A RECURSIVE SYSTEM FOR INVERSE HEAT CONDUCTION PROBLEMS

Priscila Ferreira Barbosa de Sousa, priscila@mecanica.ufu.br
Ana Paula Fernandes, apfernandes@mecanica.ufu.br
Valério Luiz Borges, vlborges@mecanica.ufu.br
Solidônio Rodrigeus de Carvalho, srcarvalho@mecanica.ufu.br
Gilmar Guimaraes, gguima@mecanica.ufu.br
Federal Universtity of Uberlandia, UFU
School of Mechanical Engineering, FEMEC
Uberlandia, MG, Brasil.

George S. Dulikravich, dulikrav@fiu.edu
Florida International University, FIU
Department of Mechanical and Materials Engineering
Miami, FL, USA

Abstract. *This work uses the idea of a recursive system in order to solve multidimensional heat conduction inverse problems. The inverse technique is based on the general linear constant-coefficient difference equation that can be applied in an important class of linear time-invariant discrete-time dynamic systems. Besides the difference equation this work also uses the idea of dynamic system in order to obtain the response frequency trough of an auxiliary problem and Green's function concept. The technique is also adapted to define a thermal model transfer function that is able to use more than one temperature information at the same time that improved the accuracy and stability of the algorithm. The modified technique is applied in two experimental and controlled cases. The inverse technique proposed has a simple algorithm that is easy to implement and gives a good estimations of results.*

Keywords*: inverse problem, observers, Green's function, multiple thermocouples, digital filters.*

1. INTRODUCTION

Inverse problems can be found in all areas of science. The advantage of inverse techniques consists in the possibility of indirectly obtaining the solution of a physical problem with partially unavailable information. For example, the determination of a surface temperature without direct access or the diagnosis of diseases using computerized tomography. In both cases, the boundaries are unknown and inaccessible. These problems can be solved using information obtained from sensors located in accessible positions.

Several techniques can be found in literature for the solution of Inverse Heat Conduction Problems (IHCP). Recently, techniques based on filters such as the use of dynamic observers (Blum and Marquardt, 1997), have also been employed for the solution of the IHCP. Methods employing observers (Blum and Marquardt, 1997; Sousa *et al*., 2006) have demonstrated their flexibility and efficiency to solve one-dimensional problems.

Sousa (2006), extended the dynamic observers technique proposed by Blum and Marquardt (1997) focused on the one-dimensional linear case and presented a technique of dynamic observers based on Green's functions that can be applied directly to solve multidimensional problems. In the dynamic observer technique, the IHCP solution algorithms are interpreted as filters passing low-frequency components of the true boundary heat flux signal while rejecting high-frequency components in order to avoid excessive amplification of measurement noise (Blum and Marquardt, 1997). In order to deal with multidimensional thermal models, Sousa proposed an alternative way of obtaining the heat transfer function, G_H, by using the Green's function concept instead of taking the Laplace transform of the spatially discretized system. This procedure allows flexibility and efficiency to solve multidimensional inverse problems.

Sousa *et al.* (2008) presented a different method of obtaining the transfer function model using analytical functions instead of numerical procedures and also defined a new concept of G_H to allow the use of more than one response temperature which simplified the procedure and improved the robustness and stability of the algorithm.

The use of Green's function and the recursive dynamic system concept is the base of the inverse procedure proposed in this work. Results of experimental tests are presented and compared with the use of dynamic observers technique developed in previous work (Sousa *et al.,* 2008).

2. FUNDAMENTALS

2.1.1. Transfer function model

The 3D-transient linear heat conduction problem shown in Fig. 1 can be described by diffusion equation as

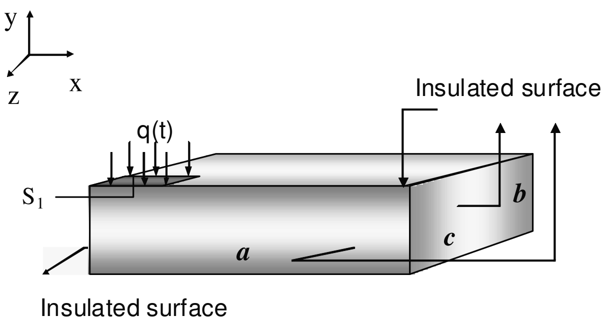


Figure 1. Three-dimensional transient thermal model

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{1a}$$

In the region R (0 < x < a, 0 < y < b, 0 < z < c) and for time t > 0, subject to boundary conditions

