ANALITICAL STUDY OF THE STEADY STATE CONDUCTION OF HEAT ON ELECTRICAL CONDUCTORS WITH AN INSULATION COVER

Abstract. *Conducting materials carrying an electrical current dissipate heat. This phenomenon is known as Joule effect that establishes the relation between the generated heat and the electrical current. In industry, there is a series of problems related to the determination of the distribution of temperature in conductors, generally in the form of cables extended or configured in coils of different geometries and other arrangements. This article presents analytical solutions to the steady state heat transfer problem of an infinite electrical cylindrical conductor with an insulation cover carryong a direct electrical current and submitted to boundary conditions of the first and third kind. Special attention is given to the influence of the electrical resistivity temperature-dependence on the temperature in the center of the cable. Following the analytical solutions, numerical results are presented and discussed using a commercial cable specification.*

Keywords*:* Electrical conductors, Joule effect, Temperature distribution, Resistivity.

1. INTRODUCTION

Thermal analysis of electrical conductors (wires and cables) carrying an electrical current is an important subject for the generation, transmission and distribution electrical energy companies. In these cases, the conductors form long transmission lines and therefore the Joule losses are very important.

Recently, with the advance of the embarked electronic systems, specially in vehicles, the great amount of cables having a few meters of length distributed in small spaces has motivated a series of studies about the ideal diameter of these cables and consequently their weight, when considering the maximum current that can be carried and the heat to be dissipated.

Electronics equipments are constituted of a set of components, amongst them, inductors. These devices, of different geometries like cylindrical (circular or rectangular section) and toroidale, heats when carrying an electrical current. The generated heat must be removed to the environment and therefore is indispensable the knowledge of the thermal load.

There is in industry many equipments that use coils of metallic wires carrying an electrical currents, that, as expected, produces heat. As it is known, elctronic/electrical equipments and coils with superheating aggravate the risk of accidents and malfunction.

In all these applications, the knowledge of the radial temperature distribution in the conductor and consequently the determination of its maximum temperature (hot-spot temperature) is very important.

Although there are in specialized literature articles about heat conduction in wires and cables (Das et al., 2001 and Hiranandani, 1991), there are few works that present closed analytical solutions (Carslaw and Jaeger, 1959).

This article presents an analytical study of the temperature behaviour in the center of cables with an insulation cover carrying an electrical direct current, considering boundary conditions of the first and third kind. Two cases will be analyzed: resistivity of the conducting material which is (a) temperature-independent and (b) temperature-linearly dependent.

2. MATHEMATICAL FORMULATION

The problem analyzed in this work consists in the determination of the radial temperature in an infinite and isotropic cylindrical electrical conductor covered by an insulation layer and carrying a direct electrical current. The analysis is based on the analytical solution of the heat conduction differential equation subject to boundary conditions of the first and third kind. The analysis shows the effect of the electrical resistivity of the material on the temperature in the center of the conductor as a function of the boundary condition imposed on the external surface of the cable. Figure 1 shows the geometry studied. It consists of an inner cylinder of radius r_1 (m) and thermal conductivity k_1 (W/mK) made of an isotropic electrical conductor material and an outer cylinder of radius r_2 (m) and thermal conductivity k_2 (W/mK) made also of an isotropic insulation material. Heat is generated in the inner cylinder and dissipated by convection from the outside surface at $r = r_2$. It is assumed that the inner and outer cylinders are in perfect thermal contact.

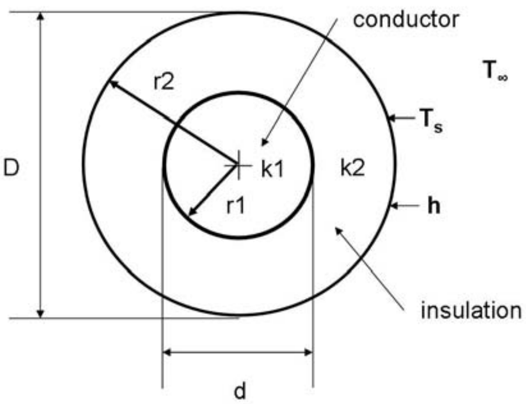


Figure 1. Geometry of the problem.

