EXACT SOLUTIONS FOR HEAT TRANSFER PROBLEMS USING DIFFERENTIAL CONSTRAINTS

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Abstract. *The analytical methods generally used to solve the advection-diffusion equation subjected to boundary conditions of second kind are based on integral transforms, separation of variables and spectral methods in general. These formulations usually produce solutions expressed in infinite series, which even when truncated to perform the post-processing tasks, have a great number of basis functions. The problem related to the number of these basis functions is concerned about the completeness of the solution obtained, since a complete space of solutions can represent any boundary condition which may be applied, as well as restrictions based on the scenario that is being studied. In most scenarios involving convection or diffusion, boundary conditions of second and third type are prescribed. In these cases, the problem of completeness can be solved using differential constraints, which are extra differential equations that particularize the space of solutions, but preserve, in the resultant subspace, the solutions which satisfy the prescribed conditions of the original problem. In this paper, exact solutions to problems in heat transfer are obtained using differential constraints that become boundary conditions of second or third kind when applied to the corresponding interfaces. This procedure produces exact solutions in a compact form whose series expansions result the same ones obtained by the application of the method of separation of variables and spectral formulations. The main advantages of applying this formulation are the high processing speed, the reduction of the amount of memory required to perform the scenario simulation and the reduced computational code.*

Keywords*: heat transfer, differential constraints, advection-diffusion equation*

1. INTRODUCTION

Problems described by the advection-diffusion equation subjected to boundary conditions of second kind are usually solved by analytical methods based on integral transforms (Cotta, 1993), separation of variables (Crank, 1975, Osizik, 2000) and spectral methods in general (Polyanin, 2004; Zwillinger, 1992). The solutions obtained are generally expressed as infinite series, which have a large number of basis functions, even when truncated to perform the post-processing tasks. The problem related to the number of these basis functions is concerned about the completeness of the solution obtained, since a complete space of solutions can represent any boundary condition that can be applied, as well as restrictions based on the scenario that is being studied.

In several scenarios involving convection or conduction, boundary conditions of second and third type are prescribed. In these cases, the problem of completeness can be solved using differential constraints, which are extra differential equations that particularize the space of solutions, but preserve, in the resultant subspace, the solutions which satisfy the prescribed conditions of the original problem.

The aim of this work is to obtain exact solutions to problems in heat transfer using differential constraints that become boundary conditions of second or third kind when applied to the interfaces. This procedure produces exact solutions in a compact form whose series expansions result the same ones obtained by the application of the method of separation of variables and spectral. The main advantages of applying this formulation are the high processing speed, the reduction of the amount of memory required to perform the scenario simulation and the reduced computational code.

This article is outlined as follows. In section 2, the method is explained. In section 3, the applications of the proposed formulation are described. Finally, in section 4, conclusion and recommendations for future work are drawn.

2. DIFFERENTIAL CONSTRAINTS

A differential constraint can be defined as any differential equation which can be employed to restrict the space solution of the original one.

Consider the problem described by the equation

$$Lf = 0 \tag{1}$$

where L is a differential operator and which is subjected to a boundary condition of second kind.

The use of the differential constraints is based on the following steps:

1. Writing a differential constraint from the boundary condition using arbitrary parameters or functions;
2. Differentiating the differential constraint in order to obtain auxiliary equations which have the same order of the original one.
3. Isolating the derivatives of higher order on the differential constraints, performing the substitution of the corresponding expressions into the original equation, which is simplified as one of its derivatives is eliminated.
4. Solving the resulting equation.
5. Performing the substitution of the solution into the original equation, as well as into the original boundary conditions.

The next section shows how this procedure can be applied to solve heat transfer problems.

3. APPLICATIONS

3.1 One-dimensional homogeneous problem in a semi-infinite medium

Consider the solution of a homogeneous heat-conduction problem for a semi-infinite region. That is, a semi-infinite region, $0 \leq x < \infty$, is initially at a temperature $F_0(x)$ and for times $t > 0$ the boundary surface at $x = 0$ dissipates heat by convection into a medium at zero temperature.

