A DISCRETE-ORDINATES SOLUTION FOR A RADIATION THERAPY PROBLEM

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Abstract. *A concise and accurate procedure for evaluating dose distribution, in a radiation therapy planning, is presented. The analytical discrete-ordinates method (ADO method) is used to develop a complete solution for a spectral dependent radiative transfer equation, in a one-dimensional medium, according to a multigroup scheme. Numerical results are presented for test problems, where the Klein-Nishina scattering kernel was used to describe the interaction processes.*

Keywords: *Radiotherapy, Discrete-ordinates method, Spectral dependence*

1. INTRODUCTION

Radiotherapy is the medical use of ionizing radiation as part, for instance, of a cancer treatment (Gale Encyclopedia of Medicine, 2006). Due to the harmful effects of ionizing radiation, it is very important to deal efficiently, as much as possible, with the problem of evaluating the radiation fluxes.

The basic model to describe radiative transfer is through the Boltzmann equation for photons and other particles (Wood, 1982), which is a very complex formulation. Still, in particular, the energy variable may be of special interest in those problems. In this sense, the search of numerical and computational tools (Attix, 1986) to handle with the radiation therapy problem is of great interest. In the context of deterministic approaches, the solution of this problem has been also investigated over the years (Sauer, 1997; Trindade, 1997; Rodrigues, 2007).

In this work, we use the ADO method (Barichello and Siewert, 1999), to solve the radiative transfer equation including the energy dependence (Siewert, 2000), in order to develop an efficient way of evaluating dose distribution in a radiotherapy problem. Numerical results for various energy levels are presented. In fact, the definition of the energy intervals is associated with the use of numerical quadrature rules: the Gauss-Legendre quadrature scheme and the Simpson's rule.

In this way, this paper is organized such that we present in the next section the formulation of the problem. Continuing, in section 3, we deal with the energy dependence of the problem. In section 4, we develop the discrete-ordinates solution for the problem of interest. Numerical results are discussed in section 5.

2. FORMULATION OF THE PROBLEM

We write the homogeneous radiative transfer equation, for the particular case of one-dimensional medium, steady-state case, as (Wood, 1982)

$$\mu\frac{\partial\Phi}{\partial x}(x,E,\vec{\Omega})+\sigma(E)\Phi(x,E,\vec{\Omega})=\int_{4\pi}\int_{0}^{\infty}\Sigma(E'\rightarrow E;\vec{\Omega}'\rightarrow\vec{\Omega})\Phi(x,E',\vec{\Omega}')dE'd\vec{\Omega}', \tag{1}$$

where $\Phi(x,E,\vec{\Omega})$ is the angular flux density of the particles, $\vec{\Omega}$ is the unitary particle direction vector, E is the energy variable and $\sigma(E)$ is the linear attenuation coefficient. Still, $\Sigma(E'\rightarrow E;\vec{\Omega}'\rightarrow\vec{\Omega})$ denotes the double-differential scattering kernel for the medium (Wood, 1982).

However, when considering the behavior of photons, in particular, it seems more convenient (Wood, 1982) to refer to the quantity called angular energy flux density, such that

$$I(\vec{r},E,\vec{\Omega})=E\Phi(\vec{r},E,\vec{\Omega})\,. \tag{2}$$

In addition, it is also more usual to work with the wavelength of the photon, λ, in Compton units, instead of using the energy variable, E. In other words,

$$\lambda=\frac{0,511}{E}\,, \tag{3}$$

where E is in Mev units.

Continuing, in regard to Eq. (1), we consider

$$\hat{\Sigma}(\lambda' \to \lambda; \vec{\Omega}' \to \vec{\Omega}) = \frac{1}{2\pi} \sum_{l=0}^{\infty} \frac{2l+1}{2} \hat{\Sigma}_l(\lambda', \lambda) P_l(\mu_0) \tag{4}$$

with

$$\hat{\Sigma}_l(\lambda', \lambda) = 2\pi \int_{-1}^{1} \hat{\Sigma}(\lambda' \to \lambda; \vec{\Omega}' \to \vec{\Omega}) P_l(\mu_0) d\mu_0 \tag{5}$$

where $P_l(\mu)$ are the Lengendre polynomials and $\mu_0 = \vec{\Omega}'.\vec{\Omega}$, is the cosine of the phase angle.

