TWO-DIMENSIONAL HEAT CONDUCTION IN WALLS OF RECTANGULAR DUCTS

Gustavo Adolfo Ronceros Rivas, gustavo@ita.br
Ezio Castejon Garcia, ezio@ita.br
Instituto Tecnológico de Aeronáutica, Praça Marechal Eduardo Gomes, 50 - Vila das Acácias
CEP 12.228-900 – São José dos Campos – SP – Brasil .

Abstract. *The present work treats about the development of a numerical method in which the main objective is analyze the two-dimensional heat conduction in plates and rectangular duct walls. Finite volumes (FV) and finite differences (FD) methods are employed; perfomance comparisons between both methods are fulfilled. For the case of rectangular ducts, four equations are solved (one for each wall of the duct) with different boundary conditions: prescribed temperature, convection and adiabatic. These four equations are coupled among themselves. Tri-diagonal Matrix Algorithm(TDMA) and/or Gauss-Seidel methods are employed to solve the equation system. A variable mesh is employed to discretize the continuo.*

Keywords*: Simulation Numerical, Rectangular Ducts, Finite Volume, Finite Difference.*

1. INTRODUCTION

Rectangular cross section duct is a common geometry employed in much practical situations in thermal systems. In these systems, one or more modes of heat transfer occur, and they are relevant. For instance, this normally occurs in compact heat exchangers, air conditioning system, cooling channels in combustion chambers, gas turbine cooling system, etc.

The heat conduction is a mode of heat transfers very important. The analysis of this mode can identify maximum and minimum temperatures, thermal gradients and heat fluxes in materials of thermal equipment. For these objectives, some methods can be used just as: experimental investigations, analytical solutions, and numerical solutions. In the present work, numerical solutions have been implemented to analyze the heat conduction. For this purpose, *Finite Volumes (FV)* and *Finite Differences (FD)* methods were employed to discretize the governing differential equations. They are made comparisons between *FV* and *FD* solutions. Also, *Tri-Diagonal Matrix Algorithm (TDMA)* and/or *Gauss- Seidel* methods are used to solve the governing equation systems, and they are made comparisons between both, too. Patankar (1980), Versteeg and Malalasekera (1995) describe the application the *FV* methods. They are very clear in the presentation of several situations of heat transfer applications, especially in the pure diffusion. Incropera and DeWitt (2003), Ferziger and Peric (2002) present interesting methodologies for the development of the finite difference methods, such as uniform and non-uniform grids, respectively.

Kakaç and Yener (1985), and Ozisik (1980) present subject important in the development of analytical methods. In the present work is shortly described the separation variable method to make a comparison with the implemented numerical solutions. This work presents the analysis of the heat conduction in two-dimensional in plates and rectangular duct walls, steady-state without internal heat sources, with different boundary conditions such as imposition of temperature, convection and adiabatic on the contour.

2. GOVERNING EQUATIONS

The heat conduction equation is a mathematical relation, expressed by a differential equation, including temperature, space coordinate and time (Kakaç and Yener, 1985). Based in the application of First Thermodynamic Law, the general equation describing the heat conduction for solids can be written in the following vector form:

$$\vec{\nabla}(k\vec{\nabla}T) + \dot{q} = \rho c_P \frac{\partial T}{\partial t} \tag{1}$$

where $\dot{q}$ is the internal heat source, ρ is density, c_p is specific heat, $\partial T / \partial t$ the thermal transient.

For rectangular coordinates, two-dimensional, isotropic material without internal heat sources and steady-state regime, the general heat conduction equation reduces to:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2}$$

2.1. Finite difference method

FD method is based on the transformation of continuous differential equations into discrete difference equations, suitable for numerical computing. The approximation of derivatives by finite differences plays a central role in finite difference methods for the numerical solution of differential equations, especially boundary value problems. A *Taylor series* can be used to calculate the value of derivatives in every point. Usually, the grid is locally structured, i.e. each grid node may be considered the origin of a local coordinate system, whose axes coincide with grid lines. Figure 1 shows two-dimensional (2D) Cartesian grid used in *FD method*.

