = 0 \\ &\rho_F \left[\frac{\partial \mathbf{v}_F}{\partial t} + (\nabla \mathbf{v}_F)\mathbf{v}_F \right] = \nabla \cdot \mathbf{T}_F + \mathbf{m}_F + \rho_F \mathbf{b}_F \end{aligned} \tag{1}$$

in which ρ_F stands for the fluid constituent mass density – representing the local ratio between the fluid constituent mass and the corresponding volume of mixture and $\mathbf{v}_F$ is the fluid constituent velocity in the mixture. In the momentum equation $\mathbf{T}_F$ represents the partial stress tensor – analogous to Cauchy stress tensor in Continuum Mechanics – associated with the fluid constituent, the body force is represented by $\mathbf{b}_F$ while $\mathbf{m}_F$ is the momentum supply acting on the fluid constituent due to its interaction with the remaining constituents of the mixture. This momentum source is an internal contribution; consequently the net momentum supply to the mixture – due to all the constituents – must be zero.

The saturation ψ is defined as the ratio between the fluid fraction φ and the porous matrix porosity ε, so that $\psi = \varphi / \varepsilon = \rho_F / (\varepsilon \rho_f)$ with $0 \le \psi \le 1$ everywhere, in which ρ_f is the actual mass density of the fluid – regarded as a single continuum, in contrast to ρ_F defined as the fluid constituent mass density.

Constitutive relations for the momentum source term and the partial stress tensor are required to build the mechanical model for a mixture representing an unsaturated flow of an incompressible Newtonian fluid through a porous matrix. Considering the normal fluid stresses dominant over shear stresses and interphase tractions (Allen, 1986), and neglecting the so-called darcian term, accounting for drag forces effect (Martins-Costa et al., 1995) the momentum source and the partial stress are given by (Williams, 1978, Saldanha da Gama and Sampaio, 1987)

$$\mathbf{m}_F = -\frac{\varepsilon^2 \mu_f \mathcal{D}}{K} \psi \nabla \psi \qquad \mathbf{T}_F = -\varepsilon^2 \bar{p} \psi^2 \mathbf{I} \tag{2}$$

where μ_f represents the fluid viscosity (measured considering a Continuum Mechanics viewpoint), K the porous matrix specific permeability, $\mathcal{D}$ a diffusion coefficient – analogous to the usual mass diffusion coefficient and $\bar{p}$ is a pressure (assumed constant while the flow is unsaturated).

Assuming all the quantities depending only on the time *t* and on the position *x* and that *u* is the only non-vanishing component of the fluid constituent velocity $\mathbf{v}_F$, then the balance equations (1) combined with the saturation definition and the constitutive relations (2) give rise to

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(\psi u) = 0 \\ &\rho_f \varepsilon \left[\psi \frac{\partial u}{\partial t} + \psi u \frac{\partial u}{\partial x} \right] = -\frac{\partial}{\partial x}\left(\varepsilon^2 \psi^2 \bar{p} \right) - \frac{\beta \rho_f^2 \varepsilon^2}{2} \frac{\partial \psi^2}{\partial x} \end{aligned} \tag{3}$$

The nonlinear system presented in equation (3) may be rewritten in a more convenient form by redefining a reference pressure p_0 as $p_0 = \bar{p} + \beta \rho_f^2 / 2$ and introducing the following dimensionless quantities

$$v = u \sqrt{\frac{\rho_f}{\varepsilon p_0}} \qquad \eta = \frac{x}{L} \qquad \tau = \frac{t}{L} \sqrt{\frac{\varepsilon p_0}{\rho_f}} \tag{4}$$

in which $\eta = x / L$ with L standing for a characteristic length:

$$\begin{aligned} &\frac{\partial \psi}{\partial \tau} + \frac{\partial}{\partial \eta}(\psi v) = 0 \\ &\frac{\partial}{\partial \tau}(\psi v) + \frac{\partial}{\partial \eta}\left(\psi v^2 + \psi^2 \right) = 0 \end{aligned} \tag{5}$$

Assuming a convenient variables redefinition: $F \equiv \psi$, $G \equiv \psi v$, the system presented in equation (5) may be rewritten as

$$\begin{aligned} &\frac{\partial F}{\partial \tau} + \frac{\partial G}{\partial \eta} = 0 \\ &\frac{\partial G}{\partial \tau} + \frac{\partial}{\partial \eta}\left(\frac{G^2}{F} + F^2 \right) = 0 \end{aligned} \tag{6}$$

in which $F = \hat{F}_n(\eta, \tau)$ and $G = \hat{G}_n(\eta, \tau)$.

