A DISCONTINUOUS STABILIZED FINITE ELEMENT METHOD FOR VISCOPLASTIC FLUIDS

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Abstract. *The first model for viscoplastic fluids was idealized by Bingham consisting in a superposition of a Newtonian term and the residual stress term, with the material flowing only when the shear stress is higher than a yield stress. This system, which presents a singularity, was handled by Glowinski to a one variable problem by using a lagrangean multiplier and solving the formulation through a regularization method. Viscoplastic flow problems of incompressible fluids can be modelled by the equations of motion, mass balance and momentum balance. Few numerical methods have been proposed for these problems and, in general, they transfer the instabilities to the boundaries, resulting in unstable pressure fields. Karam and Loula proposed a mixed stabilized finite element formulation in velocity and discontinuous pressure variables able to handle the incompressibility constraint. Although obtaining stable results for linear case, when the Bingham relation is considered it is difficult to obtain theoretically the range for the stabilizing parameters. Based on both approaches, in this work we propose a mixed regularized stabilized finite element method in velocity and discontinuous pressure to which it is possible to obtain mathematically the range of stabilizing parameters, not limited to low Bingham numbers.*

Keywords: *Viscoplasticity, Bingham fluids, Finite elements, Stabilized methods.*

1. INTRODUCTION

Viscoplasticity, as idealized by Bingham (1922), is a phenomenon characterized by the existence of a residual value for the shear stress, beyond which the material would present a viscous flow. He attempted to the fact that before flowing as in a Newtonian way, those material systems behaved as plastic solids and called them viscoplastic fluids. Defining τ_y as the residual stress, or yield stress, and μ as the plastic viscosity the first model for this behavior was

$$\begin{aligned} \tau(\mathbf{u}) = \tau_y + \mu\dot{\gamma} \quad &\Leftrightarrow \quad \tau(\mathbf{u}) > \tau_y, \\ \dot{\gamma}(\mathbf{u}) = 0 \quad &\Leftrightarrow \quad \tau(\mathbf{u}) \leq \tau_y \end{aligned} \tag{1}$$

System (1) presents a singularity, which was handled by Glowinski (1976) to a one variable problem by using a lagrangean multiplier and solving the formulation through a regularization method. Viscoplastic flow problems of incompressible fluids can be modelled by the following system of equations:

$$\begin{aligned} -\operatorname{div}(\tau(\mathbf{u})) + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \text{ in } \Omega \end{aligned} \tag{2}$$

with $\mathbf{u} = \overline{\mathbf{u}}$ on $\partial\Omega$ and $\tau(\mathbf{u})$ is given by (1); Ω and $\partial\Omega$ are the domain and its boundary, respectively. Few numerical methods have been proposed for these problems and, in general, they transfer the instabilities to the boundaries, resulting in unstable pressure fields. To solve linear incompressible fluid problems, Karam and Loula (1991) proposed a mixed stabilized finite element formulation in velocity and discontinuous pressure variables able to handle the incompressibility constraint. Although obtaining stable results for linear case, when (1) is considered it is difficult to obtain theoretically the range of the stabilizing parameters.

Based on both approaches, in this work we propose a mixed regularized stabilized finite element method in velocity and discontinuous pressure to which it is possible to obtain mathematically the range of stabilizing parameters even for high yield stress values for the generalized Stokes problem and stable pressure fields for high Bingham numbers.

2. REGULARIZED STABILIZED MIXED FORMULATION

Let $S_h^k(\Omega)$ be the finite element space of order k and class C^0, and $Q_h^l(\Omega)$ be the finite element space of order l and class C^{-1}. For the velocity and pressure fields consider the following approximation spaces

$$V_h = (S_{0h}^k(\Omega))^2 = (S_h^k(\Omega) \cap H_0^1(\Omega)) \subset V, \qquad W_h = Q_h^l(\Omega) \subset L_0^2(\Omega). \tag{3}$$

The new variational formulation for the Generalized Stokes problem with Bingham equation consists in the following saddle point problem.

Problem $\mathcal{L}_h$**:** *Find* $\{\mathbf{u_h}, p_h\} \in \mathbf{V_h} \times \mathbf{W_h}$*, such that*

$$\mathcal{L}_h(\mathbf{u}_h, q_h) \leq \mathcal{L}_h(\mathbf{u}_h, p_h) \leq \mathcal{L}_h(\mathbf{v}_h, p_h), \qquad \forall\{\mathbf{v}_h, q_h\} \in V_h \times W_h, \tag{4}$$

with the Lagrangian $\mathcal{L}_h(\mathbf{v}_h, q_h)$ *given by*

$$\mathcal{L}_h(\mathbf{v}_h, q_h) = L_h(\mathbf{v}_h, q_h) + \frac{\delta_1 h^2}{2\mu}\| - 2\mu \operatorname{div} \boldsymbol{\epsilon}(\mathbf{v}_h) + \nabla q_h - \mathbf{f}\|_h^2 + \delta_2 2\mu \| \operatorname{div} \mathbf{v}_h\|^2$$

where

$$L_h(\mathbf{v}_h, q_h) = a(\mathbf{v}_h, \mathbf{v}_h) + j_\eta(\mathbf{v}_h) + b(q_h, \mathbf{v}_h) - f(\mathbf{v}_h)$$

and

$$a(\mathbf{u}_h, \mathbf{v}_h) = 2\mu(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v}_h)), \qquad b(q_h, \mathbf{v}_h) = -(q_h, \operatorname{div} \mathbf{v}_h), \qquad j_\eta(\mathbf{v}_h) = \tau_y \int_\Omega \frac{\boldsymbol{\epsilon}(\mathbf{u}_h) \cdot \boldsymbol{\epsilon}(\mathbf{v}_h)}{\sqrt{|\boldsymbol{\epsilon}(\mathbf{u}_h)|^2 + \eta^2}} d\Omega.$$

where δ_1 and δ_2 are the stabilization parameters to be fixed, μ is the newtonian viscosity, τ_y is the yield stress, η is the regularization parameter and $\| \cdot \|_h$ is the L^2 norm taken elementwise. The saddle point generated by the $\mathcal{L}_h$ functional is characterized by

