Solution of the neutron transport equation in one-dimensional cartesian geometry for bounded and unbounded domain

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Abstract. *In this work we report an analytical solution for the time-dependent one-dimensional neutron transport equation in cartesian geometry for bounded and unbounded domain. The main idea consists in the application of the Laplace transform technique in time variable, solution of the resulting equation by the LTS*$_{\rm N}$ *method and reconstruction of the angular flux in time-variable by numerical inversion scheme. We report numerical simulations and results validations with the ones of literature.*

Keywords: *keyword 1, keyword 2, keyword 3… (up to 5 keywords)*

1. INTRODUCTION

Exists an extensive literature covering the subject of solution of the one-dimensional time-dependent transport equation in cartesian geometry, for unbounded domain. For illustration, we mention the works of [3][2][4][5][6][12][13]. However, to our knowledge, the methods appearing in those works aren't applied in the solution of the transport problems in a slab. On the other hand, recently the LTS$_{\rm N}$ method [14][9][10] solved this sort of problem in a bounded domain, we mean a slab [8]. The main idea consists of the steps: application of the Laplace transform in time variable, solution of the resulting equation by the LTS$_{\rm N}$ method and reconstruction of the angular flux in time-variable by the numerical inversion using the Gaussian quadrature scheme. We must recall that this methodology is coined as TLTS$_{\rm N}$ approach. To underline the generality of this method, in the sense it can be applied either for bounded and unbounded domain, in this work we step further by solving this time-dependent problem by this methodology for unbounded domain. For such, due the analytical feature of the solution expressed in matrix form, we just replace the boundary condition at the thickness ($x = L$) of the slab by the boundness of the angular flux at infinity. To hit this goal we vanish the sub-vector of arbitrary constants associated to the set of the positive eigenvalues. We outline the paper as follows: in section 2, we construct the LTS$_{\rm N}$ solution for either the bounded and unbounded domain and in section 3 we report numerical simulations and results validation with the ones of literature.

2. THE LTS$_{\rm N}$ SOLUTION FOR TIME-DEPENDENT PROBLEM

In order to construct the LTS$_{\rm N}$ solution for the time-dependent, one-dimensional neutron transport problem in cartesian geometry for unbounded domain, in the sequel, we briefly discuss the solution derivation of this type of problem by the LTS$_{\rm N}$ method for a slab. So far, let us consider the following isotropic transport time-dependent problem:

$$\frac{1}{v}\frac{\partial}{\partial t}\psi(t,x,\mu) + \mu\frac{\partial}{\partial x}\psi(t,x,\mu) + \sigma_t\psi(t,x,\mu) = \frac{\sigma_s}{2}\int_{-1}^{1}\psi(t,x,\mu')d\mu' + S(t,x) \qquad (1)$$

for $0 < x < d$, with the initial condition

$$\psi(0,x,\mu) = \phi(x,\mu) \qquad (2)$$

and the incident flux boundary conditions,

$$\psi(t,0,\mu) = f(t,\mu), \quad \text{for } t > 0, \;\; \mu > 0, \qquad (3)$$

and

$$\psi(t,d,\mu) = 0, \quad \text{for } t > 0, \;\; \mu < 0. \qquad (4)$$

Here, we adopt the standard notation for the parameters. Applying the Laplace transform technique in the time variable in equation (1), we come out with the equation:

$$\mu\frac{\partial}{\partial x}\Psi(p,x,\mu) + \sigma_t^p\Psi(p,x,\mu) = \frac{\sigma_s}{2}\int_{-1}^{1}\Psi(p,x,\mu')d\mu' + R(p,x,\mu), \qquad (5)$$

with the boundary condition

$$\Psi(p,0,\mu) = f(p,\mu), \quad \text{for} \quad \mu > 0, \tag{6}$$

and

$$\Psi(p,d,\mu) = 0, \quad \text{for} \quad \mu < 0. \tag{7}$$

Here $\Psi(p,x,\mu)$ denotes the Laplace transform of $\psi(t,x,\mu)$; $t \to p$, $\sigma_t^p = 1 + \frac{p}{v}$ and $R(p,x,\mu) = \frac{1}{v}\phi(x,\mu) + \bar{S}(p,x)$. The S_N approximation of the above ansatz reads like:

$$\mu_n \frac{d}{dx}\Psi_n(p,x) + \sigma_t^p \Psi_n(p,x) = \frac{\sigma_s}{2}\sum_{i=1}^{N} w_i \Psi_i(p,x) w_i + R_n(p,x), \tag{8}$$