In this way, we finally express the basic equation for the formulation of our problem as

$$\mu \frac{\partial}{\partial x} I(x, \lambda, \mu) + \sigma(\lambda) I(x, \lambda, \mu) = \sum_{l=0}^{L} \frac{2l+1}{2} \int_{-1}^{1} \int_{0}^{\lambda} \hat{\Sigma}_l(\lambda', \lambda) P_l(\mu) P_l(\mu') I(x, \lambda', \mu') d\lambda' d\mu', \tag{6}$$

for $\mu \in [-1, 1]$ the cosine of the angle between the particle direction and the x-axis. Here, we supplement the equation with the following boundary conditions

$$I(0, \lambda, \mu) = F_1(\lambda, \mu), \tag{7}$$

and

$$I(x_0, \lambda, -\mu) = F_2(\lambda, \mu)\,, \tag{8}$$

for $x \in [0, x_0]$.

Now, if the kernel in the scattering integral is related to the Klein-Nishina double-differential cross section for Compton scattering (Wood, 1982), we can write, in Eq. (6),

$$\hat{\Sigma}_l(\lambda', \lambda) = \alpha K(\lambda', \lambda) P_l(1 + \lambda' - \lambda) \tag{9}$$

with $\alpha = N_A Z \sigma_T \rho / A = 0,40061 Z\rho/A\ [cm^{-1}]$, A the atomic mass, Z the atomic number, N_A the number of atoms, σ_T the cross section, ρ the density and

$$K(\lambda', \lambda) = \begin{cases} \frac{3}{8} \frac{\lambda'}{\lambda} [\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} + 2(\lambda' - \lambda) + (\lambda' - \lambda)^2] & \lambda' \le \lambda \le \lambda' + 2 \\ 0 & \lambda \notin [\lambda', \lambda' + 2] \end{cases} \tag{10}$$

Then, the final form for Eq. (6) becomes

$$\mu \frac{\partial}{\partial x} I(x, \lambda, \mu) + \sigma(\lambda) I(x, \lambda, \mu) = \alpha \sum_{l=0}^{L} \frac{2l+1}{2} P_l(\mu) \times \int_{\lambda_0}^{\lambda_0 + 2} \int_{-1}^{1} K(\lambda', \lambda) P_l(1 + \lambda' - \lambda) P_l(\mu') I(x, \lambda', \mu') d\mu' d\lambda'. \tag{11}$$

Here, λ_0, is related to the incoming energy, E_0, by the expression

$$\lambda_0 = \frac{0,511}{E_0[Mev]}. \tag{12}$$

3. A DISCRETE APPROACH FOR THE ENERGY

In this section we develop a numerical scheme for treating the spectral dependence of the problem given by Eq. (11). In particular, two different approaches are introduced to define intervals of interest for the variable λ.

3.1 The Gauss-Legendre Quadrature Scheme

To approximate the integral term, related to the wavelenght, in Eq. (11), we first chose to map the interval $[\lambda_0, \lambda_0 + 2]$, into $[-1, 1]$, by using the transformation,

$$\lambda_* = \lambda - \lambda_0 - 1, \tag{13}$$

in order to use the well known Gauss-Legendre quadrature scheme (Atkinson, 1988). In this way, we obtain a discrete version of Eq. (11), written, for $j = 1, \cdots, M$, as (Sauer, 1997)

$$\mu\frac{\partial}{\partial x}I(x,\lambda_{*j},\mu)+\sigma(\lambda_{*j})I(x,\lambda_{*j},\mu) = \alpha\sum_{l=0}^{L}\frac{2l+1}{2}P_l(\mu)\sum_{r=1}^{M}K(\lambda_{*r}+\lambda_0+1,\lambda_{*j}+\lambda_0+1)P_l(1+\lambda_{*r}-\lambda_{*j})\varpi_r \times\int_{-1}^{1}P_l(\mu')I(x,\lambda_{*r},\mu')d\mu'. \quad (14)$$

Here the subscripts r and j are used to represent the photon's energy before and after the collisions and λ_* denotes the nodes and ϖ the weights of the quadrature scheme. We note that M will be used to define the number of intervals in which the domain of the variable λ will be discretized.