Problem $\overline{PGG}_h$**:** *Given* $\mathbf{f} \in \mathbf{V'_h}$*, find* $\{\mathbf{u_h}, p_h\} \in \mathbf{V_h} \times \mathbf{W_h}$*, such that*

$$A_h(\{\mathbf{u}_h, p_h\}; \{\mathbf{v}_h, q_h\}) + j_\eta(\mathbf{v}_h) + b(p_h, \mathbf{v}_h) + b(q_h, \mathbf{u}_h) = F_h(\mathbf{v}_h, q_h), \quad \forall\{\mathbf{v_h}, q_h\} \in \mathbf{V_h} \times \mathbf{W_h} \tag{5}$$

with

$$\begin{aligned} A_h(\{\mathbf{u}_h, p_h\}; \{\mathbf{v}_h, q_h\}) &= a(\mathbf{u_h}, \mathbf{v_h}) + \delta_2 2\mu(\operatorname{div} \mathbf{u_h}, \operatorname{div} \mathbf{v_h}) \\ &+ \frac{\delta_1 h^2}{2\mu}(-2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{u_h})) + \nabla p_h, -2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{v_h})) + \nabla q_h)_h \end{aligned}$$

$$F_h(\mathbf{v}_h, q_h) = f(\mathbf{v_h}) + \frac{\delta_1 h^2}{2\mu}(\mathbf{f}, -2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{v_h})) + \nabla q_h)_h$$

To make the analysis easier we decompose the pressure p_h at the element level into a zero mean valued function, p_h^*, and a constant by part $\overline{p}_h$, that is

$$p_h = p_h^* + \overline{p}_h, \qquad p_h^* \in W_h^*, \qquad \overline{p}_h \in \bar{W}_h \tag{6}$$

such that

$$(W_h^* \subset W_h) = \{p_h^* \in L^2 : \int_{\Omega^e} p_h^* d\Omega = 0; \qquad \nabla p_h^e = \nabla {p_h^*}^e\}; \tag{7}$$

$$(\overline{W}_h \subset W_h) = \{\overline{p}_h \in L^2 : \nabla \overline{p}_h^e = 0, \qquad \overline{p}^e = \int_{\Omega^e} p^e d\Omega / \int_{\Omega^e} d\Omega\}, \tag{8}$$

Using decomposition (6) we can restate the Problem PGG$_h$ as

Problem PGG$_h$**:** *Find* $\{\mathbf{u_h}, p_h\} \in \mathbf{V_h} \times \mathbf{W_h}$*, such that*

$$\begin{aligned} A_h^*(\{\mathbf{u}_h, p_h^*\}; \{\mathbf{v}_h, q_h^*\}) + j_\eta(\mathbf{v}_h) + b(\overline{p}_h, \mathbf{v}_h) &= F_h^*(\mathbf{v}_h, q_h^*), \quad \forall v_h \in \mathbf{V_h}, \forall q_h^* \in \mathbf{W_h^*}, \\ b(\overline{q}_h, \mathbf{u}_h) &= 0, \forall \overline{q}_h \in \overline{W}_h, \end{aligned}$$

with

$$\begin{aligned} A_h^*(\{\mathbf{u}_h, p_h^*\}; \{\mathbf{v}_h, q_h^*\}) &= a(\mathbf{u_h}, \mathbf{v_h}) + b(p_h^*, \mathbf{v}_h) + b(q_h^*, \mathbf{u}_h) \\ &+ \frac{\delta_1 h^2}{2\mu}(-2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{u_h})) + \nabla p_h^*, -2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{v_h})) + \nabla q_h^*)_h \\ &+ \delta_2 2\mu(\operatorname{div} \mathbf{u_h}, \operatorname{div} \mathbf{v_h}) \end{aligned} \tag{9}$$

$$F_h^*(\mathbf{v}_h, q_h^*) = f(\mathbf{v_h}) + \frac{\delta_1 h^2}{2\mu}(\mathbf{f}, -2\mu \operatorname{div}(\boldsymbol{\epsilon}(\mathbf{v_h})) + \nabla q_h^*)_h \tag{10}$$

$$b(\overline{q}_h, \mathbf{u}_h) = -(\overline{q}_h, \operatorname{div} \mathbf{u}_h), \tag{11}$$

3. NUMERICAL ANALYSIS BY BREZZI'S THEOREM

Identifying the subspace $\overline{K}_h \subset \mathbf{V}_h$, we may consider the reduced form,

Problem PGG$_{hR}$:*Find* $\{\mathbf{u_h}, p_h\} \in \mathbf{V_h} \times \mathbf{W_h}$, *such that*

$$\overline{A_h^*}(\{\mathbf{u}_h, p_h^*\}; \{\mathbf{v}_h, q_h^*\}) \quad = \quad F_h^*(\mathbf{v}_h, q_h^*), \quad \forall \mathbf{v}_h \in \overline{K}_h, \forall q_h^* \in W_h^*, \tag{12}$$

with

$$\overline{K}_h = \{\mathbf{v}_h \in V_h : b(\overline{q}_h, \mathbf{v}_h) = 0, \forall \overline{q}_h \in \overline{W}_h\}, \tag{13}$$

$$\overline{A_h^*}(\{\mathbf{u}_h, p_h^*\}; \{\mathbf{v}_h, q_h^*\}) = A_h^*(\{\mathbf{u}_h, p_h^*\}; \{\mathbf{v}_h, q_h^*\}) + j(\mathbf{v}_h) \tag{14}$$