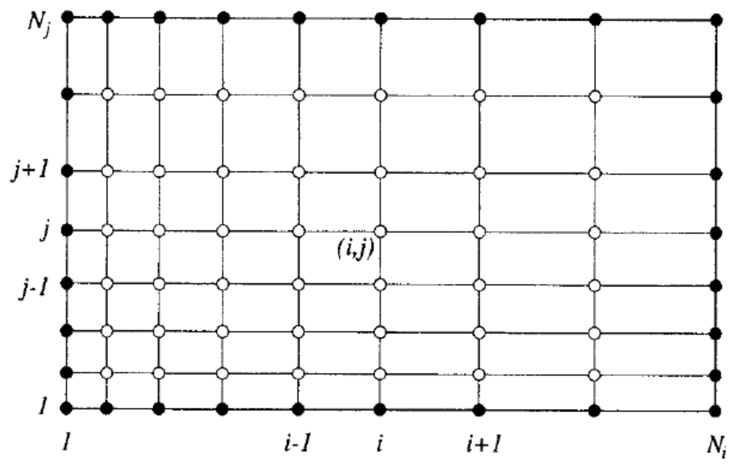

Figure 1. Two-dimensional grid for *FD method*

A continuous differentiable function, at a finite number of points near of "*x*", can be expressed as a *Taylor series*:

$$\phi(x) = \phi(x_i) + (x - x_i)\left(\frac{\partial \phi}{\partial x}\right)_i + \frac{(x - x_i)^2}{2!}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_i + \frac{(x - x_i)^3}{3!}\left(\frac{\partial^3 \phi}{\partial x^3}\right) + ... + \frac{(x - x_i)^n}{n!}\left(\frac{\partial^n \phi}{\partial x^n}\right) + E \qquad (3)$$

where *E* means of highest order terms. Replacing "ϕ" to "*T*", and "*x*" to "x_{i+1}" and "x_{i-1}", respectively, for case of uniform grid (Incropera and DeWitt, 2003):

$$T_{i+1} = T_i + \left.\frac{dT}{dx}\right|_i \Delta x + \left.\frac{d^2T}{dx^2}\right|_i \frac{\Delta x^2}{2!} + ... \quad \text{And} \quad T_{i-1} = T_i - \left.\frac{dT}{dx}\right|_i \Delta x + \left.\frac{d^2T}{dx^2}\right|_i \frac{\Delta x^2}{2!} + ... \qquad (4)$$

Ignoring the highest order terms (above of second term), and making the necessaries arrangements, it is possible to obtain the second derivate for uniform grid:

$$\left.\frac{d^2T}{dx^2}\right|_i = \frac{T_{i+1} + T_{i-1} - 2T_i}{\Delta x^2} \qquad (5)$$

For uniform grid, $\Delta x = \Delta y$, Eq. (5) into Eq. (2), it is obtained:

$$T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j} - 4T_{i,j} = 0 \qquad (6)$$

For non-uniform grid, it is possible to evaluate the first derivative based on points between x_i and x_{i+1}, and x_i and x_{i-1}, respectively (Ferziger and Peric, 2002):

$$\left(\frac{\partial \phi}{\partial x}\right)_{i+1/2} \approx \frac{\phi_{i+1}-\phi_i}{x_{i+1}-x_i}; \left(\frac{\partial \phi}{\partial x}\right)_{i-1/2} \approx \frac{\phi_i-\phi_{i-1}}{x_i-x_{i-1}} \tag{7}$$

Equation (7) represents the central finite difference at the points *"i+1/2" and "i-1/2"*. Substituting Eq. (7) into Eq. (2), it obtains the expression for the second derivative, as follow:

$$\left(\frac{\partial^2 \phi}{\partial x^2}\right) \approx \frac{\left(\frac{\partial \phi}{\partial x}\right)_{i+1/2}-\left(\frac{\partial \phi}{\partial x}\right)_{i-1/2}}{\frac{1}{2}\left(x_{i+1}-x_{i-1}\right)} \approx \frac{\phi_{i+1}(x_i-x_{i-1})+\phi_{i-1}\left(x_{i+1}-x_i\right)+\phi_i\left(x_{i+1}-x_{i-1}\right)}{\frac{1}{2}\left(x_{i+1}-x_{i-1}\right)\left(x_{i+1}-x_i\right)\left(x_i-x_{i-1}\right)} \tag{8}$$

Replacing Eq. (8) into Eq. (2), it obtains the expression algebraic for non-uniform grid. If makes $\Delta x = \Delta y$, it drives to Eq. (6) again (uniform grid).