3. NUMERICAL APPROXIMATION

The numerical approximation for the solution of the system (6) at a time instant τ_{n+1} is performed by employing Glimm's scheme, whose usual implementation requires previous knowledge of the solution of the associated Riemann problem. Essentially, a time evolution through Glimm's scheme is performed by solving the associated Riemann problem between each two consecutive steps.

The first step to implement Glimm's scheme is to approximate by piecewise constant functions the arbitrary initial condition given by a function of the position η – namely $F(\eta, 0) = F_0(\eta)$ and $G(\eta, 0) = G_0(\eta)$ – with the piecewise constant functions conveniently chosen with equal width steps. At a given time instant τ_i this approximation may be expressed as

$$\begin{aligned} &F = \hat{F}_n(\eta) \approx F_{n_i} = \hat{F}_n(\eta_i + \theta_n \Delta\eta) \\ &G = \hat{G}_n(\eta) \approx G_{n_i} = \hat{G}_n(\eta_i + \theta_n \Delta\eta) \end{aligned} \qquad \text{for} \qquad \eta_i - \frac{\Delta\eta}{2} < \eta < \eta_i + \frac{\Delta\eta}{2} \tag{7}$$

in which θ_n is a number randomly chosen in the interval $(-1/2,\ +1/2)$ and $\Delta\eta = \eta_{i+1} - \eta_i$ is the width of each step.

Combining the approximations for the initial data presented in equation (7) with problem (6) the following Riemann problem (Smoller, 1983) associated with equations (6) is obtained for each two consecutive steps:

$$\begin{aligned} &\frac{\partial F}{\partial \tau} + \frac{\partial G}{\partial \eta} = 0 \\ &\frac{\partial G}{\partial \tau} + \frac{\partial}{\partial \eta}\left(\frac{G^2}{F} + F^2 \right) = 0 \end{aligned} \quad \text{with} \quad \begin{aligned} &(F, G) = (F_{n_i}, G_{n_i}) \qquad \text{for} \quad \tau = \tau_n\ , \quad -\infty < \eta < \eta_i + \frac{\Delta\eta}{2} \\ &(F, G) = (F_{n_{i+1}}, G_{n_{i+1}}) \qquad \text{for} \quad \tau = \tau_n\ , \quad \eta_{i+1} - \frac{\Delta\eta}{2} < \eta < \infty \end{aligned} \tag{8}$$

Denoting by $\bar{F}_{n_i}$ and $\bar{G}_{n_i}$ the generalized solution of the Riemann problem stated in equations (8), the approximation for the solution of equations (6) at time τ_{n+1} is given as follows:

$$\begin{aligned} &F_{n+1} = \hat{F}_{n+1}(\eta) \approx \bar{F}_{n_i}(\eta,\ \tau_{n+1}) \\ &G_{n+1} = \hat{G}_{n+1}(\eta) \approx \bar{G}_{n_i}(\eta,\ \tau_{n+1}) \end{aligned} \qquad \text{for} \qquad \eta_i < \eta < \eta_{i+1} \tag{9}$$

It important to note that in order to prevent interactions among nearby shocks of adjacent Riemann problems, assuring uniqueness for the solution, the time step $\Delta\tau$ and, consequently, τ_{n+1}, must be chosen in such a way that the Courant-Friedrichs-Lewy condition (Smoller, 1983) is satisfied:

$$\tau_{n+1} - \tau_n \le \frac{\Delta\eta}{2|\lambda|_{\max}} \tag{10}$$

where $|\lambda|_{\max}$ is the maximum (in absolute value) propagation speed of shocks, considering all the Riemann problems at time τ_n.