Existence, uniqueness and stability of solution for Problem PGG$_{hR}$ can be obtained by verifying the following (i), (ii), and (iii) Brezzi's conditions, Brezzi (1974). Recalling the definition of the norm in the product space $V \times W$,

$\|\{v_h, q_h\}\| = \|v_h\|_1 + \|q_h\|_0$ or $\|\{v_h, q_h\}\|^2 = \|v_h\|_1^2 + \|q_h\|_0^2$.

we have:

(i) Continuity if $\overline{A_h^*}(\{\cdot,\cdot\};\{\cdot,\cdot\})$ and $b(\cdot,\cdot)$: Using Hölder-Schwarz and Minkowski inequalities, from (14) we have

$$\begin{aligned}
|\overline{A_h^*}(\mathbf{u}_h, p_h^*; \mathbf{v}_h, q_h^*)| \quad &\leq \quad 2\mu\|\boldsymbol{\epsilon}(\mathbf{u}_h)\|\|\boldsymbol{\epsilon}(\mathbf{v}_h)\| + \left|\tau_y \int_\Omega \frac{\boldsymbol{\epsilon}(\mathbf{u}_h)\cdot\boldsymbol{\epsilon}(\mathbf{v}_h)}{\sqrt{|\boldsymbol{\epsilon}(\mathbf{u}_h)|^2 + \eta^2}} dx\right| \\
&+ \quad \frac{\delta_1 h^2}{2\mu}\left(\|2\mu \operatorname{div} \boldsymbol{\epsilon}(\mathbf{u}_h)\|_h + \|\nabla p_h^*\|_h\right)\left(\|2\mu \operatorname{div} \boldsymbol{\epsilon}(\mathbf{v}_h)\|_h + \|\nabla q_h^*\|_h\right) \\
&+ \quad \|p_h^*\|\|\operatorname{div} \mathbf{v}_h\| + \|q_h^*\|\|\operatorname{div} \mathbf{u}_h\| + \delta_2 2\mu\|\operatorname{div} \mathbf{u}_h\|\|\operatorname{div} \mathbf{v}_h\|
\end{aligned}$$

Since

$$\|\mathbf{u}_h\|_1 \geq \|\boldsymbol{\epsilon}(\mathbf{u}_h)\| \geq \frac{1}{\sqrt{n}}\|\operatorname{div} \mathbf{u}_h\|, \tag{15}$$

where n is the domain dimension, and by the inverse estimates for the velocity and the pressure fields, i.e.,

$$h\|\operatorname{div} \boldsymbol{\epsilon}(\mathbf{v}_h)\|_h \leq \gamma_1\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|, \qquad h\|\nabla q_h^*\|_h \leq \gamma_2\|q_h^*\|,$$

and by considering $C_1 \leq |\boldsymbol{\epsilon}(v_h)| \leq C_2$ we obtain

$$\begin{aligned}
|\overline{A_h^*}(\mathbf{u}_h, p_h^*; \mathbf{v}_h, q_h^*)| \quad &\leq \quad 2\mu(1 + n\delta_2)\|\,\mathbf{u}_h\|_1\|\mathbf{v}_h\|_1 + \frac{\tau_y}{\sqrt{C_1^2 + \eta^2}}\|\,\mathbf{u}_h\|_1\|\mathbf{v}_h\|_1 \\
&+ \quad \frac{\delta_1}{2\mu}\left(2\mu\gamma_1\|\boldsymbol{\epsilon}(\mathbf{u}_h)\| + \gamma_2\|\,p_h^*\|\right)\left(2\mu\gamma_1\|\boldsymbol{\epsilon}(\mathbf{v}_h)\| + \gamma_2\|\,q_h^*\|\right) \\
&+ \quad \sqrt{n}\left(\|p_h^*\|\|\mathbf{v}_h\|_1 + \|q_h^*\|\|\mathbf{u}_h\|_1\right).
\end{aligned} \tag{16}$$

Using the product space norm we have the continuity through

$$|\overline{A_h^*}(\mathbf{u}_h, p_h^*; \mathbf{v}_h, q_h^*)| \quad \leq \quad \alpha_1(\tau_y)\|\{\mathbf{u}_h, p_h^*\}\|\|\{\mathbf{v}_h, q_h^*\}\|$$

with

$$\alpha_1(\tau_y) = \max\left\{2\mu(1 + n\delta_2) + \frac{\tau_y}{\sqrt{C_1^2 + \eta^2}}, \delta_1\gamma_1, \frac{\delta_1\gamma_2}{2\mu}, \sqrt{n}\right\} \tag{17}$$

independent of h but depending on τ_y.

The continuity of $b(\cdot,\cdot)$ comes out from the application of the Hölder inequality and the use of (15) on $b(\overline{q}_h, \mathbf{u}_h)$, yielding

$$|b(\overline{q}_h, \mathbf{u}_h)| \leq \sqrt{n}\|\overline{q}_h\|_W\|\mathbf{u}_h\|_V. \tag{18}$$