The mathematical formulation of this problem can be described as

$$\alpha \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} \qquad \text{in } 0 < x < \infty, \ t > 0 \tag{2}$$

$$\frac{\partial f}{\partial x} = hf \qquad \text{at } x = 0 \tag{3}$$

$$f = F_0(x) \qquad \text{at } t = 0. \tag{4}$$

The first stage consists of writing a differential constraint which can be reduced to a boundary condition of the third kind such as

$$\frac{\partial f}{\partial x} = mf. \tag{5}$$

The result of differentiating Eq. (5) with respect to x is

$$\frac{\partial^2 f}{\partial x^2} = m \frac{\partial f}{\partial x}, \tag{6}$$

which can be placed into Eq. (2) which becomes

$$\alpha \left[m \frac{\partial f}{\partial x} \right] - \frac{\partial f}{\partial t} = 0. \tag{7}$$

From Eq. (5), it is possible to rewrite Eq. (7) as

$$\alpha [m(mf)] - \frac{\partial f}{\partial t} = 0 \tag{8}$$

which is a first order partial differential equation whose solution is

$$f(x,t) = F_1(x) e^{\alpha m^2 t}, \tag{9}$$

where $F_1(x)$ is an arbitrary function.

In order to specify $F_1(x)$, Eq. (9) is placed into Eq. (2) and the result is the following ordinary differential equation

$$\frac{dF_1}{dx} - m^2 F_1 = 0 \qquad , \tag{10}$$

whose general solution is given by

$$F_1(x) = Ae^{mx} + Be^{-mx} . \tag{11}$$

The application of Eq. (11) into Eq. (9) yields

$$f(x,t) = e^{\alpha m^2 t}\left[Ae^{mx} + Be^{-mx}\right]. \tag{12}$$

The boundary condition at $x = 0$ can be applied into Eq. (12). As a result, the value of the arbitrary constant B is determined, that is,

$$B = 0 \tag{13}$$

and Eq. (12) becomes

$$f(x,t) = Ae^{\alpha m^2 t} e^{mx} . \tag{14}$$

The initial condition specifies the constant A, so

$$A = F_0(x) e^{-mx} \tag{15}$$

and finally the solution to the problem described by Eq. (2), Eq. (3) and Eq. (4) is

$$f(x,t) = F_0(x) e^{\alpha m^2 t} . \tag{16}$$

3.2 Two-dimensional homogeneous problem

Consider a region $0 \le x < \infty$ and $0 \le y < \infty$ that is initially at a temperature $f_i(x, y)$. For times, $t > 0$ the boundary at $y = 0$ is kept insulated and the boundary at $x = 0$ dissipates heat by convection with a heat transfer coefficient h into a medium at zero temperature. There is no heat generation in the medium.

The mathematical formulation of this problem is given as

$$\alpha\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial f}{\partial t} \qquad \text{in } 0 < x < \infty,\ 0 < y < \infty,\ t > 0 \tag{17}$$

$$\frac{\partial f}{\partial x} = hf \qquad \text{at } x = 0 \tag{18}$$

$$\frac{\partial f}{\partial y} = 0 \qquad \text{at } y = 0 \tag{19}$$

$$f = f_i(x, y) \qquad \text{at } t = 0 . \tag{20}$$

The following differential constraints are defined:

$$\frac{\partial f}{\partial x} = mf \tag{21}$$

$$\frac{\partial f}{\partial y} = a(x, y)\,. \tag{22}$$

Differentiating Eq. (21) in relation to x and Eq. (22) in relation to y and placing the results into Eq. (17) yield:

$$\alpha\left(m^2 f + \frac{\partial a}{\partial y}\right) = \frac{\partial f}{\partial t}\,. \tag{23}$$

It is possible to obtain an expression to $f(x, y, t)$ by solving Eq. (23), that is

$$f(x, y, t) = -\frac{1}{m^2}\frac{\partial a}{\partial y} + F_1(x, y)e^{\alpha m^2 t}\,, \tag{24}$$

where $F_1(x, y)$ is an arbitrary function.