$$\begin{aligned} -k\left.\frac{\partial \theta}{\partial y}\right|_{y=b} &= q(t) \; on \; S_1 \; (0 \le x \le x_H ,\; 0 \le z \le z_H) \\ -k\left.\frac{\partial \theta}{\partial y}\right|_{y=b} &= 0 \; on \; S_2 \; (x, z \in \left. S\right|_{(x,z)} \notin S_1) \end{aligned} \tag{1b}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = \left.\frac{\partial \theta}{\partial x}\right|_{x=a} = \left.\frac{\partial \theta}{\partial z}\right|_{z=o} = \left.\frac{\partial \theta}{\partial z}\right|_{z=c} = \left.\frac{\partial \theta}{\partial y}\right|_{y=0} = \left.\frac{\partial \theta}{\partial y}\right|_{y=b} = 0 \tag{1c}$$

and initial condition

$$\theta(x, y, z, 0) = T(x, y, z, 0) - T_0 \tag{1d}$$

Here, S is defined by $(0 \le x \le a,\, 0 \le z \le c)$ and x_H and z_H are the boundaries of S_1 where the heat flux is applied.

The solution of Eqs. (1) can be given in terms of Green's function as in Beck *et al.* (1992)

$$\theta(x, y, z, t) = \int_{\tau=0}^{t} \left[G_H(x, y, z, t-\tau) \;\; q(\tau) \right] d\tau \tag{2}$$

where

$$G_H(x, y, z, t-\tau) = \frac{\alpha}{k} \int_0^{x_h} \int_0^{z_h} \left. G_H{}^{+}(x, y, z, t-\tau) \right|_{y'=0} dx' dz' \tag{3}$$

and $G_H{}^{+}(x, y, z, t-\tau)$ represents the Green's function of the thermal problem given by Eqs.(1).

Equation (2) reveals that an equivalent thermal model can be associated with a dynamic model. It means, a response of the input/output system can be associated to Eq. (2) in the Laplace domain as the convolution product (Özisik, 1993)

$$\theta(x, y, z, t) = G_H(x, y, z, t-\tau) * q(\tau) \tag{4}$$

This dynamic system can be represented as shown in Fig. (2). Equation (4) can also be evaluated in the Laplace domain as a single product (Bendat and Piersol, 1986) as follows

$$\overline{\theta}(x,y,z,s) = \overline{G}_H(x,y,z,s)\ \overline{q}(s) \tag{5}$$

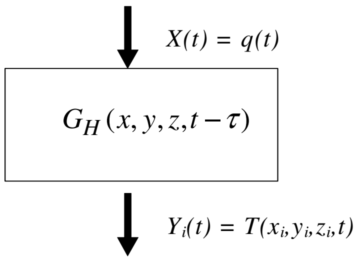


Figure 2. Dynamic thermal model system

where the Laplace transform of a *F(t)* function is defined by

$$L[F(t)] = \overline{F}(s) = \int_t^{\infty} e^{-st'} F(t')dt'$$

If heat transfer function, $\overline{G}_H(x,y,z,s)$, and temperatures response $\overline{\theta}(x,y,z,s)$ are available, Eq.(5) can be used in an inverse procedure to estimate the heat flux input, $\overline{q}(s)$. One way to estimate this heat flux input is to apply digital filter techniques that will be described in the next section.

2.1.2. Obtaining the estimators

The thermal model (Fig. 1 and 2) can also be represented by Figure 3 that considers the discrete nature of the dynamic system and treats the problem as a time-invariant dynamic systems (LTI) (Porakis and Manolakis, 1996):