with to the boundary condition

$$\Psi_n(p,0) = f_n(p), \quad \text{for} \quad \mu_n > 0, \tag{9}$$

and

$$\Psi_m(p,d) = 0, \quad \text{for} \quad \mu_n < 0. \tag{10}$$

Here, μ_n are the N roots of the N^{th} degree Legendre Polynomial, ordering in a decrease order, $-1 < \mu_N < ... < \mu_{N/2+1} < 0 < \mu_{N/2} < ... < \mu_N < 1$, $\Psi_n(p,x)$ is the transformed angular flux at the discrete direction μ_n, and $R_n(p,x)$ is the transformed source term. Recasting equation (8) in matrix form, we have:

$$\frac{d}{dx}\mathbf{\Psi}(p,x) - \mathbf{A}\mathbf{\Psi}(p,x) = \mathbf{R}(p,x) \tag{11}$$

where $\mathbf{A}(p)$ a N order matrix whose entries are

$$a_{ij} = \begin{cases} \dfrac{\sigma_s w_j}{2\mu_i} - \dfrac{\sigma_t^p}{\mu_i} & if \ i = j, \\ \dfrac{\sigma_s w_j}{2\mu_i} & if \ i \neq j. \end{cases} \tag{12}$$

and boundary conditions:

$$\Psi_1(x) = \mathbf{f} \quad and \quad \Psi_2(x) = 0 \tag{13}$$

Here $\Psi_1(x)$ and $\Psi_2(x)$ denote the $N/2$ order vectors for respectively the positive and negative μ directions. The well known LTS_N solution for (11) problem has the form:

$$\begin{pmatrix} \mathbf{\Psi_1}(p,x) \\ \mathbf{\Psi_2}(p,x) \end{pmatrix} = \begin{pmatrix} \mathbf{X_{11}}(p,x) & \mathbf{X_{12}}(p,x) \\ \mathbf{X_{21}}(p,x) & \mathbf{X_{22}}(p,x) \end{pmatrix} \begin{pmatrix} \mathbf{e}^{\mathbf{D}^+ x} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{\mathbf{D}^- x} \end{pmatrix} \begin{pmatrix} \xi_\mathbf{1}(p) \\ \xi_\mathbf{2}(p) \end{pmatrix} + \begin{pmatrix} \mathbf{H_1}(p,x) \\ \mathbf{H_2}(p,x) \end{pmatrix} \tag{14}$$

Here $\mathbf{D}^{\pm}$ are respectively the positive and negative eigenvalues diagonal matrices of $N/2$ order and $\mathbf{X}(p)$ is the eigenvector matrix of $\mathbf{A}(p)$ appearing in equation (12). The particular solution $\mathbf{H}(p,x)$ is written as:

$$\mathbf{H}(p,x) = \mathbf{H}^+(p,x) + \mathbf{H}^-(p,x) = \int_L^x \mathbf{B}^+(p,x-\zeta)\mathbf{R}(p,\zeta)d\zeta + \int_0^x \mathbf{B}^-(p,x-\zeta)\mathbf{R}(p,\zeta)d\zeta. \tag{15}$$

where

$$\mathbf{B}(p,x) = \mathbf{X}(p)\mathbf{e}^{\mathbf{D}(p)x}\mathbf{X}^{-1}(p), \tag{16}$$

At this point we are in position to construct the solution for the unbounded domain. For such, we replace the boundary condition given by equation (4) by the boundness of the angular flux at infinity. We fulfill this condition forcing that the $N/2$ component of the unknown vector ξ_1, appearing in equation (14), are identically null. Then applying the boundary condition (3) at $x = 0$, we determine the remaining unknown sub-vector ξ_2 by solving the resulting linear system. Once the vector of arbitrary constants are determinate, the final solution is expressed like:

$$\mathbf{\Psi}_1(p,x) = \mathbf{x}_{12}(p,x)\mathbf{e}^{\mathbf{D}^- x}\xi_2 + \mathbf{H}_1^-(p,x) \qquad if \quad \mu > 0 \tag{17}$$

and

$$\mathbf{\Psi}_2(p,x) = \mathbf{X}_{22}(p,x)\mathbf{e}^{\mathbf{D}^- x}\xi_2 + \mathbf{H}_2^-(p,x) \qquad\qquad if \quad \mu < 0 \tag{18}$$