4. ALTERNATIVE PROCEDURE: APPROXIMATE RIEMANN SOLVER

The alternative procedure employed in this work – the approximate Riemann solver – consists of replacing the classical exact solutions of the Riemann problem (8) by generalized approximations in order to advance in time through Glimm's scheme. These generalized approximations consist of weak solutions for the associated Riemann problem within a space of piecewise constant functions with a maximum of two jumps – built by assuming that two given states are connected by a discontinuity which may not satisfy the entropy conditions.

A generalized solution for the Riemann problem described in equations (8), depending on (η, τ), may be expressed as a function of a similarity variable $\xi = \eta / \tau$ (Smoller, 1983; Saldanha da Gama, 1990) being constructed by connecting the left (*L*) and right (*R*) states to an intermediate state (*) by rarefactions or shocks as follows: (F_L, G_L) $\rightarrow$ (F_*, G_*) $\rightarrow$ (F_R, G_R). So, the associated Riemann problem to system (8) may be expressed as:

$$\left.\begin{aligned} \frac{\partial F}{\partial \tau} + \frac{\partial G}{\partial \eta} &= 0 \\ \frac{\partial G}{\partial \tau} + \frac{\partial}{\partial \eta}\left(\frac{G^2}{F} + p(F)\right) &= 0 \end{aligned}\right\} \quad \tau > \bar{\tau}\,, \qquad -\infty < \eta < \infty$$

$$\begin{aligned} (F, G) &= (F_L, G_L) \qquad \text{for} \quad \tau = \bar{\tau}\,, \quad \eta < \bar{\eta} \\ (F, G) &= (F_R, G_R) \qquad \text{for} \quad \tau = \bar{\tau}\,, \quad \eta > \bar{\eta} \end{aligned} \tag{11}$$

The eigenvalues of (11) are $\lambda_j = G/F + (-1)^j (\tilde{p}')^{1/2} = \tilde{v} + (-1)^j (\tilde{p}')^{1/2}$, $j = 1, 2$ (in increasing order). The genuinely nonlinear hyperbolic system (11) is a Riemann problem. Its generalized solution depends on the ratio $(\eta - \bar{\eta})/(\tau - \bar{\tau})$, being reached by connecting the left state (F_L, G_L) and the right state (F_R, G_R) to an intermediate state (F_*, G_*). When the corresponding eigenvalues λ_j are increasing functions of $(\eta - \bar{\eta})/(\tau - \bar{\tau})$ between two given states, they are connected by a continuous solution – a *j*-rarefaction (Lax, 1971; John, 1974). Associated to a *j*-rarefaction, there exists a Riemann invariant – a constant between two states – given by:

$$R_j = v - (-1)^j \int \frac{\sqrt{\tilde{p}'}}{\rho} d\rho = \text{constant} \qquad j = 1, 2 \tag{12}$$

When the eigenvalues λ_j are decreasing functions of $(\eta - \bar{\eta})/(\tau - \bar{\tau})$, the states are connected by a *j*-shock (a discontinuous solution) with speed s_j. To preserve uniqueness of the weak solutions, the entropy condition must be verified. The following Rankine-Hugoniot jump conditions, associated with equation (11), must be satisfied for any two given states connected by a *j*-shock with speed s_j:

$$s_j = \frac{[G]}{[F]} = \frac{\left[G^2/F + \tilde{p}\right]}{[G]} \tag{13}$$

in which s_j represents the speed of discontinuity propagation and $[f]$, the jump of a quantity *f*.