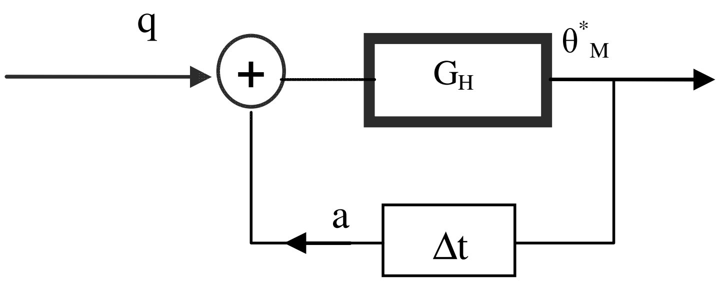


Figure 3. A single input/output recursive system

In this case the model of Figure 3 is a class of LTI that can be characterized by the general linear constant-coefficient difference equations (Porakis and Manolakis, 1996)

$$\theta_M^*(n) = +\sum_{k=0}^{M} b_k \hat{q}(n-k) - \sum_{k=1}^{N} a_k \theta_M^*(k-i) \tag{15}$$

where M and N denote, respectively, the number of frequency components to signals of heat flux and temperature and the coefficients $\{a_k\}$ and $\{b_k\}$ are constant parameters that specify the system and are independent of $\theta_M^*(n)$ and $\hat{q}(n)$. If a z-transform of a discrete-time signal $\hat{q}(n)$ is defined as a power series

$$\hat{q}(z) = \sum_{n=-\infty}^{\infty} \hat{q}(n)\, z^{-n}$$

where z is a complex variable, then such a class of linear time-invariant discrete-time systems given by Eq.(15) are also characterized by the rational system function (Porakis and Manolakis, 1996)

$$G_H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-1}} \tag{16}$$

which is a ratio of two polynomials in z^{-1}. In Equation (16), the coefficients $\{a_k\}$ and $\{b_k\}$ are identified by the frequency response characteristic of the system, G_H.

In Equation (15) the unknown heat flux *q(s)* is applied to the conductor (system), G_H, and results in a measurement signal θ^*_M corrupted by noise N, so that the corresponding temperature is

$$\theta^*_M = \theta_M + N = G_H \hat{q} + N \tag{17}$$

The estimated value $\hat{q}$ is then computed from the input data θ^*_M . Equation (15) characterizes the behavior of the solution algorithm. The next section shows the process of obtaining G_H and consequently the coefficients identification $\{a_k\}$ and $\{b_k\}$.

3. IDENTIFICATION OF HEAT TRANSFER FUNCTION, G_H, USING GREEN'S FUNCTION

As shown, the solution of Eqs. (1) can be given in terms of Green's function by Eq.(2) so that

$$T(\mathrm{x}, y, z, t) = \int_{\tau=0}^{t} \left[G_H(x, y, z, t/\tau)\ q(\tau) \right] d\tau \tag{18}$$

The Green's function is available for the homogeneous version associated with the problem defined by Eqs. (1). Although the analytical Green's function is available and exists (Beck et al., 1992) it will not be used in this work. On the contrary, the solution of the problem defined by Eqs. (1) will be performed numerically.

As already mentioned, Eq. (18) can be written in Laplace domain as

$$\overline{T}(\mathrm{x}, y, z, s) = \overline{G}_H(x, y, z, s)\ \ \overline{q}(s) \tag{18}$$

The heat transfer function $\overline{G}_H(x, y, z, s)$ can then be obtained *via* an auxiliary problem which is a homogenous version of the problem defined by Eq.(1) for the same region with a zero initial temperature and unit impulsive source located at the region of the original heating. This means that the auxiliary thermal problem can be described as

$$\frac{\partial^2 T^+}{\partial x^{+2}} + \frac{\partial^2 T^+}{\partial y^{+2}} + \frac{\partial^2 T^+}{\partial z^{+2}} = \frac{\partial T^+}{\partial t^+} \tag{19a}$$

in the region *R* and $t^+ > 0$, subjected to the boundary condition

$$-k \left. \frac{\partial T^+}{\partial y^+} \right|_{y=0} = 1 \ on\ S_1 \qquad -k \left. \frac{\partial T^+}{\partial y^+} \right|_{y=0} = 0 \ on\ S_2 \tag{19b}$$

$$\left. \frac{\partial T^+}{\partial x^+} \right|_{x^+=0} = \left. \frac{\partial T^+}{\partial x^+} \right|_{x^+=1} = \left. \frac{\partial T^+}{\partial z^+} \right|_{z^+=o} = \left. \frac{\partial T^+}{\partial z^+} \right|_{z^+=1} = \left. \frac{\partial T^+}{\partial y^+} \right|_{y^+=1} = 0 \tag{19c}$$

and the initial condition

$$T^+\left(x^+, y^+, z^+, 0\right) = 0 \tag{19d}$$

Similarly, the auxiliary thermal problem solution can be derived using Green's function and the convolution properties as

$$T^+\left(x^+, y^+, z^+, t^+\right) = G_H(x^+, y^+, z^+, t^+ - \tau^+) * 1 \tag{20}$$