The equation for the conduction of heat in an isotropic medium is the following (Baehr and Stephan, 1994):

$$\nabla.[k(T)\nabla T] + \dot{q}(\mathbf{x}, T, t) = \rho c(T)\frac{\partial T}{\partial t}, \tag{1}$$

where T is the temperature (K), t the time (s), k is the thermal conductivity (W/mK), ρ the density (kg/m^3), c the specific heat (J/kgK), $\dot{q}$ the energy generation rate per unit of volume (W/m^3) and **x** is the position vector.

In cases of the thermal conductivity and the specific heat are temperature-independent, Eq.(1) becomes:

$$\alpha\nabla^2 T + \frac{\dot{q}(\mathbf{x}, T, t)}{\rho c} = \frac{\partial T}{\partial t}, \tag{2}$$

where α(T) = k/ρc(t) is the thermal diffusivity (m^2/s).

When the energy generation rate per unit of volume is independent of the time and position, Eq.(2) yields:

$$\alpha\nabla^2 T + \frac{\dot{q}(T)}{\rho c} = \frac{\partial T}{\partial t}. \tag{3}$$

For an infinite cylindrical conductor assuming that $\partial T/\partial \phi = \partial T/\partial z = 0$ and steady state conditions, Eq.(3) becomes:

$$\frac{\alpha}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}(T)}{\rho c} = 0. \tag{4}$$

For a conductor of resistivity $\rho_e(T)$ (Ω m) carrying an electrical current of density J (A/m^2), the energy generation rate per unit of volume is given by $\rho_e(T)J^2$. In cases of the resistivity of the material is a linear function of the temperature, one has:

$$\dot{q}(T) = \rho_e(T)J^2 = a + bT = \rho_o(1 - \alpha_L T_o)J^2 + \rho_o\alpha_L J^2 T, \tag{5}$$

with $a = \rho_o(1 - \alpha_L T_o)J^2$ and $b = \rho_{oL} J^2$, where ρ_o (Ωm) is the resistivity and α_L (1/K) is the temperature coefficient, both at T= T_o.

3. GENERAL SOLUTIONS

3.1. Temperature - independent resistivity

There are two regions to be considered: the electrical conductor ($r \le r_1$) and the insulation cover ($r_1 \le r \le r_2$). For the first region, the differential equation and their solution are:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right)+\frac{\rho_e J^2}{k_1}=0 \tag{6}$$

$$T_1 = A + B\ln(r) - \frac{\rho_e J^2}{4k_1}r^2 , \tag{7}$$

where A and B are integration constants depending on the boundary conditions at $r = 0$ and $r = r_1$. Otherwise, for the second region, one obtains readly the following differential equation and their solution:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dr}\right)=0 \tag{8}$$

$$T_2 = C + D\ln(r) , \tag{9}$$

where C and D are integration constants depending on the boundary conditions at $r = r_1$ and $r = r_2$.

3.2. Temperature – linear dependent resistivity

In this case, the mathematical formulation for the electrical conductor region is given by the following differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right)+\frac{a+bT_1}{k_1}=0 . \tag{10}$$

The solution of the above equation is:

$$T_1 = AJ_0(\lambda r) + BN_0(\lambda r) - \frac{a}{b} , \tag{11}$$

where $\lambda = \sqrt{b/k_1}$, A and B are the integration constants depending on the boundary conditions at $r = 0$ and $r = r_1$ and J_0 and N_0 are the Bessel functions of first and second kind, both of zero order.

On the other hand, for the insulation cover region, the differential equation and their solution are:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dr}\right)=0 \tag{12}$$

$$T_2 = C + D\ln(r) , \tag{13}$$

where C and D are the integration constants depending on the boundary conditions at $r = r_1$ and $r = r_2$.