Now, we are able to reconstruct the angular flux using the definition of the Laplace transform inversion for the transformed angular flux (14), we mean:

$$\mathbf{\Psi}(t,x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{\mathbf{\Psi}}(p,x)e^{pt}dp. \tag{19}$$

We must underline that the above ansatz is an analytical solution for the problem (8), in the sense that no approximation is made along its derivation. To overcome the drawback of solving analytically the line integral in equation (19), we solve it numerically. Bearing in mind the exponential behavior of the solution, the line integral appearing in equation (19) is well approximated by the Gaussian quadrature scheme

$$\mathbf{\Psi}(t,x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{\mathbf{\Psi}}(p,x)e^{pt}dp \approx \sum_{m=1}^{\mathcal{M}} a_m \frac{p_m}{t}\bar{\mathbf{\Psi}}\left(\frac{p_m}{t}, x\right) \tag{20}$$

where a_m e p_m are the roots and weights of the Gaussian quadrature. We must recall that the equation (20) is exact if the integrand is a polynomial of degree $2\mathcal{M} - 1$ in the variable $1/p$. Finally, we must remark that the $\mathrm{TLTS_N}$ solution for a bounded domain (slab) is expressed by equation (14) meanwhile for unbounded domain by equations (17) e (18).

3. NUMERICAL RESULTS

To show the aptness of the $\mathrm{TLTS_N}$ method we apply this methodology to solve the following time-dependent transport problem in unbounded domain: reflexive and vacuum boundary condition at $x = 0$; σ_s takes the values 0.9, 0.8 and 0.3; $\sigma_t = 1.0cm^{-1}$ and $v = 10^6 cm/sec$. On the other hand, the initial condition $\phi(x,\mu)$ is written as the solution of the following stationary problem:

$$\mu\frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu) = \frac{\omega}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + Q(x), \tag{21}$$

with

$$Q(x) = \begin{cases} 1, & \text{if} \quad 0 < x < 10, \\ 0, & \text{if} \quad x > 10, \end{cases} \tag{22}$$

and subject to the boundary condition

$$\psi(0,\mu) = \psi(0,-\mu), \;\; \mu > 0 \qquad \text{and} \qquad \lim_{x\to\infty}\psi(x,\mu) = 0. \tag{23}$$

Here we must mention that we solve the initial condition problem as a two-layer semi-infinite slab. In fact we consider that the first layer has a constant source and the second one is a homogeneous medium, we mean, without source. We also apply the idea described in this work to solve the stationary problem by the $\mathrm{LTS_N}$ method.

In table 1, we report the numerical results attained by the $\mathrm{TLTS_N}$ method, proceeding likewise the stationary problem. We also present validation of the results encountered comparing the $\mathrm{TLTS_N}$ results with the ones got by the $\mathrm{TLTS_N}$ approach assuming now a bounded domain, we mean a slab with increasing thickness $40 \leq L \leq 80$. The reason for this procedure comes from the fact that for large thickness it is well known, that the slab solution coincides with the ones of semi-infinite medium.

Table 1. Numerical Results for $TLTS_N$ with $\sigma_s = 0.9$

N	L	$x = 0$	$x = 10$
4	40	1.27498×10^{-1}	3.15297×10^{-4}
	50	1.28973×10^{-1}	3.19445×10^{-4}
	60	1.28127×10^{-1}	3.21779×10^{-4}
	70	1.28127×10^{-1}	3.21779×10^{-4}
	80	1.28127×10^{-1}	3.21779×10^{-4}
10	40	1.28159×10^{-1}	3.19215×10^{-4}
	50	1.29171×10^{-1}	3.19823×10^{-4}
	60	1.29615×10^{-1}	3.22104×10^{-4}
	70	1.29615×10^{-1}	3.22104×10^{-4}
	80	1.29615×10^{-1}	3.22104×10^{-4}
50	40	1.28742×10^{-1}	3.19478×10^{-4}
	50	1.29196×10^{-1}	3.19880×10^{-4}
	60	1.29328×10^{-1}	3.22132×10^{-4}
	70	1.29328×10^{-1}	3.22132×10^{-4}
	80	1.29328×10^{-1}	3.22132×10^{-4}
100	40	1.28743×10^{-1}	3.19478×10^{-4}
	50	1.29196×10^{-1}	3.19988×10^{-4}
	60	1.29328×10^{-1}	3.22136×10^{-4}
	70	1.29328×10^{-1}	3.22136×10^{-4}
	80	1.29328×10^{-1}	3.22136×10^{-4}