(b) K_h-ellipticity of $\overline{A_h^*}(\{\cdot,\cdot\};\{\cdot,\cdot\})$:
Since

$$\begin{aligned}\overline{A_h^*}(\{\mathbf{v}_h,q_h^*\};\{\mathbf{v}_h,q_h^*\}) &= 2\mu\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|^2+\tau_y\int_\Omega\frac{\boldsymbol{\epsilon}(\mathbf{v}_h)\cdot\boldsymbol{\epsilon}(\mathbf{v}_h)}{\sqrt{|\boldsymbol{\epsilon}(\mathbf{v}_h)|^2+\eta^2}}dx\\ &+ \frac{\delta_1h^2}{2\mu}\|-2\mu\,\mathrm{div}\,\boldsymbol{\epsilon}(\mathbf{v}_h)+\nabla q_h^*\|_h^2+\delta_2 2\mu\|\,\mathrm{div}\,\mathbf{v}_h\|^2-2(q_h^*,\mathrm{div}\,\mathbf{v}_h)\end{aligned}$$

and considering the bounds of $\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|$ so that

$$\tau_y\int_\Omega\frac{\boldsymbol{\epsilon}(\mathbf{v}_h)\cdot\boldsymbol{\epsilon}(\mathbf{v}_h)}{\sqrt{|\boldsymbol{\epsilon}(\mathbf{v}_h)|^2+\eta^2}}dx\geq\frac{\tau_y}{\sqrt{C_2^2+\eta^2}}\int_\Omega\boldsymbol{\epsilon}(\mathbf{v}_h)\cdot\boldsymbol{\epsilon}(\mathbf{v}_h)dx.$$

one obtains

$$\begin{aligned}\overline{A_h^*}(\{\mathbf{v}_h,q_h^*\};\{\mathbf{v}_h,q_h^*\}) &\geq 2\mu\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|^2+\frac{\tau_y}{\sqrt{C_2^2+\eta^2}}\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|^2\\ &+ \frac{\delta_1h^2}{2\mu}\|-2\mu\,\mathrm{div}\,\boldsymbol{\epsilon}(\mathbf{v}_h)+\nabla q_h^*\|_h^2+\delta_2 2\mu\|\,\mathrm{div}\,\mathbf{v}_h\|^2-2(q_h^*,\mathrm{div}\,\mathbf{v}_h).\end{aligned}$$

Using the triangle inequality, $\|a-b\|^2\geq(\|a\|-\|b\|)^2$, we may write

$$\begin{aligned}\overline{A_h^*}(\{\mathbf{v}_h,q_h^*\};\{\mathbf{v}_h,q_h^*\}) &\geq 2\mu\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|^2+\frac{\tau_y}{\sqrt{C_2^2+\eta^2}}\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|^2+\frac{\delta_1h^2}{2\mu}4\mu^2\|\,\mathrm{div}\,\boldsymbol{\epsilon}(\mathbf{v}_h)\|_h^2\\ &+ \frac{\delta_1h^2}{2\mu}\|\nabla q_h^*\|_h^2-2\frac{\delta_1h^2}{2\mu}\int_\Omega|2\mu\,\mathrm{div}\,\boldsymbol{\epsilon}(v_h)\cdot\nabla q_h^*|dx\\ &+ \delta_2 2\mu\|\,\mathrm{div}\,\mathbf{v}_h\|^2-2(q_h^*,\mathrm{div}\,\mathbf{v}_h).\end{aligned}$$

Applying Körn inequality, $\|\boldsymbol{\epsilon}(v_h)\|^2\geq C\|v_h\|_1^2$, using the inverse estimate $h\|\,\mathrm{div}\,\boldsymbol{\epsilon}(v_h)\|_h\leq\gamma_1\|\boldsymbol{\epsilon}(v_h)\|$ in a borrowed part of $\|v_h\|_1^2$, we have

$$\begin{aligned}\overline{A_h^*}(\{\mathbf{v}_h,q_h^*\};\{\mathbf{v}_h,q_h^*\}) &\geq \mu C\|\mathbf{v}_h\|_1^2+\mu\frac{h^2}{\gamma_1^2}\|\,\mathrm{div}\,\boldsymbol{\epsilon}(\mathbf{v}_h)\|_h^2+\frac{\tau_y}{\sqrt{C_2^2+\eta^2}}\frac{h^2}{\gamma_1^2}\|\,\mathrm{div}\,\boldsymbol{\epsilon}(\mathbf{v}_h)\|_h^2\\ &+ \delta_1h^2 2\mu\|\,\mathrm{div}\,\boldsymbol{\epsilon}(\mathbf{v}_h)\|_h^2+\frac{\delta_1h^2}{2\mu}\|\nabla q_h^*\|_h^2\\ &- 2\delta_1h^2\int_\Omega|\,\mathrm{div}\,\boldsymbol{\epsilon}(v_h)\cdot\nabla q_h^*|dx+\delta_2 2\mu\|\,\mathrm{div}\,\mathbf{v}_h\|^2-2(q_h^*,\mathrm{div}\,\mathbf{v}_h).\end{aligned}$$

Using the the Young inequality, with arbitrary ξ, int the product term $(\cdot,\cdot)_h$, we have

$$\begin{aligned}\overline{A_h^*}(\{\mathbf{v}_h,q_h^*\};\{\mathbf{v}_h,q_h^*\}) &\geq \mu C\|\mathbf{v}_h\|_1^2+\left[\frac{\mu}{\gamma_1^2}+2\mu\delta_1+\frac{\tau_y}{\sqrt{C_2^2+\eta^2}}\frac{1}{\gamma_1^2}-\delta_1\xi\right]h^2\|\,\mathrm{div}\,\boldsymbol{\epsilon}(\mathbf{v}_h)\|_h^2\\ &+ \delta_1h^2\left(\frac{1}{2\mu}-\frac{1}{\xi}\right)\|\nabla q_h^*\|_h^2+\delta_2 2\mu\|\,\mathrm{div}\,\mathbf{v}_h\|^2-2(q_h^*,\mathrm{div}\,\mathbf{v}_h).\end{aligned}$$