4. PARTICULAR SOLUTIONS

4.1. Temperature - independent resistivity

Boundary conditions of the first kind

Here, for $r = r_1$, one has $T_1 = T_2$. In addition to this condition, it follows:

$$k_1 \frac{dT_1}{dr}\Big|_{r_1} = k_2 \frac{dT_2}{dr}\Big|_{r_1} \ . \tag{14}$$

On the other hand, for $r = r_2$, one has $T_2 = T_S$, where T_S is the temperature on the surface of the cable. Therefore, the solutions are:

$$T_1 = T_s + \frac{\rho_e J^2 r_1^2}{4k_1}\left[1 + \frac{2k_1}{k_2}\ln\left(\frac{r_2}{r_1}\right) - \left(\frac{r}{r_1}\right)^2\right], \tag{15}$$

$$T_2 = T_s + \frac{\rho_e J^2 r_1^2}{2k_2}\ln\left(\frac{r_2}{r}\right). \tag{16}$$

Boundary conditions of the third kind

In this case, for $r = r_1$, one has $T_1 = T_2$. Besides,

$$k_1 \frac{dT_1}{dr}\Big|_{r_1} = k_2 \frac{dT_2}{dr}\Big|_{r_1} \tag{17}$$

and

$$2\pi r_1 k_2 \frac{dT_2}{dr}\Big|_{r=r_2} = 2\pi r_1 h(T_\infty - T_s), \tag{18}$$

where T_∞ is the air temperature. Therefore, the solutions are:

$$T_1 = T_\infty + \frac{\rho_e J^2 r_1^2}{2k_2}\left[\frac{k_2}{hr_2} - \ln\left(\frac{r_1}{r_2}\right)\right] + \frac{\rho_e J^2}{4k_1}\left(r_1^2 - r^2\right), \tag{19}$$

$$T_2 = T_\infty + \frac{\rho_e J^2 r_1^2}{2k_2}\left[\frac{k_2}{hr_2} - \ln\left(\frac{r}{r_2}\right)\right]. \tag{20}$$

4.2. Temperature – linear dependent resistivity

Boundary conditions of the first kind

Here, for $r = r_1$, one has $T_1 = T_2$. In addition to this condition, yields:

$$k_1 \frac{dT_1}{dr}\Big|_{r_1} = k_2 \frac{dT_2}{dr}\Big|_{r_1} \ . \tag{21}$$

On the other hand, for $r = r_2$, one has $T_2 = T_S$. In this way, the solutions are:

$$T_1 = \frac{\left(T_S + \frac{a}{b}\right)J_0(\lambda r)}{J_0(\lambda r_1) - \frac{k_1}{k_2} r_1 \lambda J_1(\lambda r_1)\ln\left(\frac{r_2}{r_1}\right)} - \frac{a}{b}, \tag{22}$$

$$T_2 = T_s - \frac{\left(T_s + \frac{a}{b}\right)\ln\left(\frac{r_2}{r}\right)}{\ln\left(\frac{r_2}{r_1}\right) - \frac{k_2}{k_1 \lambda r_1}\frac{J_0(\lambda r_1)}{J_1(\lambda r_1)}}. \tag{23}$$

Boundary conditions of the third kind

In this case, for $r = r_1$, one has $T_1 = T_2$. Besides, one has also the following conditions:

$$k_1 \frac{dT_1}{dr}\Big|_{r_1} = k_2 \frac{dT_2}{dr}\Big|_{r_1}, \tag{24}$$

$$-2\pi r_2 k_2 \frac{dT_2}{dr}\Big|_{r_2} = 2\pi r_2 h\left(T_s - T_\infty\right). \tag{25}$$