2.2. Finite volume method

Equation (1), in steady-state and two-dimensional, can derive in the following:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+S=0 \tag{9}$$

where k is the diffusion coefficient and S is the source term.

The procedure to discretize Eq. (9), using *FV method*, is the following (Versteeg and Malalasekera., 1995):

Grid generation: division of the two-dimensional domain into discrete control volumes, conform shown in the Fig. (2).

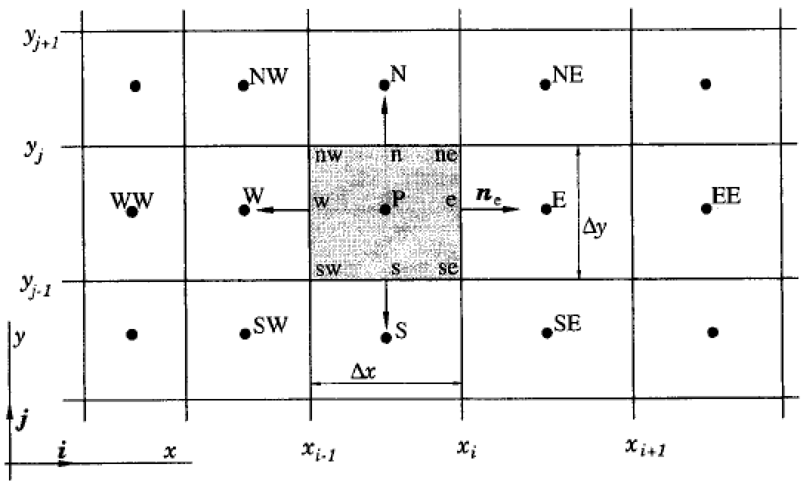

Figure 2. Two-dimensional grid for *FV method*.

Discretization: integration of the governing equation over a control volume to yield a discretized equation at its nodal point *P*, to obtain:

$$\int_{\Delta V}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx.dy+\int_{\Delta V}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dx.dy+\int_{\Delta V}S_T dV=0 \tag{10}$$

For uniform grid $A_e = A_w = \Delta y$ and $A_n = A_s = \Delta x$. Thus we obtain:

$$\left[k_e A_e \left(\frac{\partial T}{\partial x}\right)_e - k_w A_w \left(\frac{\partial T}{\partial x}\right)_w\right] + \left[k_n A_n \left(\frac{\partial T}{\partial y}\right)_n - k_s A_s \left(\frac{\partial T}{\partial y}\right)_s\right] + \overline{S}\Delta V = 0 \quad (11)$$

If $\overline{S}\Delta V = S_u + S_P T_P$, and making the heat flux across the east face *q"*= $k_e A_e \left.\frac{\partial T}{\partial x}\right|_e = k_e A_e \frac{(T_E - T_P)}{\delta x_{PE}}$, and making similar form for the other faces, Eq. (11) can be rearranged to obtain the general discretized equation for interior nodes:

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + S_u \quad (12)$$

where the coefficients above are presented in the Tab. 1.

Table 1. Coefficients of the Eq. (12).

a_W	a_E	a_S	a_N	a_P
$\frac{k_w A_w}{\delta x_{WP}}$	$\frac{k_e A_e}{\delta x_{PE}}$	$\frac{k_s A_s}{\delta y_{SP}}$	$\frac{k_n A_n}{\delta y_{PN}}$	$a_W + a_E + a_S + a_N - S_P$

Besides:

- sets coefficient $a_f = 0$, where the underwritten *"f"* represents the volume on the contour;
- for source contribution: $S_u = \frac{2k_f A_f}{\Delta\psi} T_f$ and $S_P = -\frac{2k_f A_f}{\Delta\psi}$, even so $\Delta\psi$ can be Δx or Δy ;
- for prescribed heat flux q''_f : $S_u + S_P T_P = q_f$ (13)

<u>Solution of the equation system:</u> The linear algebraic equations result in a system, in which have to be solved to obtain the distribution of the property *"T"* for each nodal point. To solve this problem, two methods are employed: *Tri-diagonal Matrix Algorithm (TDMA, two-dimensional)* and/or *Gauss-Seidel*.

3. RESULTS

In this item, comparisons between the *FV and FD method* developments are done. The language programming used for these was *Compaq Visual FORTRAN.* Uniform grid as well as non-uniform has been implemented and the results have shown very good concordance. Also, these results have been compared with commercial software. Figure 3 shows a gross uniform grid for a classic problem using *FV method* to analyze the temperature distribution in a given plate.