The initial condition can be applied to Eq. (24), and as a consequence $F_1(x, y)$ is determined

$$F_1(x, y) = f_i(x, y) + \frac{1}{m^2}\frac{\partial a}{\partial y}\,. \tag{25}$$

The expression for $F_1(x, y)$ can be placed into Eq. (24) and consequently

$$f(x, y, t) = -\frac{1}{m^2}\frac{\partial a}{\partial y} + \left[f_i(x, y) + \frac{1}{m^2}\frac{\partial a}{\partial y}\right]e^{\alpha m^2 t}\,. \tag{26}$$

Differentiating Eq. (21) with respect to y and Eq. (22) with respect to x, we find the mixed derivatives that must be equal and the result is

$$\frac{\partial a}{\partial x} - ma = 0\,. \tag{27}$$

Solving Eq. (25), we get

$$a(x, y) = b(y)e^{mx} \tag{28}$$

where $b(y)$ is an arbitrary function.

Inserting Eq. (28) into the solution given by Eq. (26)

$$f(x, y, t) = -\frac{1}{m^2}e^{mx}\frac{db}{dy} + \frac{1}{m^2}e^{mx}\frac{db}{dy}e^{\alpha m^2 t} + f_i(x, y)e^{\alpha m^2 t} \tag{29}$$

and differentiating Eq. (29) with respect to y, we obtain

$$\frac{\partial f}{\partial y} = -\frac{1}{m^2}e^{mx}\frac{d^2 b}{dy^2} + \frac{1}{m^2}e^{mx}e^{\alpha m^2 t}\frac{d^2 b}{dy^2} + \frac{\partial f_i}{\partial y}e^{\alpha m^2 t} \tag{30}$$

and applying it into Eq. (22); the result is

$$-\frac{1}{m^2}e^{mx}\frac{d^2b}{dy^2}+\frac{1}{m^2}e^{mx}e^{\alpha m^2 t}\frac{d^2b}{dy^2}+\frac{\partial f_i}{\partial y}e^{\alpha m^2 t}=b(y)e^{mx}. \quad (31)$$

In order to ensure the equality in Eq. (31) holds true, we should take

$$\frac{1}{m^2}e^{mx}\frac{d^2b}{dy^2}=b(y)e^{mx} \quad (32)$$

$$\frac{1}{m^2}e^{mx}e^{\alpha m^2 t}\frac{d^2b}{dy^2}+\frac{\partial f_i}{\partial y}e^{\alpha m^2 t}=0. \quad (33)$$

Solving the ordinary differential equation given by Eq. (32), we find

$$b(y)=c_1 sen(my)+c_2\cos(my). \quad (34)$$

Differentiating Eq. (34) twice with respect to y and substituting the result into Eq. (33) produce

$$\frac{1}{m^2}e^{mx}\left[-c_1 m^2 sen(my)-m^2 c_2\cos(my)\right]+\frac{\partial f_i}{\partial y}=0 \quad (35)$$

that can be solved in order o obtain f_i

$$f_i(x,y)=\frac{e^{mx}}{m}\left[-c_1\cos(my)+c_2 sen(my)\right]+F_1(x) \quad (36)$$

Inserting this result into Eq. (29) and rearranging its terms, we get

$$f(x,y,t)=-\frac{e^{mx}}{m}\left[c_1\cos(my)-c_2 sen(my)\right]+F_1(x)e^{\alpha m^2 t}. \quad (37)$$

Applying Eq. (37) into the first differential constraint given by Eq. (21) results in a stronger restriction than if it was substituted into the original equation, so we obtain

$$\frac{dF_1}{dx}-mF_1=0 \quad (38)$$

which can be solved by the method of separation of variables and yields

$$F_1=c_0 e^{mx}. \quad (39)$$

Then, Eq. (37) becomes

$$f(x,y,t)=\frac{e^{mx}}{m}\left[-c_1\cos(my)-c_2 sen(my)\right]+c_0 m e^{\alpha m^2 t}. \quad (40)$$

3.3 Factorization of Differential Equations

Differential constraints can also be used to establish the factorized form a differential equation. The factors obtained are easier to solve by direct substitution.