We now let,

$$K_{rj} \equiv K(\lambda_{*r}+\lambda_0+1,\lambda_{*j}+\lambda_0+1) \quad (15)$$

and we define the $M \times M$ matrix $\mathbf{C}_l$, with elements given by

$$(C_l)_{i,j} = \alpha K_{ji}\varpi_j P_l(1+\lambda_{*j}-\lambda_{*i}), \qquad i,j = 1,2,\ldots,M \quad (16)$$

such that we can write Eq. (14) in a matrix form,

$$\mu\frac{\partial}{\partial x}\hat{\mathbf{I}}(x,\mu)+\mathbf{\Sigma}\hat{\mathbf{I}}(x,\mu) = \frac{1}{2}\sum_{l=0}^{l}(2l+1)P_l(\mu)\mathbf{C}_l\int_{-1}^{1}P_l(\mu')\hat{\mathbf{I}}(x,\mu')d\mu', \quad (17)$$

for $x \in (0,x_0)$ and $\mu \in [-1,1]$. Here the components of the $M \times 1$ vector $\hat{\mathbf{I}}(x,\mu)$ are defined by $I(x,\lambda_{*j},\mu)$ and $\mathbf{\Sigma}$ is a diagonal matrix with elements $\sigma_j = \sigma(\lambda_{*j})$. The boundary conditions, Eqs. (7) and (8) are also (consistently) written in a matrix form, as

$$\hat{\mathbf{I}}(0,\mu) = \mathbf{F}_1(\mu) \quad (18)$$

and

$$\hat{\mathbf{I}}(x_0,-\mu) = \mathbf{F}_2(\mu), \quad (19)$$

for $\mu \in (0,1]$.

3.2 The Simpson's Rule

Here we follow analogous procedure, and use the Simpson's rule (Atkinson, 1988) to approximate the (wavelenght) integral term, in Eq. (11), such that the formulation for the problem is

$$\mu\frac{\partial}{\partial x}I(x,\lambda_j,\mu)+\sigma(\lambda_j)I(x,\lambda_j,\mu) = \alpha\frac{\Delta}{3}\sum_{l=0}^{L}\frac{2l+1}{2}P_l(\mu)\sum_{r=1}^{M}c_rK(\lambda_r,\lambda_j)P_l(1+\lambda_r-\lambda_j) \times\int_{-1}^{1}P_l(\mu')I(x,\lambda_r,\mu')d\mu'. \quad (20)$$

where, by definition,

$$\Delta = \frac{2}{m}, \quad (21)$$

$$c_1 = c_{m+1} = 1, \quad (22)$$

$$c_r = \begin{cases} 4 & for\ \ r\ \ even \\ 2 & for\ \ r\ \ odd, \end{cases} \quad (23)$$

$$\lambda_r = \lambda_0 + k\Delta, k = 0,1,2,...,m, \quad (24)$$

for $m = M-1$, where M is the odd number of energy groups. We note that, this scheme has the advantage of including the intervals limits in the quadrature scheme. Here we define the $\mathbf{C}_l$ matrix elements as

$$(C_l)_{i,j} = \alpha\frac{\Delta}{3}K(\lambda_j,\lambda_i)c_jP_l(1+\lambda_j-\lambda_i), \qquad i,j = 1,2,\ldots,M \quad (25)$$

such that, the final form of the problem, when using Simpson's rule, is the same as before – given by Eq. (17) with boundary conditions defined in Eqs. (18) and (19).