Choosing ξ such that

$$\frac{\mu}{\gamma_1^2}+2\mu\delta_1+\frac{\tau_y}{\sqrt{C_2^2+\eta^2}}\frac{1}{\gamma_1^2}-\delta_1\xi=0, \tag{19}$$

we have

$$\overline{A_h^*}(\{\mathbf{v}_h,q_h^*\};\{\mathbf{v}_h,q_h^*\})\geq\mu C\|\mathbf{v}_h\|_1^2+\frac{\delta_1h^2}{2\mu}\frac{\chi(\tau_y)}{\chi(\tau_y)+2\delta_1}\|\nabla q_h^*\|_h^2+\delta_2 2\mu\|\,\mathrm{div}\,\mathbf{v}_h\|^2-2(q_h^*,\mathrm{div}\,\mathbf{v}_h)$$

where

$$\chi(\tau_y)=\frac{1}{\gamma_1^2\mu}\left(\mu+\frac{\tau_y}{\sqrt{C_2^2+\eta^2}}\right). \tag{20}$$

Aplying again Young's inequality, now to the last term, with with arbitrary ζ we have

$$\overline{A_h^*}(\{\mathbf{v}_h, q_h^*\}; \{\mathbf{v}_h, q_h^*\}) \geq \mu C\|\mathbf{v}_h\|_1^2 + \frac{\delta_1 h^2}{2\mu}\frac{\chi(\tau_y)}{\chi(\tau_y)+2\delta_1}\|\nabla q_h^*\|_h^2 + [\delta_2 2\mu - \zeta]\,\|\operatorname{div}\mathbf{v}_h\|^2 - \frac{1}{\zeta}\|q_h^*\|^2.$$

Choosing ζ such that, $2\mu\delta_2 - \zeta \Rightarrow \xi' = 2\mu\delta_2$, and aplying the Poincaré inequality elementwise to the pressure,

$$h^2\|\nabla q_h^*\|^2 \geq \rho\|q_h^*\|^2, \tag{21}$$

we obtain

$$\overline{A_h^*}(\{\mathbf{v}_h, q_h^*\}; \{\mathbf{v}_h, q_h^*\}) \geq \alpha_2(\tau_y)\left(\|v_h\|_1^2 + \|q_h^*\|^2\right) \geq \alpha_2(\tau_y)\|\{v_h, q_h^*\}\|^2 \tag{22}$$

with

$$\alpha_2(\tau_y) = \min\left\{\mu C, \frac{1}{2\mu}\left(\frac{\delta_1\beta\chi(\tau_y)}{\chi(\tau_y)+2\delta_1} - \frac{1}{\delta_2}\right)\right\}. \tag{23}$$

independent of h. Since $\alpha_2(\tau_y)$ must be positive, it is sufficient that δ_1, δ_2 and τ_y satisfy

$$\frac{1}{2\mu}\left(\frac{\delta_1\beta\chi(\tau_y)}{\chi(\tau_y)+2\delta_1} - \frac{1}{\delta_2}\right) > 0. \tag{24}$$

(c) LBB Condition: For $b(\overline{q}_h, \mathbf{v}_h)$ given by (11) and $\{\mathbf{v}_h, \overline{q}_h\} \in V_h \times \overline{W}_h$, there exists a constant $\beta_h \geq 0$, independent of h, such that

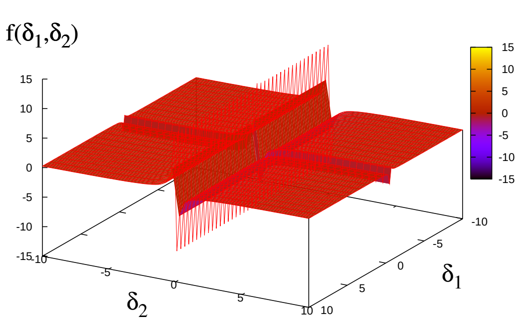
$$\sup_{\mathbf{v}_h \in V_h} \frac{|b(\overline{q}_h, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_V} \geq \beta_h\|\overline{q}_h\|_W, \forall\overline{q}_h \in \overline{W}_h. \tag{25}$$

This result may be seen in (Girault and Raviart, 1986). With conditions (i), (ii) and (iii) satisfied, in accordance with Brezzi's theorem, Problem PGG$_h$ has a unique solution.

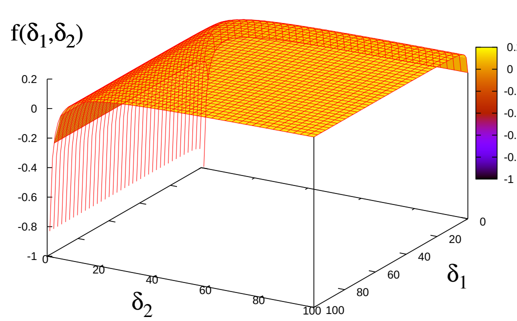
From the coercivity of $\overline{A_h^*}(\{\mathbf{v}_h, q_h^*\}; \{\mathbf{v}_h, q_h^*\})$ it came out a sufficient condition, eq.(24), that can be used to estimate δ_1 and δ_2 for a given τ_y. Denoting by $f(\delta_1, \delta_2)$ the relation between δ_1 , δ_2 and τ_y in eq.(24), we have, then.

$$f(\delta_1, \delta_2) = \frac{\delta_1\beta\chi(\tau_y)}{\chi(\tau_y)+2\delta_1} - \frac{1}{\delta_2} = \frac{(\delta_1\delta_2\beta - 1)\chi(\tau_y) - 2\delta_1}{\delta_2(\chi(\tau_y)+2\delta_1)} > 0. \tag{26}$$

As an example to estimate the parameters, and considering common constants for the inverse inequalities (C_2, γ_1 and β, Figure (1(a)) shows values of $f(\delta_1, \delta_2)$ for several combinations of δ_1 and δ_2 for τ_y=100.