Consider the following differential constraints

$$D_m \frac{\partial}{\partial x} f(x, y) - u(x, y) f(x, y) - a(x, y) = 0 \tag{41}$$

$$D_m \frac{\partial}{\partial y} f(x, y) - v(x, y) f(x, y) - b(x, y) = 0 \tag{42}$$

which can be used to obtain the two-dimensional advective-diffusion equation.

Deriving Eq. (41) with respect to x yields

$$D_m \frac{\partial^2 f}{\partial x^2} - \frac{\partial u}{\partial x} f - u \frac{\partial f}{\partial x} - \frac{\partial a}{\partial x} = 0\,, \tag{43}$$

and the differentiation of Eq. (42) with respect to y results

$$D_m \frac{\partial^2 f}{\partial y^2} - \frac{\partial v}{\partial y} f - v \frac{\partial f}{\partial v} - \frac{\partial b}{\partial y} = 0\,. \tag{44}$$

The addition of Eq. (43) and Eq. (44) produces the following advection-diffusion equation

$$D_m \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right] - u \frac{\partial f}{\partial x} - v \frac{\partial f}{\partial y} - \frac{\partial u}{\partial x} f - \frac{\partial v}{\partial y} f - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} = 0 \tag{45}$$

whose source term is

$$Q = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\,. \tag{46}$$

In order to solve Eq. (45) it is enough to impose the condition of equality of the mixed derivatives of Eq. (41) and Eq. (42). Differentiating Eq. (41) with respect to y and Eq. (42) with respect to x yield

$$D_m \frac{\partial^2 f}{\partial y \partial x} - f \frac{\partial u}{\partial y} - u \frac{\partial f}{\partial y} - \frac{\partial a}{\partial y} = 0 \tag{47}$$

and

$$D_m \frac{\partial^2 f}{\partial x \partial y} - f \frac{\partial v}{\partial x} - v \frac{\partial f}{\partial x} - \frac{\partial b}{\partial x} = 0\,. \tag{48}$$

The equality of Eq. (47) and Eq. (48) gives

$$f \frac{\partial u}{\partial y} + u \frac{\partial f}{\partial y} + \frac{\partial a}{\partial y} = f \frac{\partial v}{\partial x} + v \frac{\partial f}{\partial x} + \frac{\partial b}{\partial x}\,. \tag{49}$$

It is possible to obtain expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from Eq. (41) and Eq. (42), that is,

$$\frac{\partial f}{\partial x} = \frac{uf + a}{D_m} \tag{50}$$

and

$$\frac{\partial f}{\partial y} = \frac{vf + b}{D_m}, \tag{51}$$

placing them into Eq. (49) results

$$f\frac{\partial u}{\partial y} + u\left[\frac{vf + b}{D_m}\right] + \frac{\partial a}{\partial y} = f\frac{\partial v}{\partial x} + v\left[\frac{uf + a}{D_m}\right] + \frac{\partial b}{\partial x}. \tag{52}$$

Equation (52) gives the following an expression to f

$$f = \frac{\frac{va}{D_m} + \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} - \frac{ub}{D_m}}{\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right]}. \tag{53}$$

The function $a(x, y)$ can be determined from Eq. (46) if $b(x, y)$ is known, so

$$a = \int\left(Q - \frac{\partial b}{\partial y}\right)dx - a_0(y), \tag{54}$$

where $a_0(y)$ is an arbitrary function.

4. CONCLUSION

Differential constraints are used to obtain exact solutions to problems in heat transfer The procedure yields exact solutions in a compact form whose series expansions result the same ones obtained by the application of the method of separation of variables and spectral formulations. The main advantage of the proposed method relies on the computational features of the corresponding code. The low processing time required obtaining the solutions with Maple XI and the small amount of memory needed in most calculations, the solutions can be implemented in a code written in procedural language, avoiding the use of numerical methods for simulating realistic scenarios. The research is currently focused in the formulation of analytical procedures for dealing with the pollutant dispersion occurring during the transport of chemicals in water bodies.

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