The figure 4 show Results Comparison of error the *FV, FD methods* with one Commercial Software in the center the plate along the axis X as well as Gross Grid and Refined Grid, at the same time show the one difference the *TDMA and Gauss Seidel methods.*

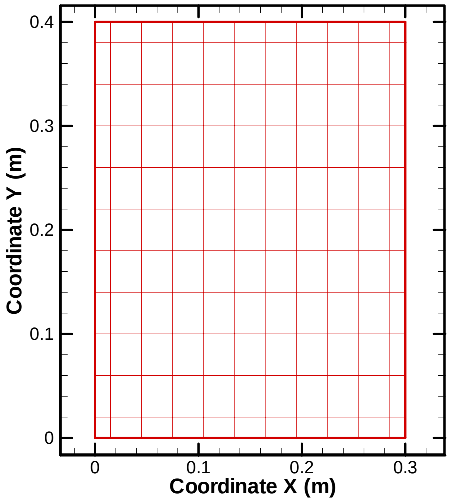

Figure 3. Uniform Grid for *FV method* in the plate analysis

Figure 4 shows result comparisons among *FV, FD methods* and Commercial Software. This presents the error in which is defined as the result difference between the analyzed developed program and the commercial software in which this one was used as a reference. The analysis are done at the center of the plate, *Y = 0.2 m*, along of the *X*-axis. Figure 4a presents the errors for 12x12 point gross grid as following: *FD Method* plus Gauss-Seidel, *FV Method* plus TDMA and *FD Method* plus Gauss-Seidel. Looking this Fig. 4a, we can say that the two *FV methods* seem to be better than *FD method.* Figure 4b presents the same methods in a 32x42 point refined grid; in this we can see that all errors have been reduced more than ten times, and *FV methods* have continued to have better behavior than *FD*. Thus, a comparing between the *FV methods* in refined grid, we can see that the *FV+TDMA* had the best performance. For this reason, the *FV Method plus TDMA* was chosen to solve the two-dimensional heat conduction problems in plates and in rectangular duct walls.

Figure 5 compares the thermal profile results among *FV* and *FD method*, and Commercial Software. All them show good concordances themselves for grid of 32x42 points. The prescribed temperature for the North, South, East and West were 373K, 673K, 473K, 573K, respectively.

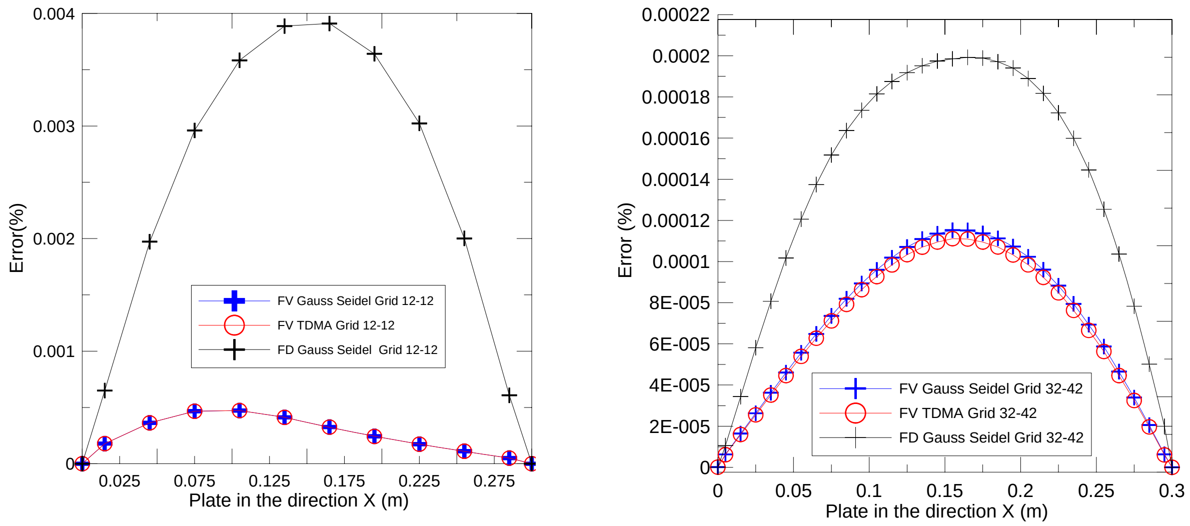

Figure 4. Results Comparison of error the *FV, FD methods* with one Commercial Software. in the center the plate along the axis X: (a) Gross Grid; (b) Refined Grid.