Once the Laplace transform of 1 is defined as

$$L[1] = \frac{1}{s} \tag{21}$$

then

$$\overline{T}^+\left(x^+, y^+, z^+, s\right) = \overline{G}_H(x^+, y^+, z^+, s)\ \frac{1}{s} \tag{22}$$

If the dynamic system is linear, invariant and physically invariable, the response function $\overline{G}_H(x, y, z, s)$ is the same and independent of the input/output pairs. It can be obtained by

$$\overline{G}_H(x, y, z, s) = s\ \overline{T}^+\left(x, y, z, s\right) \tag{23}$$

In order to complete the $\overline{G}_h(x, y, z, s)$ identification, the $\overline{T}^+(x, y, z, s)$ must be obtained at a specific position r_i = $(x_i,\ y_i,\ z_i)$. A simple and efficient procedure is proposed here to obtain $T^+(r_i, s)$. If Eq.(22) represents a cross correlation function of the two functions of stationary random process $\boldsymbol{s}$ and $\overline{T}^+\left(x, y, z, s\right)$, then $\overline{G}_H(x, y, z, s)$ will be independent of the absolute time $\boldsymbol{t}$ and will depend only on the time separation $\boldsymbol{t_a}$. In this case, the function $T^+(r_i, s)$ can be fitted by a polynomial in the sampled interval [0, t_a] as

$$T^+\left(r_i, t\right) = a_0 + a_1\ t + a_2\ t^2 + a_3\ t^3 + \cdots \tag{24}$$

where $\boldsymbol{a_i}$ are the polynomial coefficients. Then , taking the Laplace transform of Eq.(24) gives

$$\overline{T}^+\left(r_i, s\right) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2s^3} + \frac{a_3}{6s^4} + \cdots \tag{25}$$

Thus, from Eq. (21), G_H can be written as

$$\overline{G}_H(r_i, s) = s\ \overline{T}^+\left(r_i, s\right) = \ = s\left[\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2s^3} + \frac{a_3}{6s^4} + \cdots + \right] \tag{26}$$

or $$\overline{G}_H(r_i, s) = a_0 + \frac{a_1}{s} + \frac{a_2}{2s^2} + \frac{a_3}{6s^3} + \cdots + \tag{27}$$

It can be observed that from the theory of partial fractions, if $\overline{G}_H(r_i, s)$ is expressible in partial fractions as in Eq. (27) its inversion is readily obtained by using the Laplace transform table. Since, Eq. (27) does not present any pole for s > 0, then its inversion is stable. This fact guarantees robustness to the inverse algorithm given by Eqs. (15). Another advantage of this procedure is that the same procedure can be used indistinctly by one, two or three-dimensional models, provided the only active boundary condition is the unknown heat source.

The next section shows the results obtained with this technique for two experimental controlled cases.