The approximate Riemann solver employed (Saldanha da Gama and Martins-Costa, 2008) consists of assuming the solution within a space of piecewise constant functions for system (11) in which any two states are always connected by a discontinuity, i. e.: $(F_L, G_L) \rightarrow$ 1-shock $\rightarrow (F_*, G_*) \rightarrow$ 2-shock $\rightarrow (F_R, G_R)$. This approximation no longer requires considering the original four possible solutions which characterize the Riemann problem exact solution, i.e.: $(F_L, G_L) \underset{Raref \text{ or } Shock}{\Longrightarrow} (F_*, G_*) \underset{Raref \text{ or } Shock}{\Longrightarrow} (F_R, G_R)$. On the other hand, the entropy conditions are not ensured. It is to be noted that the conservation laws are satisfied in a weak sense.

The (generalized) solution of (11), within a space of piecewise constant functions, is reached as follows

$$(F, G) = \begin{cases} (F_L, G_L) & \text{if} \quad -\infty < (\eta - \bar{\eta})/(\tau - \bar{\tau}) < s_1 \\ (F_*, G_*) & \text{if} \quad s_1 < (\eta - \bar{\eta})/(\tau - \bar{\tau}) < s_2 \\ (F_R, G_R) & \text{if} \quad s_2 < (\eta - \bar{\eta})/(\tau - \bar{\tau}) < \infty \end{cases} \tag{14}$$

5. RESULTS

All depicted results have been obtained by employing Glimm's scheme with 300 steps for each time advance. Each considered case is presented in a set composed by six lines and two columns of graphs. Each line represents a distinct time instant – the first one being the initial condition, while each column corresponds to the behavior of a distinct variable – namely the saturation and the fluid constituent velocity. In all depicted results the vertical axis corresponds to the numerical value assumed by ψ and v, the horizontal one being the spatial coordinate η, as depicted in the schematic diagram presented in figure 1, representing saturation and fluid constituent velocity variation with position for a given time instant, to be employed in figures 2 to 5. The qualitative results exhibited in Figures 2 to 5 – referred as "Exact solution" and "Riemann solver" were obtained by employing a convenient normalization, in such a way that zero and unit displayed values for the variables associated with ψ and v correspond to the minimum and maximum actual values in which ψ and v are redefined as:

$$\frac{\psi-(\psi)_{\min}}{(\psi)_{\max}-(\psi)_{\min}}\rightarrow\psi \qquad \frac{v-(v)_{\min}}{(v)_{\max}-(v)_{\min}}\rightarrow v \tag{15}$$

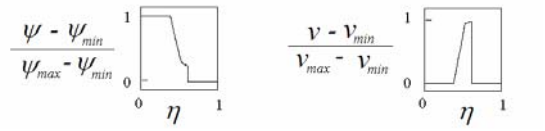

Figure 1. Schematic diagram of the figures, representing saturation and fluid constituent velocity variation with position, for a given time instant.

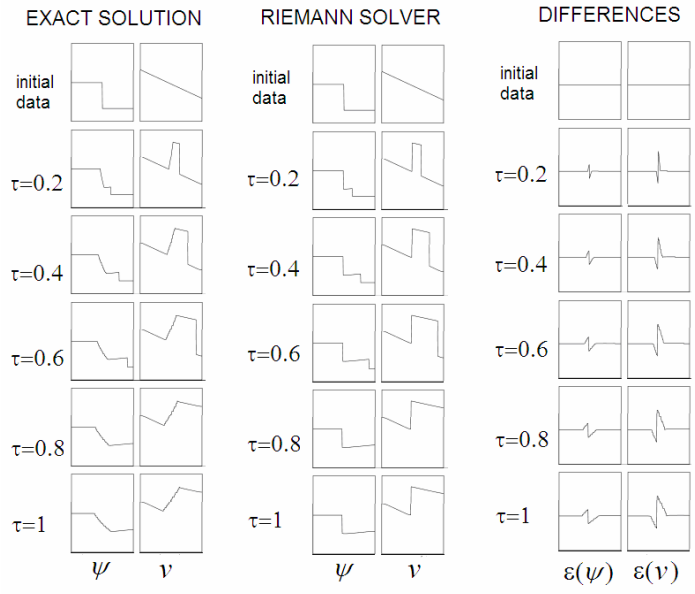

Figure 2. Saturation and fluid constituent velocity variation with position – initial data: step function for saturation with $\psi_L > \psi_R$ and linear decreasing velocity.