(a)



(b)

Figure 1. $f(\delta_1, \delta_2)$ (a) in all region (b) only at the first quadrant, with $C_2 = 100.0, \eta = 10^{-8}, \gamma_1 = 1.0, \beta = 0.3, \mu = 1.0$ and $\tau_y = 100.0$.

Figure (1(b)) shows these values for δ_1 and δ_2 positives. In Figures (2(a)) and (2(b)) curves of $f(\delta_1, \delta_2)$ are shown when δ_2 and δ_1 are fixed, respectively. From these Figures we can see that there is a wide range of combinations between the two parameters that satisfies the stability condition eq.(24).

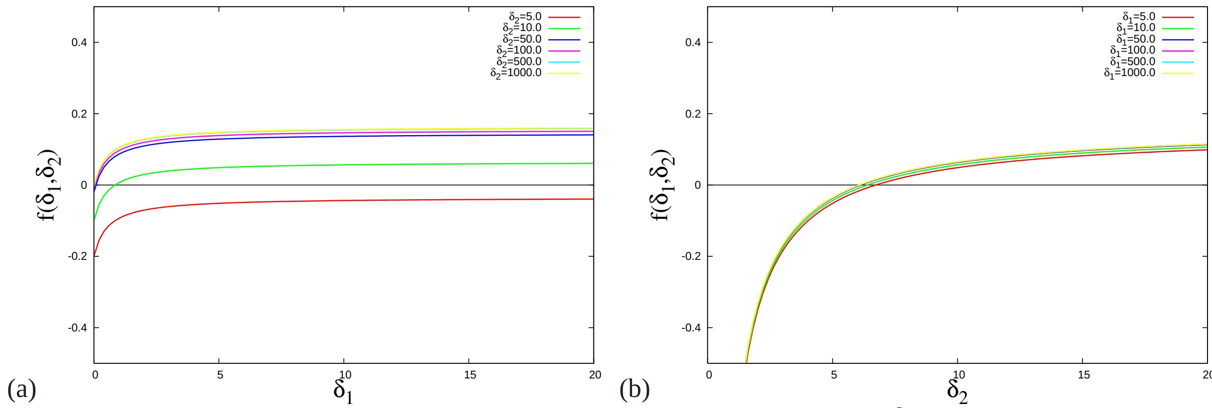


Figure 2. $f(\delta_1, \delta_2)$ (a) with δ_2 fixed (b) with δ_1 fixed, with $C_2 = 100.0, \eta = 10^{-8}, \gamma_1 = 1.0, \beta = 0.3, \mu = 1.0$ e $\tau_y = 10.0$.

4. ERROR ESTIMATES

The continuity of $A_h^* : (V_h \times W_h) \times (V_h \times W_h) \rightarrow \mathbb{R}$ established before is not valid in the continuous space $V \times W$ because the inverse estimates do not hold for infinite dimensional spaces making difficult to obtain error estimates in the usual product space norm. For this reason we use the following equivalent mesh-dependent norm that was defined in Karam-Loula (1991):

$$\|\{v_h, q_h\}\|_{V\times W,h} = \|\{v_h, q_h\}\|_{V\times W} + \sup_{v_h, q_h^* \in V_h \times W_h^*} \frac{\delta_1 h^2}{2\mu} \frac{(\Lambda u + B' p^*, \Lambda v_h + B' q_h^*)}{h \|\Lambda v_h + B' q_h^*\|} \tag{27}$$

with

$$\|\{v_h, q_h\}\|_{V\times W} = \|v_h\|_V + \|q_h\|_W \qquad \Lambda u + B' p^* = -2\mu \operatorname{div} \boldsymbol{\epsilon}(u) + \nabla p^* \tag{28}$$

that satisfies the following lemma

Lemma 4.1 *Let $v_h \in V_h$ and $q_h \in W_h$; The norms $\|\{v_h, q_h\}\|_{V\times W}$ and $\|\{v_h, q_h\}\|_{V\times W,h}$ are equivalents, i.e., there exists $M_h < \infty$, such that*

$$\|\{v_h, q_h\}\|_{V\times W} \leq \|\{v_h, q_h\}\|_{V\times W,h} \leq M_h \|\{v_h, q_h\}\|_{V\times W} \tag{29}$$

with

$$M_h = 1 + \max\{\gamma_1, \gamma_2\} \tag{30}$$

We can also show that the bilinear form $\overline{A_h^*}(\{\cdot,\cdot\};\{\cdot,\cdot\})$ is continuous with respect to the mesh-dependent norm, that is

$$\overline{A_h^*}(\{\cdot,\cdot\};\{\cdot,\cdot\}) \leq C_{1h} \left[\|\{v_h, q_h\}\|_{V\times W} \|\{v_h, q_h\}\|_{V\times W,h}\right], \tag{31}$$

with

$$C_{1h} = \max\{2\mu(1 + n\delta_2) + \frac{\tau_y}{\sqrt{C_1^2 + \eta^2}}, \delta_1 \gamma_1, \frac{\delta_1 \gamma_2}{2\mu}, \sqrt{n}\}. \tag{32}$$

By the consistence of PGG$_h$, using conditions (i), (ii) and (iii) above and inequality eq.(31) we get the following approximation estimate result

Theorem 4.1 *The Problem PGG$_h$ has unique solution and*

$$\|\{u - u_h, p - p_h\}\|_{V\times W,h} \leq \Psi_h \|\{u - v_h, p - q_h\}\|_{V\times W,h} \tag{33}$$

with Ψ_h independent of h given by

$$\Psi_h = \max\{\left(1 + \frac{C_{1h}(\tau_y)}{\beta_h}\right)\left(1 + \frac{M_h^2 C_{1h}(\tau_y)}{\gamma} Z\right), \frac{1}{\beta_h}\left(C_{2h} + \beta_h + \frac{M_h^2 C_{2h}}{\gamma}(1 + \beta_h)\right)\} \tag{34}$$

being

$$Z = M_h \left(1 + \frac{C_{2h}}{\beta_h}\right). \tag{35}$$

For S_h^k and Q_h^l as defined we may apply inverse estimates and the results of the finite element interpolation theory (Ciarlet, 1959) to obtain the following error estimate,

$$\|\{u - u_h, p - p_h\}\|_{V\times W,h} \leq C_1(u)h^k + C_2(p)h^{l+1} \tag{36}$$