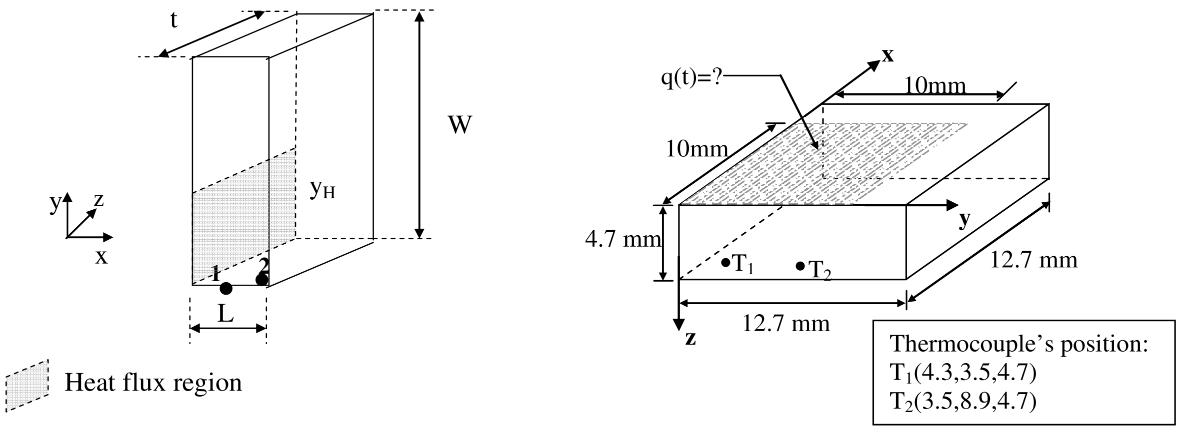
3. RESULTS AND DISCUSSION

3.1. Experimental test

The new technique proposed here is now verified by estimating a heat flux imposed in two controlled experiments (Test 1 and 2). Figure 4 presents the geometry and position of the thermocouples located in two AISI 304 samples.

The AISI304 stainless steel sample used in test 1 has thickness of 6 mm and lateral dimensions of 50 x 138 mm while in test 2 it has thickness of 4.7 mm and lateral dimensions of 10 x 12.7 mm. Both samples initially in thermal equilibrium at T_0 are submitted to a unidirectional and uniform heat flux. The heat flux is supplied by a 318 Ω electrical resistance heater, covered with silicone rubber, with lateral dimensions of 50 x 50 mm and 10 x 10 mm, respectively for test 1 and test 2. Both electrical heaters have 0.3 mm of thickness. The temperatures are measured using surface thermocouples (type k). The signals of tension and current temperatures are acquired by a data acquisition system HP Series 75000 with voltmeter E1326B controlled by a personal computer.

Two thermocouples were brazed on each sample as shown in Figure 4 and Table 1.



a) b)

Figure 4. Experimental scheme for a) Test 1; b) Test 2

Table 1. Geometry and thermocouple position for Test 1

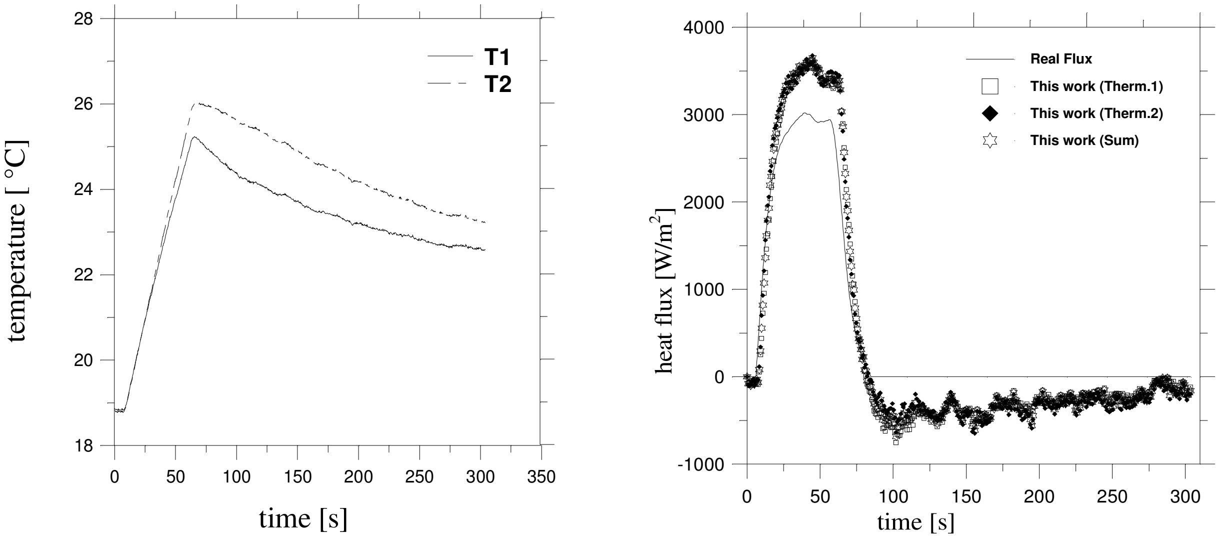
y_H = 0.05m	L = 0.006m	W = 0.138 m	T = 0.05m
position	$x_i\ 10^{-3}$[m]	$y_i 10^{-3}$[m]	$z_i 10^{-3}$[m]
1	3.1	0.0	0.0
2	5.2	0.0	22.2

3.2.1. 2D Experimental test

Figure 5a shows the temperature evolution for the two thermocouples in test 1. The estimated heat flux using the technique described here is shown in Fig. 5b. In order to compare the results, experimentally obtained temperatures were compared with estimated temperatures using the heat flux estimated by dynamic observer technique (Sousa *et al.,* 2008).