4. THE DISCRETE-ORDINATES SOLUTION

To develop a discrete-ordinates solution for the problem defined by Eqs. (17) to (19), we follow Siewert (Siewert, 2000) and rewrite them as

$$\mu_i \frac{d}{dx}\hat{\mathbf{I}}(x,\mu_i) + \mathbf{\Sigma}\hat{\mathbf{I}}(x,\mu_i) = \frac{1}{2}\sum_{l=0}^{L}(2l+1)P_l(\mu_i)\mathbf{C}_l \sum_{s=1}^{N} w_s \mathbf{I}_{l,s}(x) \tag{26}$$

and

$$-\mu_i \frac{d}{dx}\hat{\mathbf{I}}(x,-\mu_i) + \mathbf{\Sigma}\hat{\mathbf{I}}(x,-\mu_i) = \frac{1}{2}\sum_{l=0}^{L}(-1)^l(2l+1)P_l(\mu_i)\mathbf{C}_l \sum_{s=1}^{N} w_s \mathbf{I}_{l,s}(x), \tag{27}$$

for $i = 1, \cdots, N$, and where

$$\mathbf{I}_{l,s}(x) = P_l(\mu_s)[\hat{\mathbf{I}}(x,\mu_s) + (-1)^l \hat{\mathbf{I}}(x,-\mu_s)]. \tag{28}$$

Here, we note that the (arbitrary) N quadrature nodes $\{\mu_s\}$ and weights $\{w_s\}$ are to be defined in the interval $[0,1]$.

Seeking for exponential solutions, we write

$$\hat{\mathbf{I}}(x,\pm\mu_i) = \mathbf{\Phi}(\nu,\pm\mu_i)e^{-x/\nu} \tag{29}$$

and we substitute this expression into Eqs. (26) and (27) to obtain

$$\mu_i(-\frac{1}{\nu})\mathbf{\Phi}(\nu,\mu_i)e^{-x/\nu} + \mathbf{\Sigma}\mathbf{\Phi}(\nu,\mu_i)e^{-x/\nu} = \frac{1}{2}\sum_{l=0}^{L}(2l+1)P_l(\mu_i)\mathbf{C}_l \sum_{s=1}^{N} w_s \mathbf{\Phi}_{l,s}(\nu)e^{-x/\nu} \tag{30}$$

and

$$-\mu_i(-\frac{1}{\nu})\mathbf{\Phi}(\nu,-\mu_i)e^{-x/\nu} + \mathbf{\Sigma}\mathbf{\Phi}(\nu,-\mu_i)e^{-x/\nu} = \frac{1}{2}\sum_{l=0}^{L}(-1)^l(2l+1)P_l(\mu_i)\mathbf{C}_l \sum_{s=1}^{N} w_s \mathbf{\Phi}_{l,s}(\nu)e^{-x/\nu}\,. \tag{31}$$

After some algebraic manipulation, we write, for $i = 1, \cdots, M$, the system

$$(\mathbf{\Sigma} - \frac{\mu_i}{\nu}\mathbf{I})\mathbf{\Phi}(\nu,\mu_i) = \frac{1}{2}\sum_{l=0}^{L}(2l+1)P_l(\mu_i)\mathbf{C}_l \sum_{s=1}^{N} w_s \mathbf{\Phi}_{l,s}(\nu)\,, \tag{32}$$

$$(\mathbf{\Sigma} + \frac{\mu_i}{\nu}\mathbf{I})\mathbf{\Phi}(\nu,-\mu_i) = \frac{1}{2}\sum_{l=0}^{L}(-1)^l(2l+1)P_l(\mu_i)\mathbf{C}_l \sum_{s=1}^{N} w_s \mathbf{\Phi}_{l,s}(\nu), \tag{33}$$

where

$$\mathbf{\Phi}_{l,s} = P_l(\mu_s)[\mathbf{\Phi}(\nu,\mu_s) + (-1)^l \mathbf{\Phi}(\nu,-\mu_s)] \tag{34}$$

and, here, $\mathbf{I}$ is the $M \times M$ identity matrix.