It is important to observe that the results referred as "Differences" in figures 2 to 5 have not been normalized, in order to better visualize the differences between both strategies employed to deal with the associated Riemann problem. Essentially, the results are defined by

$$\varepsilon(\psi) = (\psi)_{\text{approx}} - (\psi)_{\text{exact}}$$
$$\varepsilon(v) = (v)_{\text{approx}} - (v)_{\text{exact}} \tag{16}$$

with the subscript "approx" referring to the solution obtained employing the approximate Riemann solver to build Glimm's scheme while the subscript "exact" refers to the solution obtained employing the Riemann problem exact solution.

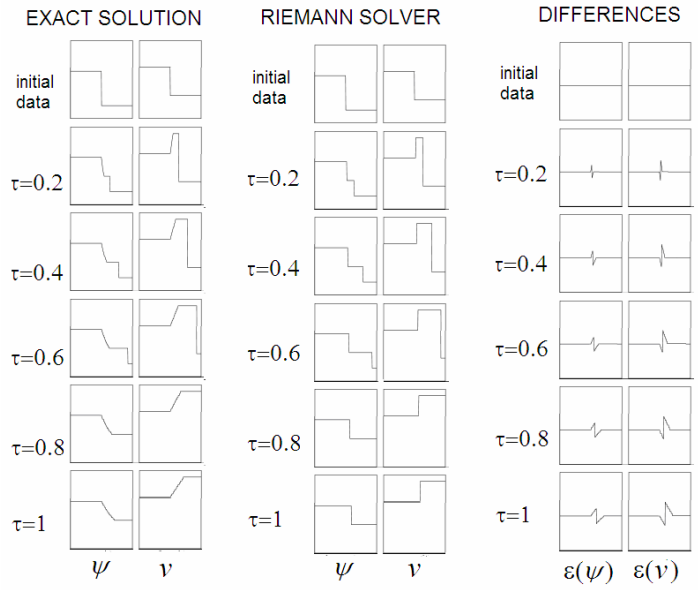

Figure 3. Saturation and fluid constituent velocity variation with position – initial data: step function for saturation and for velocity with $\psi_L > \psi_R$ and $v_L > v_R$.

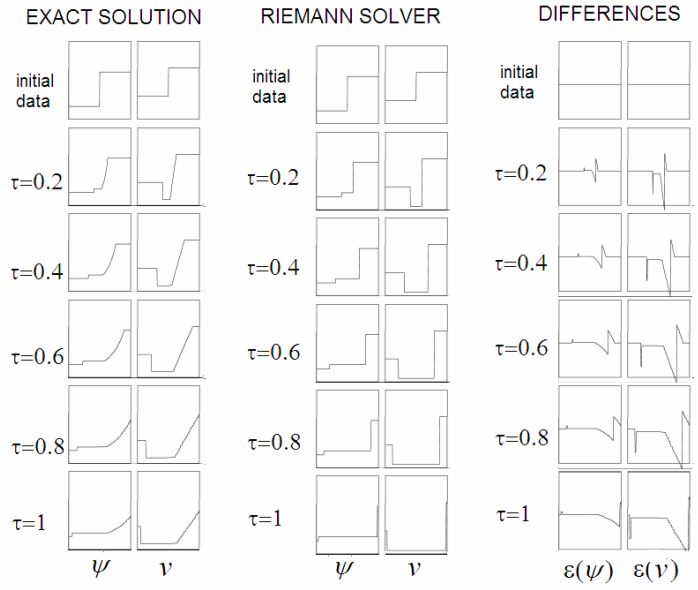

Figure 4. Saturation and fluid constituent velocity variation with position – initial data: step function for saturation and for velocity with $\psi_L < \psi_R$ and $v_L < v_R$.