Figure 5b presents estimation results by considering experimental data using only one position (position 1 and 2) and considering both thermocouple data.

It can be observed that the technique proposed here over-estimates the real heat flux for this 2D case. However, the behavior of heat flux estimates is very good and the technique appears promising. One possibility to improve the accuracy could be to add some regularization terms in order to obtain more accurate and stable results.

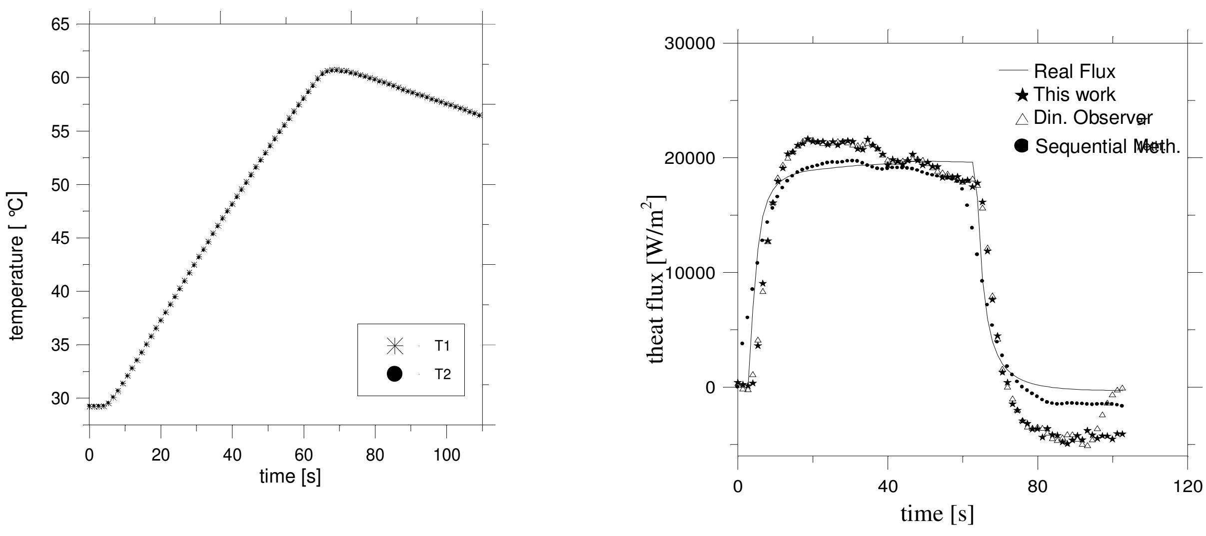


a) b)

Figure 5. Experimental data for test 1: a) experimental temperature evolution; heat flux estimation: b) heat flux evolution estimation using the technique proposed here.

3.2.2. 3D Experimental test

Similarly signals evolution of the two thermocouples and estimation results for the test 2 are shown in Fig. 6. In order to compare the inverse technique proposed here, the heat flux imposed in test 2 is also estimated using the sequential function specification method described by Beck *et al.* (1992) and using the dynamic observer techniques based on Green's functions (Sousa *et al.,* 2008). This comparison is shown in Fig. 6b. Residuals are presented in Fig. 7. The results show a difference less than 0.3 C that represents the uncertainty of thermocouple measurements.



a) b)

Figure 6. Experimental data for test 2: a) experimental temperature evolution; heat flux estimation: b) Heat flux estimation using the technique proposed here and dynamic observer methods.

The great advantage of the technique proposed here is the easy and fast numerical implementation for any 1D, 2D or 3D model. The simplicity and direct algorithm also give this procedure a great potential in inverse techniques application.