We now introduce the MN vectors:

$$\mathbf{\Phi}_+(\nu) = [\mathbf{\Phi}^T(\nu,\mu_1), \mathbf{\Phi}^T(\nu,\mu_2), ..., \mathbf{\Phi}^T(\nu,\mu_N)]^T \tag{35}$$

and

$$\mathbf{\Phi}_-(\nu) = [\mathbf{\Phi}^T(\nu,-\mu_1), \mathbf{\Phi}^T(\nu,-\mu_2), ..., \mathbf{\Phi}^T(\nu,-\mu_N)]^T, \tag{36}$$

where T denotes the transpose operation, and the $(MN \times MN)$ square matrices

$$\mathbf{W} = diag\{..., w_i\mathbf{I}, ...\}\,, \tag{37}$$

$$\mathbf{M} = diag\{..., \mu_i\mathbf{I}, ...\}\,, \tag{38}$$

$$\mathbf{D} = diag\{..., \mathbf{\Sigma}, ...\} \tag{39}$$

and the $(MN \times M)$ matrix

$$\mathbf{\Pi}_l = [P_l(\mu_1)\mathbf{I}, P_l(\mu_2)\mathbf{I}, ..., P_l(\mu_N)\mathbf{I}]^T. \tag{40}$$

In this way, Eqs. (32) and (33) can now be written as

$$(\mathbf{D} - \frac{1}{\nu}\mathbf{M})\mathbf{\Phi}_+(\nu) = \frac{1}{2}\sum_{l=0}^{L}(2l+1)\mathbf{\Pi}_l\mathbf{C}_l\mathbf{G}_l(\nu) \tag{41}$$

and

$$(\mathbf{D} + \frac{1}{\nu}\mathbf{M})\mathbf{\Phi}_-(\nu) = \frac{1}{2}\sum_{l=0}^{L}(-1)^l(2l+1)\mathbf{\Pi}_l\mathbf{C}_l\mathbf{G}_l(\nu), \tag{42}$$

with

$$\mathbf{G}_l(\nu) = \mathbf{\Pi}_l^T\mathbf{W}[\mathbf{\Phi}_+(\nu) + (-1)^l\mathbf{\Phi}_-(\nu)]\,. \tag{43}$$

We now let

$$\mathbf{U} = \mathbf{\Phi}_+(\nu) + \mathbf{\Phi}_-(\nu) \tag{44}$$

and

$$\mathbf{V} = \mathbf{\Phi}_+(\nu) - \mathbf{\Phi}_-(\nu) \tag{45}$$

so that we can take the sum and the difference of Eqs. (41) and (42) to obtain

$$\mathbf{EX} = \frac{1}{\nu}\mathbf{Y} \tag{46}$$

and

$$\mathbf{HY} = \frac{1}{\nu}\mathbf{X}\,, \tag{47}$$

where

$$\mathbf{E} = (\mathbf{D} - \frac{1}{2}\sum_{l=0}^{L}(2l+1)\mathbf{\Pi}_l\mathbf{C}_l[1 + (-1)^l]\mathbf{\Pi}_l^T\mathbf{W})\mathbf{M}^{-1}\,, \tag{48}$$

$$\mathbf{H} = (\mathbf{D} - \frac{1}{2}\sum_{l=0}^{L}(2l+1)\mathbf{\Pi}_l\mathbf{C}_l[1 - (-1)^l]\mathbf{\Pi}_l^T\mathbf{W})\mathbf{M}^{-1}\,, \tag{49}$$

$$\mathbf{X} = \mathbf{MU} \tag{50}$$

and

$$\mathbf{Y} = \mathbf{MV}\,. \tag{51}$$

In other words, we obtain two eigenvalue problems

$$(\mathbf{HE})\mathbf{X} = \lambda\mathbf{X} \tag{52}$$

and

$$(\mathbf{EH})\mathbf{Y} = \lambda\mathbf{Y} \tag{53}$$

where $\lambda = 1/\nu^2$.

The eigenvalues λ_j and the corresponding eigenvectors $\mathbf{X}(\lambda_j)$, $j = 1, 2, ..., MN$ are used to express

$$\mathbf{\Phi}_+(\nu_j) = \frac{1}{2}\mathbf{M}^{-1}(\mathbf{I} + \nu_j\mathbf{E})\mathbf{X}(\lambda_j) \tag{54}$$

and

$$\mathbf{\Phi}_{-}(\nu_j) = \frac{1}{2}\mathbf{M}^{-1}(\mathbf{I} - \nu_j\mathbf{E})\mathbf{X}(\lambda_j), \tag{55}$$

for $j = 1, 2, .., MN$. The matrix $\mathbf{I}$ is the $MN \times MN$ identity matrix.