(37)

or

$$\|\{u - u_h, p - p_h\}\|_{V\times W,h} \leq C_h(\{u, p\})h^s, \tag{38}$$

with $s = \min\{k, l+1\}$. Which is optimal for the velocity and suboptimal, with gap equal to 1, for the pressure field, when $k = l = 1$. Optimal rates for both fields are obtained with $k = l + 1$.

5. NUMERICAL RESULTS

In order to confirm the performance of the above formulation we present numerical results for a classical example, the driven cavity flow problem shown in Figure 3, we focuse attention on the stability aspect for several combinations of δ_1 and δ_2 values even for high yield stress values.

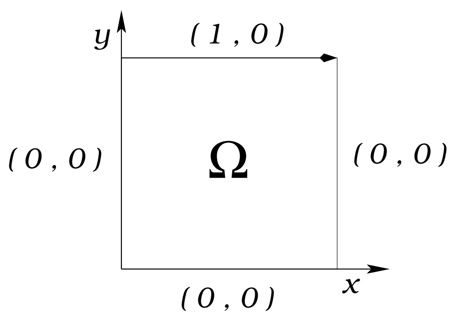


Figure 3. Problema da Cavidade.

For this analysis, we adopted a uniform mesh with 16×16 elements, continuous biquadratic interpolation for the velocity and discontinuous biquadratic interpolation for the pressure. The regularization parameter was fixed in $\eta = 10^{-12}$ for all tests.

Figure 4 presents the results for $\delta_1 = 5$ and $\delta_2 = 10$ for a series of τ_y ranging from 1.0 to 50.0 following Figures 4a-d. It is possible to observe the expected increase in the viscoplastic effect when τ_y increases, from from the graphics of the velocities, with this effect being noticed even for $\tau_y = 1$, although presenting a small yielded zone that increses for $\tau_y = 5$ and is very pronounced when τ_y is higher than 10.

Following the graphics for the pressure fields, we may observe that the results are stable for all the τ_y values. The range of pressure values for $\tau_y = 1 and 5$ being closer to each other and when τ_y is higher than 10 the pressure range values are very different than those for lower yield stresses.

Figure 5 shows the same behaviour as Figure 4, but for a bigger difference between the stabilizing parameters set as $\delta_1 = 10$ and $\delta_2 = 100$, showing that the results are not affected by these parameters.