Therefore, the solutions are:

$$T_1 = \frac{\left(T_\infty + \frac{a}{b}\right)J_0(\lambda r)}{J_0(\lambda r_1) - r_1 \lambda \frac{k_1}{k_2}\left[\frac{k_2}{hr_2} + \ln\left(\frac{r_2}{r_1}\right)\right]J_1(\lambda r_1)} - \frac{a}{b}, \tag{26}$$

$$T_2 = T_\infty - \frac{\left(T_\infty + \frac{a}{b}\right)\left[\frac{k_2}{hr_2} + \ln\left(\frac{r_2}{r}\right)\right]}{\ln\left(\frac{r_2}{r_1}\right) + \frac{k_2}{hr_2} - \frac{k_2 J_0(\lambda r_1)}{k_1 r_1 \lambda J_1(\lambda r_1)}}. \tag{27}$$

5. NUMERICAL EVALUATION

Table 1 shows the thermophysics, electrical and geometric properties of the materials used in this work.

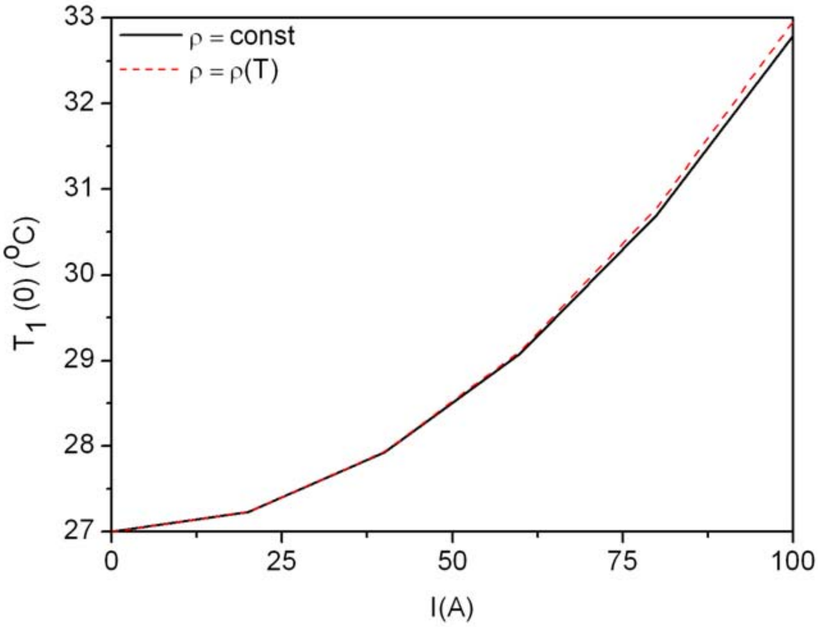
Tabela 1. Copper and the PVC datas.

Parameters	Copper	PVC	Ref
Diamenter (mm)	3.9	x	Pirelli
Thickness (mm)	x	1	Pirelli
Thermal conductivity (W/mK)	401	0.17	Incropera
Resistivity at 27°C (Ωm)	1.72×10^{-8}	x	Schuster
Temperature coefficient (/°C)	0.004	x	Schuster
Melting point (°C)	1084	80	wikipedia