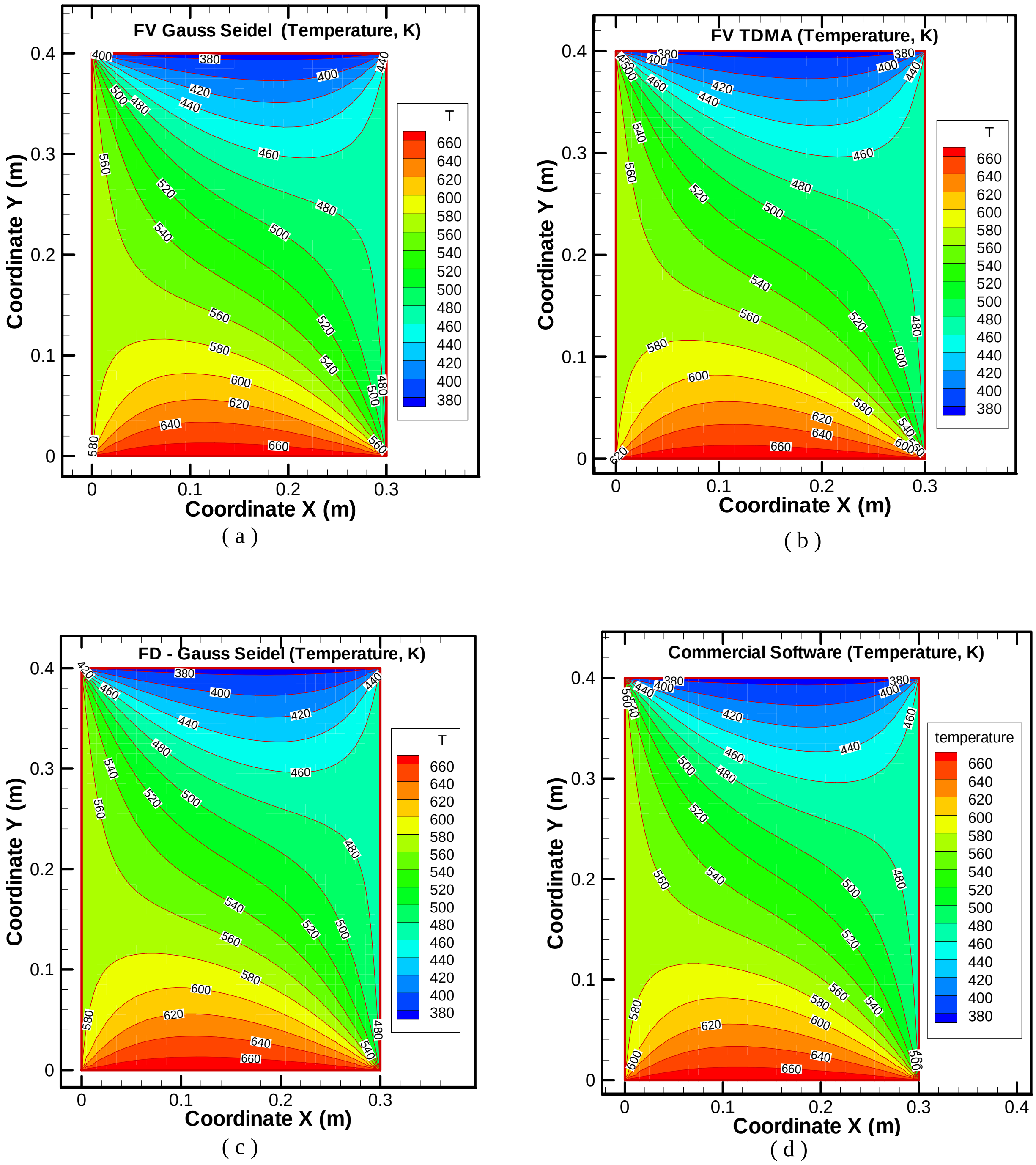

Figure 5. Result Comparisons of thermal profiles among *FV* and *FD method*, and a Commercial Software. Analysis for plates: (a) Finite Volume and Gauss Seidel; (b) Finite Volume and TDMA ; (c) Finite Differences and Gauss Seidel; (d) Commercial Software.