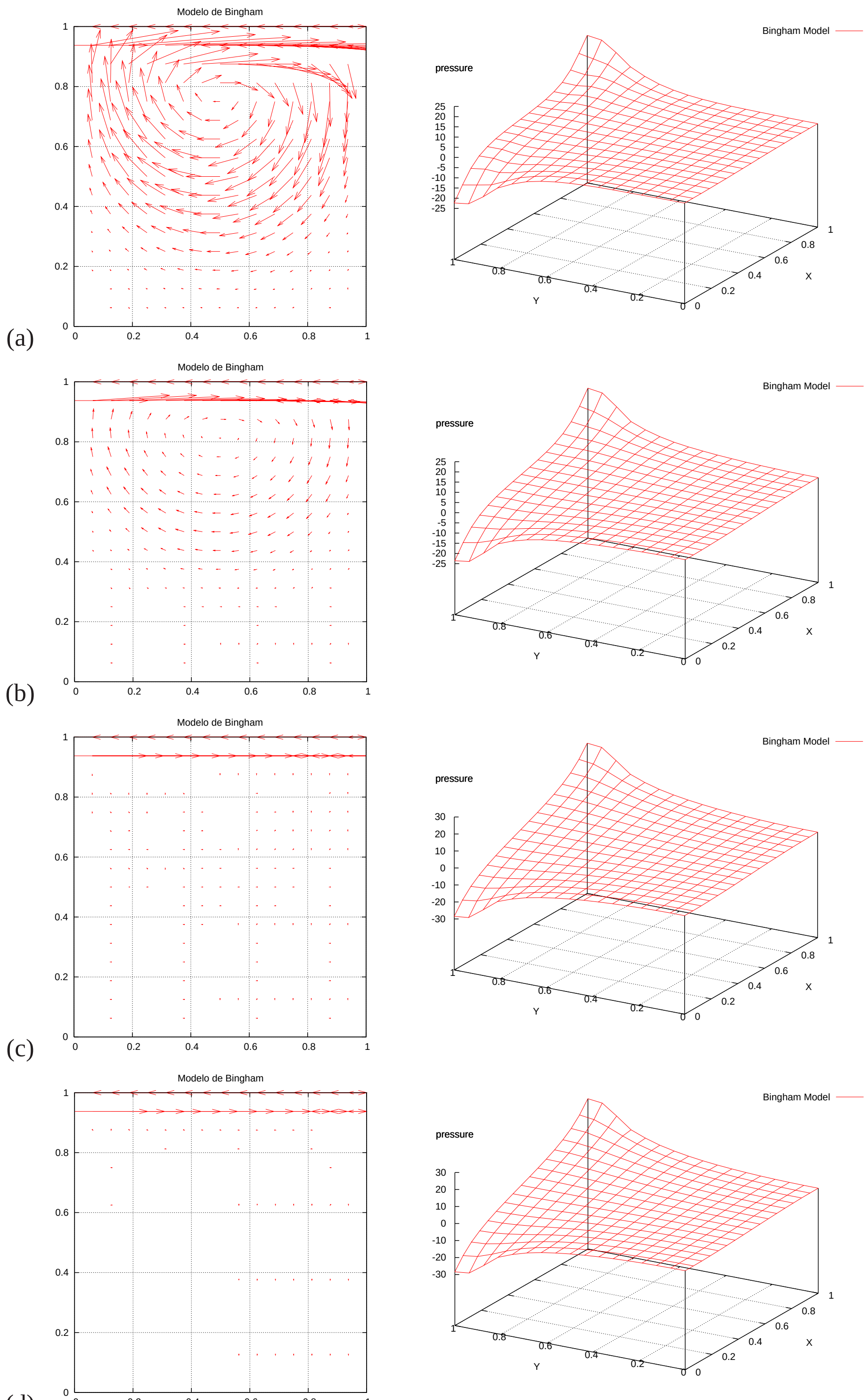


Figure 4. Velocity field and pressure value using $\delta_1 = 5.0$ and $\delta_2 = 10.0$ for (a)$\tau_y = 1.0$ (b)$\tau_y = 5.0$ (c)$\tau_y = 10.0$ (d)$\tau_y = 50.0$.

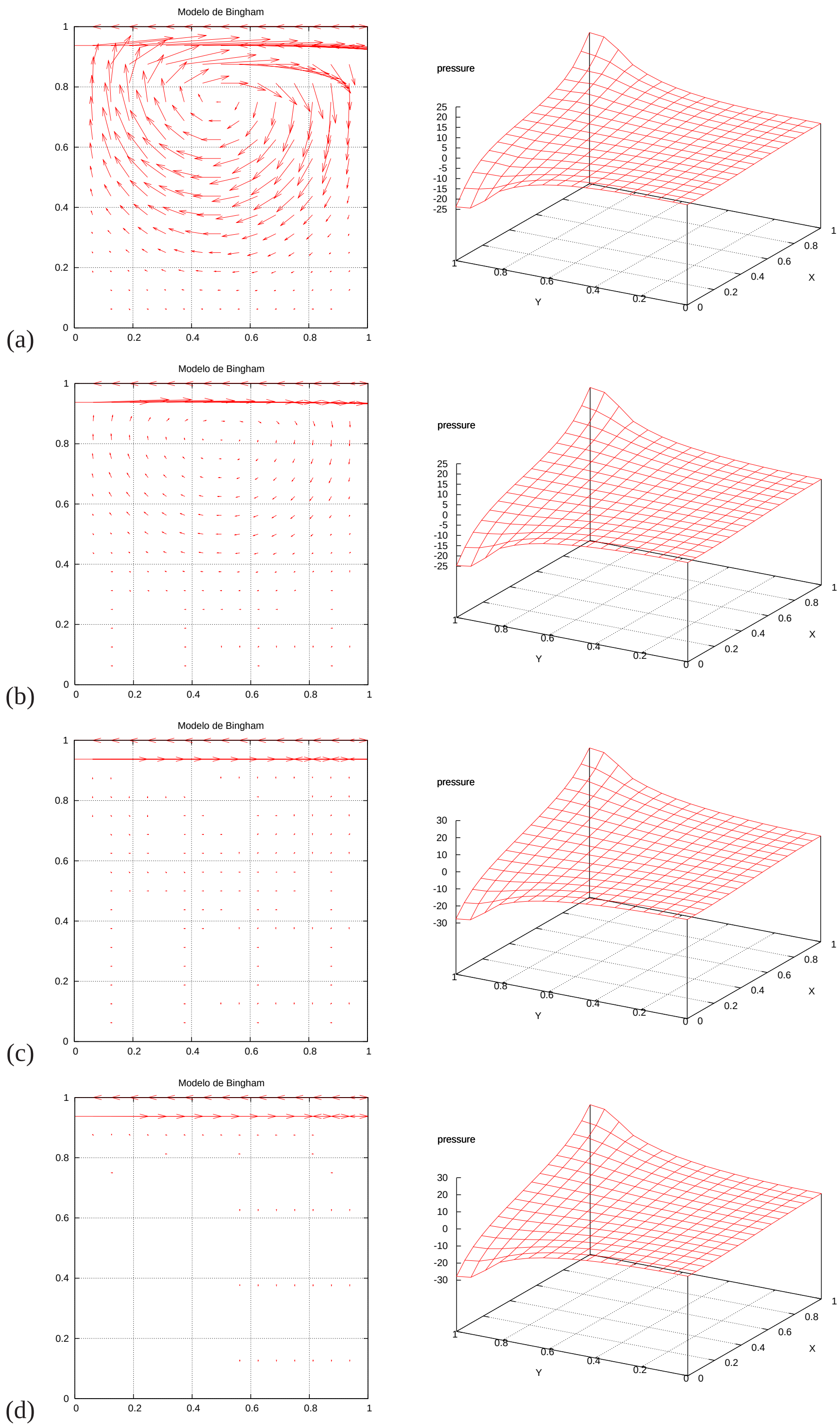


Figure 5. Velocity field and pressure value using $\delta_1 = 5.0$ and $\delta_2 = 100.0$ for (a)$\tau_y = 1.0$ (b)$\tau_y = 5.0$ (c)$\tau_y = 10.0$ (d)$\tau_y = 50.0$.

6. CONCLUSIONS

The regularized stabilized formulation in velocity and discontinuous pressure proposed in this work allows same order interpolation and satisfies Brezzi's theorem. Stable results have been generated for a wide range of the stabilizing parameters for high yield stress values even for the pressure. Optimal order of convergence is obtained as proved here in the numerical analysis.

7. ACKNOWLEDGEMENTS

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