Table 2. Numerical Results for $TLTS_N$ with $\sigma_s = 0.8$

N	L	$x = 0$	$x = 10$
4	40	1.32586×10^{-1}	3.16077×10^{-4}
	50	1.32957×10^{-1}	3.16124×10^{-4}
	60	1.33115×10^{-1}	3.17381×10^{-4}
	70	1.33115×10^{-1}	3.17381×10^{-4}
	80	1.33115×10^{-1}	3.17381×10^{-4}
10	40	1.32851×10^{-1}	3.16872×10^{-4}
	50	1.33146×10^{-1}	3.16266×10^{-4}
	60	1.34247×10^{-1}	3.17422×10^{-4}
	70	1.34247×10^{-1}	3.17422×10^{-4}
	80	1.34247×10^{-1}	3.17422×10^{-4}
50	40	1.32855×10^{-1}	3.16885×10^{-4}
	50	1.33169×10^{-1}	3.16298×10^{-4}
	60	1.34278×10^{-1}	3.17425×10^{-4}
	70	1.34278×10^{-1}	3.17425×10^{-4}
	80	1.34278×10^{-1}	3.17425×10^{-4}
100	40	1.32855×10^{-1}	3.16886×10^{-4}
	50	1.33169×10^{-1}	3.16298×10^{-4}
	60	1.34278×10^{-1}	3.17426×10^{-4}
	70	1.34278×10^{-1}	3.17426×10^{-4}
	80	1.34278×10^{-1}	3.17426×10^{-4}

Table 3. Numerical Results for TLTS_N with $\sigma_s = 0.3$

N	L	$x = 0$	$x = 10$
4	40	1.36158×10^{-3}	3.24761×10^{-6}
	50	1.35241×10^{-3}	3.22489×10^{-6}
	60	1.34986×10^{-3}	3.21915×10^{-6}
	70	1.34986×10^{-3}	3.21915×10^{-6}
	80	1.34986×10^{-1}	3.21915×10^{-6}
10	40	1.36824×10^{-3}	3.24458×10^{-6}
	50	1.35146×10^{-3}	3.22350×10^{-6}
	60	1.34326×10^{-3}	3.21453×10^{-6}
	70	1.34326×10^{-3}	3.21453×10^{-6}
	80	1.34326×10^{-3}	3.21453×10^{-6}
50	40	1.36846×10^{-3}	3.24432×10^{-6}
	50	1.35088×10^{-3}	3.22069×10^{-6}
	60	1.34305×10^{-3}	3.21450×10^{-6}
	70	1.34305×10^{-3}	3.21450×10^{-6}
	80	1.34305×10^{-3}	3.21450×10^{-6}
100	40	1.36846×10^{-3}	3.24430×10^{-6}
	50	1.35088×10^{-3}	3.22069×10^{-6}
	60	1.34305×10^{-3}	3.21451×10^{-6}
	70	1.34305×10^{-3}	3.21451×10^{-6}
	80	1.34305×10^{-3}	3.21451×10^{-6}

Table 4. Numerical Results for TLTS_N with $x = 10$

σ_s	N	$x = 10$
0.9	4	3.21779×10^{-4}
	10	3.22104×10^{-4}
	50	3.22132×10^{-4}
	100	3.22136×10^{-4}
0.8	4	3.17381×10^{-4}
	10	3.17422×10^{-4}
	50	3.17425×10^{-4}
	100	3.17426×10^{-4}
0.3	4	3.21915×10^{-6}
	10	3.21456×10^{-6}
	50	3.21450×10^{-6}
	100	3.21451×10^{-6}

4. CONCLUSION

From the analysis of the above good results, besides the analytical character of the solution as well its aptness to solve transport problem with large N, (N up 2000) we are confident to affirm that the TLTS_N technique is a promising and robust approach to handle time-dependent problem in one-dimensional cartesian geometry either for bounded and unbounded domain. Furthermore, we must remember that the proved convergence of this method allow us to generate benchmark results for this sort of problem. To show the generality of this method, we focus our future attention to the task of extension of this methodology to solve the two-dimensional stationary transport problem in cartesian geometry for the angular flux, as well for one-dimensional time-dependent radiative transfer problems without azimuthal geometry.

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6. REFERENCES

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