The Figures 6 and 7 shows the non-uniform grid for rectangular duct wall analysis: left side for gross grid; right side for refined grid, respectively. The adopted strategy was to couple the four walls (as four plates) and solving one by one. When the temperature distribution for one wall was being calculated, another two worked as boundary condition to the first one. After the convergence for this one, now that worked as boundary condition to calculate the thermal profile to other one. And then, the process has continued until to obtain the final convergence. Once obtained all thermal profiles for four walls, these worked as boundary condition to solve energy equation of the internal duct fluid; but this is not the subject of this present paper. This is the main reason to utilize one non-uniform grids, once that to obtain the fluid thermal profiles are necessary fine grids to calculate thermal gradient near of the walls.

This method of work, with the duct formatted with four coupled walls (or plates), permits to build and analyze a duct with four different materials, or different thermal conductivity. This situation demands attention on the contact among of the neighboring plates.

Figure 7 presents some calculated thermal profiles for duct walls. At left side of the Fig. 7 presents the output results from the FORTRAN developed program (*FV Method plus TDMA*). At right side presents the same profile obtained by commercial software.

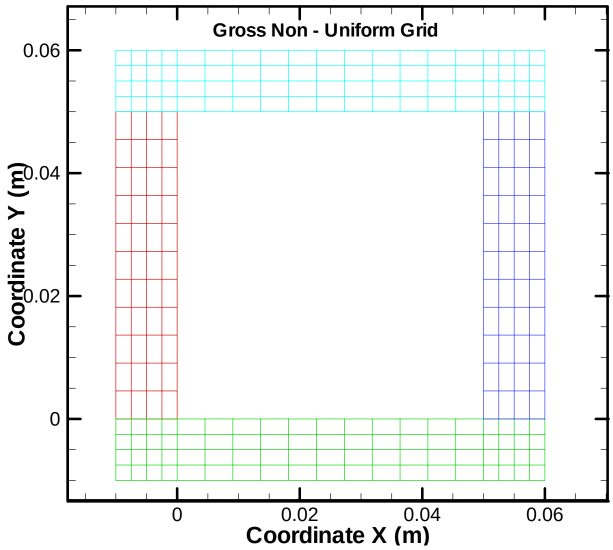

Figure 6. Gross Non-Uniform Grid for Rectangular Duct Wall Analysis.

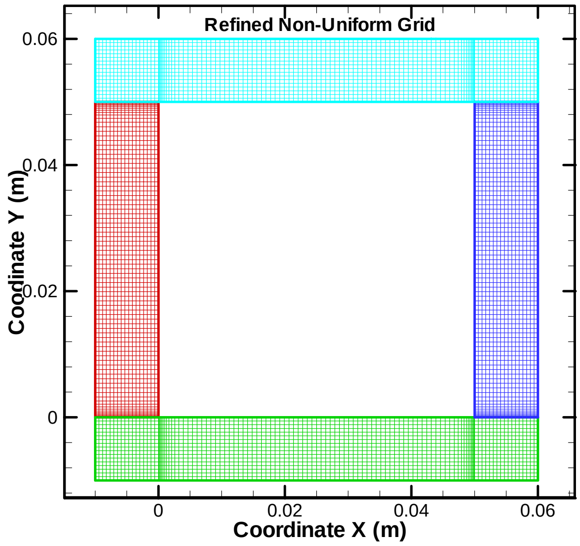

Figure 7. Refined Non-Uniform Grid for Rectangular Duct Wall Analysis.

The prescribed temperatures for the rectangular duct in the external surface of the duct, to the North, South, East and West were of 673K, 773K, 673K, and 773K, respectively. The gross grid was of 5 x 12 points for example for the plate West (Fig. 6), for a refined grid was of 18 x 83 points for example for the plate West (Fig. 7).

.