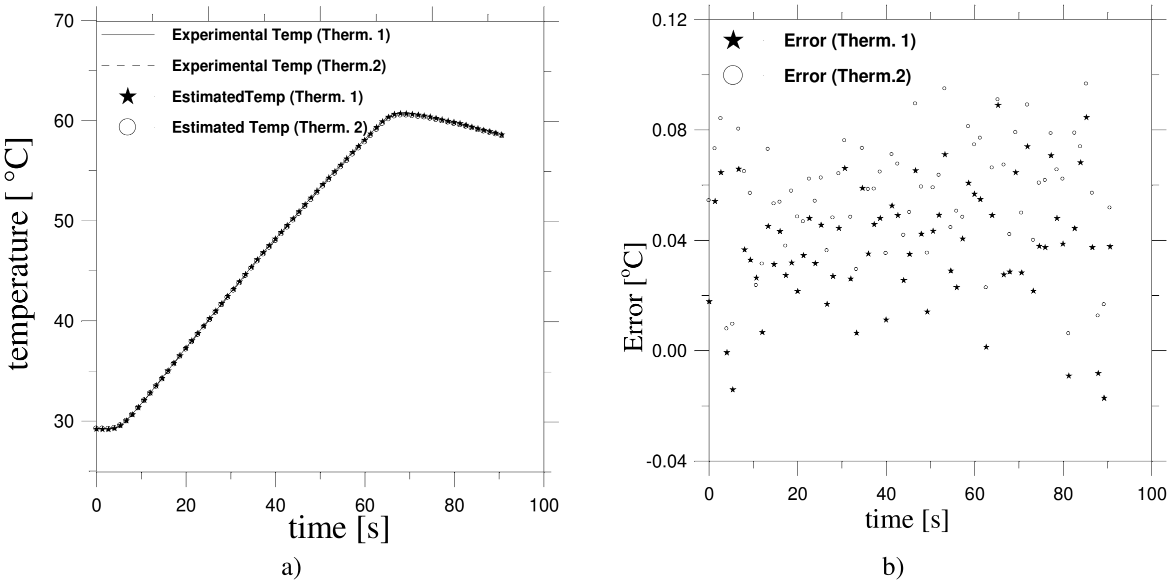


Figure 7. a) Measured and calculated temperature using the heat flux estimated; b) errors= (T. M. –T. C.)

4. CONCLUSIONS

This work has proposed a recursive system with Green's function concept in order to solve a three-dimensional inverse heat conduction problem. The good results have shown the great potential of the technique proposed. The great advantage comparing with other inverse techniques is the easy algorithm implementation. For example, if comparing with the dynamic observers that are no needs of previous analysis in order to design the Chebyshev filter. Although previous tests (not shown here) have demonstrate some instability for simulated cases the technique appears to be a great initial guess and can be improved by using regularization techniques.

5. ACKNOWLEDGEMENTS

The first five authors are thankful to Fapemig and CNPq for financial support of this research.

6. REFERENCES

Beck J. V.; Blackwell J. C., 1985, " Inverse Heat Conduction". 2.ed., N.Y., John Wiley and Sons. 300p.

Beck J. V., Cole K. D., Haji-Sheik A, Lithouki B., 1992, " Heat Conduction Using Green's Function", Hemisphere Publishing Corporation, USA, p. 523.

Bendat J. S., Piersol A. G., 1986, "Analysis and Measurement Procedures", Wiley-Intersience, 2nd ed., USA, p. 566.

Blum, J. W., Marquardt, W., 1997, "An optimal solution to inverse heat conduction problems based on frequency-domain interpretation and observers", Numerical Heat Transfer, Part B: Fundamentals, 32, December 1997, 453-478.

Özisik M. N. 1993, "Heat Conduction", John Wiley & Sons, New York.

Sousa, P. F. B., 2006, "Development of a technique based on Green's Functions and Dynamic Observers to be Applied in Inverse Problems", Dissertation of Master's degree, Federal University of Uberlândia, Uberlândia, Mg. Brazil, (in Portuguese).

Proakis, J. G., Manolakis, D. M., 1996, "Digital Signal Processing, Principles, Algorithms, and Applications", 3Ed., Prentice Hall, New Jersey, USA, 967p.

Sousa P. F. B, Fernandes A. P., Guimaraes G., 2007, "A procedure to optimaze dynamic observers by tuning parameters", Proceedings of the 19th International Congress of Mechanical Engineering, Brasília, DF Brazil.

Sousa P. F. B, Fernandes A. P., Borges V. L., Dulikravich G. S., Guimaraes G., 2008, "Dynamic Observer Method Based on Modified Green's Functions for Robust and More Stable Inverse Algorithims", Proceedings of the 2008 International Design Engineering Technical Conference & Computers and Information in Engineering Conference IDETC/CIE 2008, New York City, NY, USA.

7. RESPONSIBILITY NOTICE