If we now define

$$\hat{\mathbf{I}}_{+}(x) = [\hat{\mathbf{I}}^T(x, \mu_1), \hat{\mathbf{I}}^T(x, \mu_2), ..., \hat{\mathbf{I}}^T(x, \mu_N)]^T\,, \tag{56}$$

$$\hat{\mathbf{I}}_{-}(x) = [\hat{\mathbf{I}}^T(x, -\mu_1), \hat{\mathbf{I}}^T(x, -\mu_2), ..., \hat{\mathbf{I}}^T(x, -\mu_N)]^T \tag{57}$$

we can express the discrete-ordinates solution to the homogeneous problem, given by Eqs. (26) and (27) as

$$\hat{\mathbf{I}}_{+}(x) = \sum_{j=1}^{MN}[A_j\mathbf{\Phi}_{+}(\nu_j)e^{-x/\nu_j} + B_j\mathbf{\Phi}_{-}(\nu_j)e^{-(x_0-x)/\nu_j}] \tag{58}$$

and

$$\hat{\mathbf{I}}_{-}(x) = \sum_{j=1}^{MN}[A_j\mathbf{\Phi}_{-}(\nu_j)e^{-x/\nu_j} + B_j\mathbf{\Phi}_{+}(\nu_j)e^{-(x_0-x)/\nu_j}] \tag{59}$$

where the arbitrary constants A_j and B_j are to be determined from the linear system we obtain from the boundary conditions (evaluated at the quadrature points).

5. NUMERICAL RESULTS

5.1 Quantities of Interest

The amount of energy deposited by the radiation is referred to as the absorbed dose (Wood, 1982),

$$D_T(x, \lambda_i) = E_i\mu_{ai}\hat{\mathbf{\Phi}}(x, \lambda_i)\,, \tag{60}$$

which is defined in terms of the scalar flux of energy (i), $\hat{\mathbf{\Phi}}(x, \lambda_i)$, as (Dudersdadt, 1979)

$$\hat{\mathbf{\Phi}}(x, \lambda_i) = \int_{-1}^{1} I(x, \lambda_i, \mu)d\mu. \tag{61}$$

Here, $\mu_{ai} \equiv \mu_a(\lambda_i)$ is the absorption coefficient of the medium (in cm^2/g) for the energy $E_i(Mev)$, given by Eq. (3).

In addition, if we look back to Eq. (60) and we consider the contribution of all the energy levels, we define the total ionizing dose (TID), expressed as

$$D_T(x) = \sum_{i=0}^{M}\mu_{ai}E_i\hat{\mathbf{\Phi}}(x, \lambda_i)\,. \tag{62}$$

5.2 Computational Aspects and Numerical Results

We implemented two independent FORTRAN programs, for each problem, to evaluate the solution developed by the ADO method. One implementation was executed in a 1.73GHz Centrino Duo notebook computer, taking less than 5s to generate the results. The nuclear coefficients of the materials were taken from the site (http://physics.nist.gov), and estimated, when necessary, by linear interpolation. In order to have some confidence in our results, we first reproduced results available in the literature. Results for the scalar flux were compared with previous work by Sauer (Sauer, 1997), from where we found good agreement, by the possible graphic analysis. Results for the dose were compared with the work by (Trindade, 1997), having an excellent agreement. Both previous works used another approach (the LTS_N and LTP_N methods). However, those results were obtained for a very low order of quadrature approximation, in regard to the angular variable (N). So, here, we present, in Tables 1 to 4, results obtained for $N = 20$, which we believe to be correct in all digits listed (plus or minus one in the last digit) – noting, however, that most nuclear coefficients have only 4 digits of precision. In fact, we have varied the N in order to find the digits which are preserved.