The results presented in figures 2 to 5 compare the behavior of saturation and fluid constituent velocity for distinct initial values considering the implementation of Glimm's scheme obtained both by the standard procedure – the exact solution of the associated Riemann problem and the approximate Riemann solver. In all these mentioned figures, the first two columns refer to Glimm's scheme marching with the exact solution of the associated Riemann problem, while the third and fourth columns present results obtained employing the approximate Riemann solver presented in equation (14) and the difference between these two approaches is shown in the last two columns.

Figure 2 allows observing small differences between both methodologies employed to solve the associated Riemann problem: as expected the smooth connections (rarefaction connections) presented in the exact solution give rise to differences in the saturation and velocity, depicted in the last two columns. A slight increase of these differences is noticed as with the time increases. An analogous conclusion is obtained by observing figure 3. On the other hand, important differences are observed in figure 4, particularly for the velocities, whenever the Riemann problem exact solution presents states connected by rarefactions. It is interesting to notice the excellent agreement presented in figure 5, due to the states being connected by shocks in the Riemann problem exact solution.

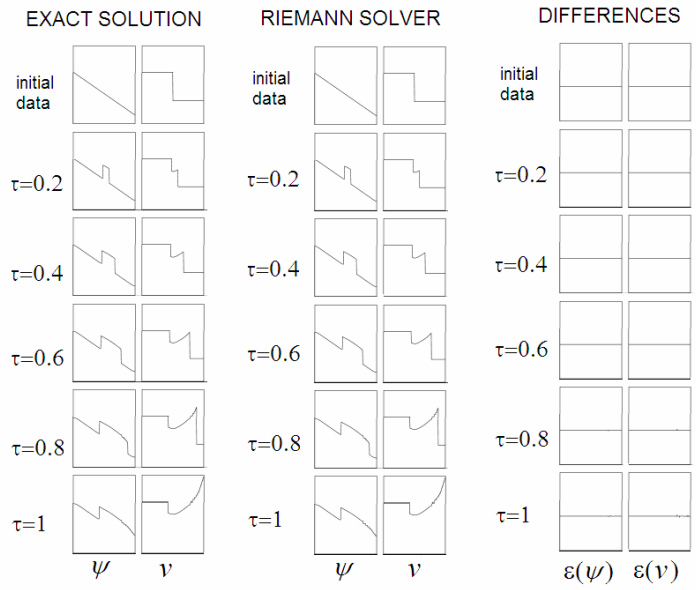

Figure 5. Saturation and fluid constituent velocity variation with position – initial data: linear decreasing saturation and step function for velocity with $v_L > v_R$.

6. CONCLUIDING REMARKS

Glimm's method is a convenient tool for solving nonlinear hyperbolic problems, presenting low storage costs and low computational effort, when compared to other numerical procedures to approximate nonlinear problems. Among the numerical methodologies currently employed to treat discontinuous problems, Glimm's scheme is the one that better preserves the shock identity. Regarding its accuracy, the method is free from numerical dissipation – preserving shock waves magnitude and presenting a deviation from the correct position smaller than the width of each step – preserving shock waves position. Besides, if the width of the steps tends to zero, Glimm's approximation tends to the exact solution of the problem. However, two important shortcomings of this methodology must be mentioned: the former its applicability being restricted to one-dimensional problems and the latter its implementation requiring previous knowledge of a solution of the associated Riemann problem – usually not an easy task.

A Riemann solver was employed to implement Glimm's method for advancing in time. The problem was also approximated by employing the usual methodology for implementing Glimm's scheme – a complete solution of the associated Riemann problem. Comparison of qualitative results employing these two solutions has shown very good agreement, considering distinct initial data.

In order to better compare the results between both methodologies to deal with the associated Riemann problem, an important simplification was introduced in the mechanical model for the flow through porous media – the Darcian term has been neglected. Also, a Cartesian geometry was considered, giving rise to a homogeneous hyperbolic system. A non-homogeneous system could have been solved by employing an operator splitting technique (see Martins-Costa and

Saldanha da Gama, 2005 and references therein), which decomposes the operator by splitting away its hyperbolic part from the purely time evolutionary one, so that the simultaneous problem is approximated by a sequential one.

7. ACKNOWLEDGEMENTS

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