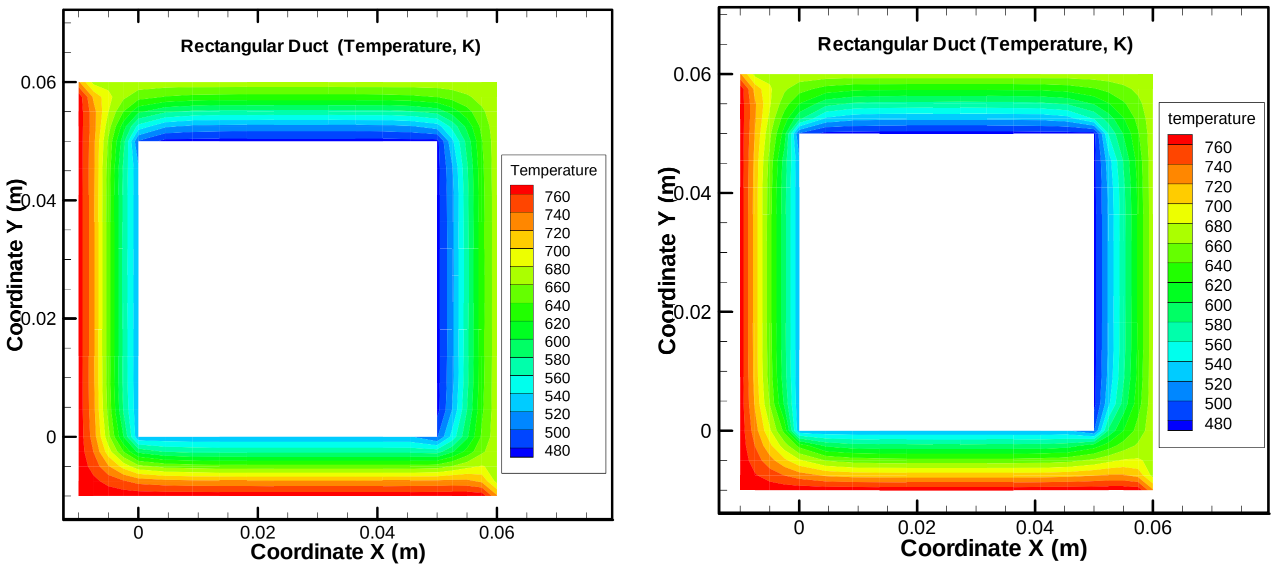

Figure 8. Thermal Profile Comparisons in the Duct Walls: a) at left: result from of the developed Fortran Program using *FV method*; b) at right: commercial software result.

4. CONCLUSIONS

There have been done comparisons between two numerical approaches of discretization, *FV and FD methods*, with two equations systems solutions: Gauss Seidel and Tri-Diagonal Matrix Algorithm (TDMA). They were implemented in the Fortran programming language. The objective was develop methods for calculate the distributions of temperature in a plate, with uniform and non-uniform grids. For the case of prescribed temperature, the method FV using TDMA, is going to represent a solution more close to a commercial software, that was adopted like a reference.

The redefined non- uniform grid, Fig. 7, can be used for the coupled of the fluid-solid. The redefined grids in the corners are ideal to simulate a viscous flow.

Still, it can be taken advantage of the analysis code for the heat conduction in thermal contact between plates with different materials, or be, with different thermal conductivities. This situation is very common in thermal systems.

As final conclusion, it can be cited that for certain specifics problems, as is the case of rectangular ducts, the use of a very refined non-uniform grid has good contribution for the calculations of thermal gradients.

5. ACKNOWLEDGEMENTS

The authors thank the “Coordenação de Aperfeiçoamento de Pessoal de Nível Superior”(CAPES).

6. REFERENCES

Ferziger, J.H.and Peric.M., 2002, “Computational Methods for Fluid Dynamics”, Third Edition, Ed.Springer-Verlag, Berlin, Germany, 423p.

Incropera, F.P.and DeWitt, D.P., 2003, “Fundamentos de Transferência de calor e de massa”, Fifth Edition, Ed.LTC, Rio de Janeiro, Brazil,698p.

Kakaç, S.and Yener, Y., 1985, “Heat Conduction”, Second Edition, Ed.Hemisphere Publishing Corporation, Washington, United States, 397p.

Ozisik, M.N., 1980, “Heat Conduction”, Ed. John Wiley & Sons, New York, United States, xxxp.

Patankar, S.V., 1980, “Numerical Heat Transfer and Fluid Flow”, Ed.Hemisphere Publishing Corporation, Washington, United States, 311p.

Versteeg, H.K.and Malalasekera.W., 1995, “An Introduction to Computational Fluid Dynamics”, Ed.Longman, London, England, 257p.

7. RESPONSIBILITY NOTICE