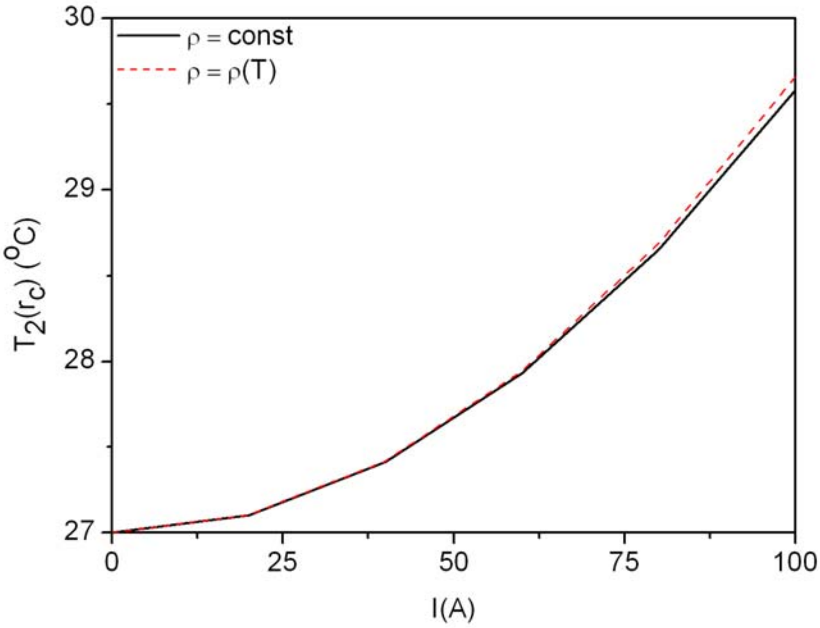
The convection heat transfer coefficient was calculated using, for the Nusselt number, the correlation proposed by Churchill and Chu (1975). The thermophysical properties of the air have been calculated with the data given by VDI Wärmeatlas (1984) after application of a polynomial fitting. The influence of the radiation heat transfer on the cable temperature will be investigated in a next work.

Figure 2 shows the temperature rise in the center of the cable ($r = 0$) (Fig.2a) and in the middle point of the insulation cover ($r = r_c = (r_1+r_2)/2$) (Fig.2b) as a function of the electrical current for the boundary condition of the first kind.

As one can see, the effect of the temperature-dependence of the electrical resistivity on the temperature in the center of the cable and, consequently, in the middle point of the insulation cover, is not important in the range from 0 to 100A. The physical explanation to this behaviour is that the boundary condition of the first kind imposes a limit on the growth of the temperature (see this at 60A, where $T_1(0)–T_\infty$ is 29°C), as a result of the great heat dissipation to the environment. The smallest the temperature variation, the smallest the influence of temperature on the resistivity. The same phenomenon does not occur in the case of the boundary condition of the third kind.



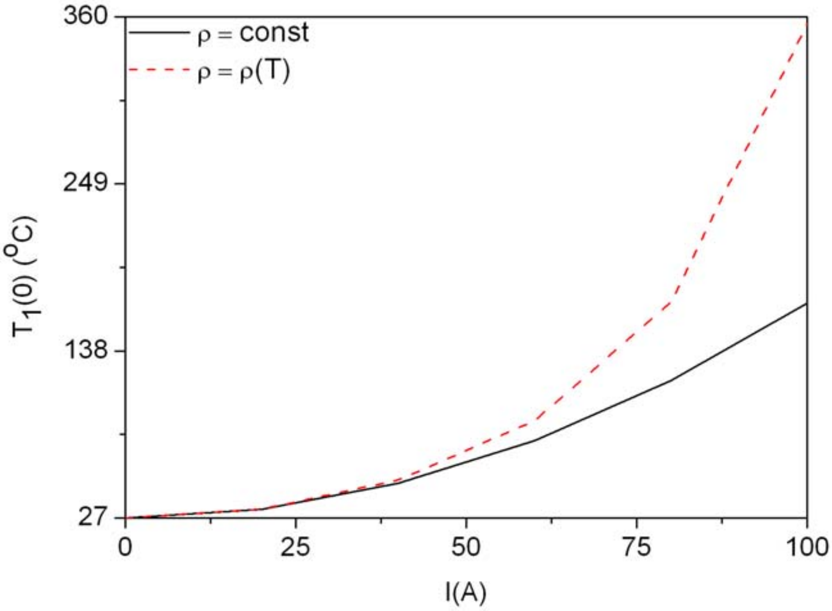
(a)