We consider a water layer with $x_0 = 20\ cm$ length, and the following boundary conditions:

$$I(0, \lambda_0, \mu) = 1 \quad \text{for } \mu > 0, \tag{63}$$

$$I(0, \lambda_i, \mu) = 0 \quad \text{for } \mu > 0 \quad \text{and } \ i = 1, 2, ..., M, \tag{64}$$

and

$$I(20, \lambda_i, \mu) = 0 \quad \text{for } \mu < 0, \tag{65}$$

for $i = 0, 1, 2, ..., M$.

Scalar fluxes, evaluated by Gauss Quadrature, for four groups of energy and incoming energy of $2Mev$, are shown in Table 1, and can be analyzed in more detail in Figure 1:

Table 1. Scalar fluxes for four groups of energy using Gauss quadrature

x (cm)	$\hat{\mathbf{\Phi}}_1(x)$	$\hat{\mathbf{\Phi}}_2(x)$	$\hat{\mathbf{\Phi}}_3(x)$	$\hat{\mathbf{\Phi}}_4(x)$
0	1.268882	0.205868	0.215991	0.206270
10	0.568027	0.315000	0.365761	0.346884
20	0.234137	0.125293	0.133550	0.121321

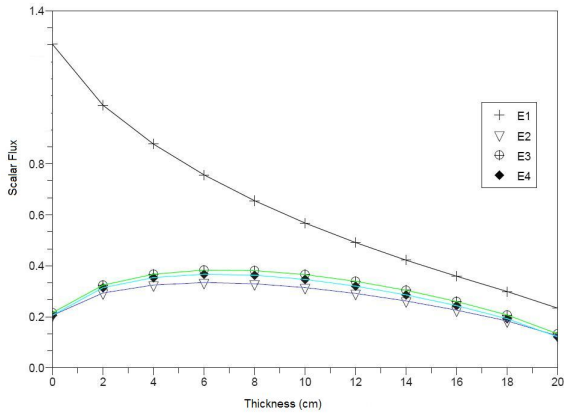

Figure 1. Scalar fluxes for four groups of energy using Gauss Quadrature

In Tables 2 and 3 we list results for the total dose, respectively, for 4 and 5 groups of energy.

Table 2. Dose: Gauss quadrature, 4 groups ($M = 4$), $E_0 = 2Mev$

x (cm)	$D_T(x)$
0	5.57272E-002
4	4.57239E-002
10	3.37190E-002
16	2.23056E-002
20	1.34764E-002

Table 3. Dose: Simpson's rule, 5 groups ($M = 5$) , $E_0 = 2Mev$

x (cm)	$D_T(x)$
0	6.37998E-002
4	4.70484E-002
10	3.28848E-002
16	2.20162E-002
20	1.46594E-002

We also consider the problem where the incoming energy was $1Mev$ in a $10cm$ deep water layer. Results, for (TID), for 8 and 10 groups of energy, obtained by using Gauss quadrature to represent the energy levels, are shown in Table 4. The goal here was to briefly analyze the influence of the number of energy levels in the evaluation of the total dose.

Table 4. Dose: Gauss quadrature, 8 and 10 groups, $E_0 = 1Mev$

x (cm)	$D_T(x)$ Gauss(8)	$D_T(x)$ Gauss(10)
0	3.27436E-002	3.26298E-002
5	1.65091E-002	1.58718E-002
10	8.77348E-003	8.47728E-003

6. CONCLUDING REMARKS

The ADO solution has shown to be concise and accurate, which allowed us to perform a very wide series of simulations, in order to analyze the choices of different quadrature schemes and levels of energy. This work is, in fact, part of other project where matrix problems as the one defined in Eq. (17), are part of a more general formulation where the spectral dependence is proposed to be continuous. To validate the present formulation was one of the aspects we sought here. We also note that, although the problems here solved were treated by different approaches earlier, the ADO approach reduces the order of the eigenvalue systems associated with the developed solution, which can mean significant improvement in terms of efficiency, when more complex problems will be addressed as extension of this work.

7. ACKNOWLEDGEMENT

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