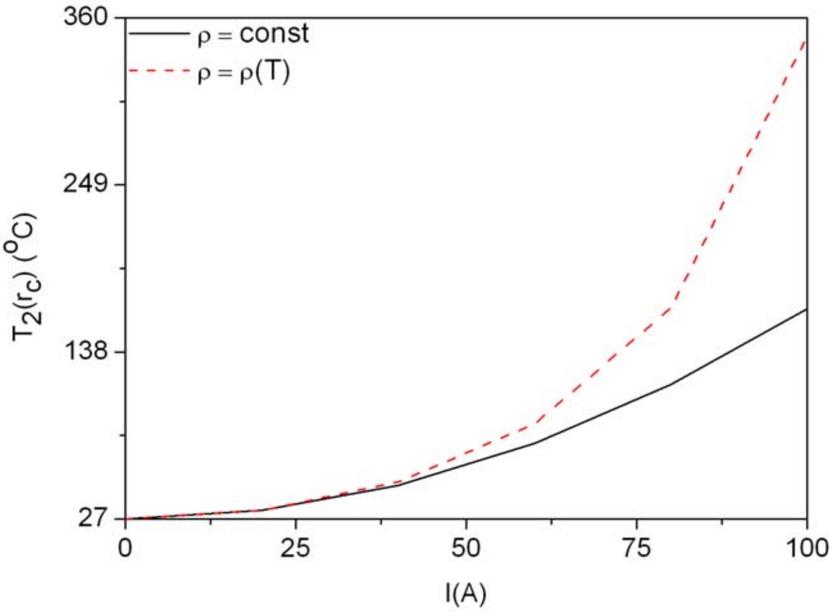
(b)

Figure 2. Temperature rise in the center of the cable (a) and in the middle point of the insulation cover (b) as a function of the electrical current for the boundary conditions of the first kind.

Figure 3 shows the temperature rise in the center of the cable (Fig.3a) and in the middle point of the insulation cover (Fig.3b) as a function of the electrical current for the boundary conditions of the third kind. In this case, the cable heating is high and, therefore, the influence of resistivity is more evident.



(a)



(b)

Figure 3. Temperature rise in the center of the cable (a) and in the middle point of the insulation cover (b) as a function of the electrical current for the boundary conditions of the third kind.

In this case, as was already pointed out, the effect of the temperature-dependence of the electrical resistivity on the temperature in the center of the cable and thus, in the middle point of the insulation cover, is significant in the range from 50 to 100A. The physical explanation to this behaviour is that the boundary condition of the third kind does not limit the growth of the temperature (see this at 60A, where $T_1(0)–T_\infty$ is 62°C). It should be noted that in the boundary condition of the first kind, the temperature at the insulation cover surface was fixed and for this reason, less heat is dissipated during the natural convection process. The highest the temperature variation, the highest the influence of temperature on the resistivity.

6. CONCLUSIONS

The conduction of heat problem analyzed in this work consists in the determination of the steady state radial temperature profile in an infinite and isotropic cylindrical electrical conductor covered with an insulation layer and carrying a direct electrical current. The analysis is based on the analytical solution of the heat conduction differential equation subject to boundary conditions of the first and third kind and temperature-dependent electrical resistivity.

From the results obtained in this work, the following conclusions and remarks could be drawn:

1. The effect of the electrical resistivity temperature-dependence of the conductor material on the temperature in the center of the conductor as well as on the middle point of the insulation cover depends on the kind of the boundary condition imposed in the external surface of the cable.
2. Under the thermal viewpoint, the insulation cover material imposes a limit on the maximum electrical current conducted by the cable.
3. In the determination of the maximum current that can be carried by an electrical conductor on natural convection regime (boundary condition of the third kind), it is important to take into account the electrical resistivity temperature-dependence. As one can verify in Fig.3b, for 75A approximately, the temperature in the middle point of the insulation cover is 80°C if one consider the resistivity temperature-independent. Otherwise, if the resistivity is considered temperature-dependent, this temperature is 94°C. In this work, the insulation cover is made of PVC, which according to Table 1 has a melting point of 80°C. Therefore, the maximum current should be less than 75A.

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