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FULLY NONLINEAR INTERACTIONS BETWEEN TWO COLLIDING SOLITARY WAVE TRAINS

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Abstract. *In this article we investigate the head-on collision between 2 solitary wave trains. First we write the full irrotational nonlinear problem using a Hamiltonian formulation of classical mechanics. The equations are solved in a periodic domain using a pseudo-spectral method for the spatial dependency, and the time-evolution is done by a fourth order Runge-Kutta method. The simulations confirm the prediction that a dispersive tail results from the nonlinear interactions between every two waves. For some specific cases an unseen recurrence pattern takes place where the energy from the dispersive tail is pumped back into the solitary waves.*

Keywords: *Solitary Wave, computational fluid mechanics, numerical model*

1. INTRODUCTION

It is well known that standard KdV equations (Korteweg and deVries, 1895) have solutions which are permanent form waves known as solitons or solitary waves that can travel to either direction in a one-dimensional space. When these waves collide, uniform phase shifts along the waves' profiles occur but the final forms of the waves after the interaction are preserved. This is because these KdV equations are low order in nonlinearity: $\delta = O(\mu^2)$, and $O(\mu^4)$ terms are neglected (μ is the water depth - wavelength ratio and δ is the wave height - water depth ratio). When the governing equations include higher order terms, a dispersive tail may appear propagating behind the colliding waves, due to corrections in the phase shifts of each interacting wave that are nonuniform along the wave profile. A detailed study of interactions involving the collision between two solitary waves can be found in (Mirie and Su, 1980) and (Su and Mirie, 1980).

In this paper, we investigate the collision between two identical infinite trains of solitary waves (as opposed to only two solitary waves) traveling in opposite directions.

2. THEORETICAL BACKGROUND

2.1 Equations

We consider the irrotational flow of an inviscid and incompressible fluid. This is the case of waves propagating in the horizontal direction over a flat bottom in a nonrotating frame of reference. The mean depth is h, the free surface is located at $z = \eta$, where $z = 0$ is at the mean water level. The irrotational flow is such that the velocity field can be derived from a potential ϕ:

$$\mathbf{u} = \nabla\phi, \tag{1}$$

where $\mathbf{u} \equiv (u, v, w)$ is the velocity vector, and, in cartesian coordinates:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{2}$$

is the gradient operator.

The conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

implies that ϕ must satisfy Laplace's equation everywhere:

$$\nabla^2\phi = 0. \tag{4}$$

At the horizontal bottom, we have simply

$$w = \frac{\partial\phi}{\partial z} = 0, \quad z = -h \tag{5}$$

We have a kinematic surface boundary condition (surface is material):

$$w = \frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y}, \quad z = \eta. \tag{6}$$

and a dynamic boundary condition (unsteady Bernoulli equation):

$$\frac{\partial \phi}{\partial t} + gz + \frac{p(z)}{\rho} + \frac{1}{2}\left|\nabla\phi\right|^2 = 0, \quad z = \eta. \tag{7}$$

At the free surface we take the pressure field to be zero without loss of generality.

The system above can be transformed in a Hamiltonian formulation which gives (Zacharov, 1968):

$$\frac{\partial \psi}{\partial t} = -g\eta + \frac{1}{2}\left(|\nabla_2\psi|^2\right) + \frac{1}{2}\left(1 + |\nabla_2\eta|^2\right)\left(\frac{\partial\phi}{\partial z}\right)^2_{z=\eta}, \tag{8}$$

$$\frac{\partial \eta}{\partial t} = -\nabla_2\psi\cdot\nabla_2\eta + \left(1 + |\nabla_2\eta|^2\right)\left(\frac{\partial\phi}{\partial z}\right)_{z=\eta}, \tag{9}$$

where $\psi = \phi_{z=\eta}$, ∇_2 is the horizontal gradient, and ϕ is subject to Laplace's equation as a constraint. According to the Hamiltonian formulation, the system above is a set of canonical equations (Hamilton's canonical equations) for the generalized momentum η and generalized coordinate ψ. The system is the variational (continuous systems) analog of the Hamiltonian formulation of classical mechanics for a system of particles.

2.2 Numerical solution

The Hamiltonian equations for a two-dimensional system (x, z) can be found by setting $\nabla_2 \equiv \frac{\partial}{\partial x}$. The domain in question is illustrated in figure 2, and is periodic in the x direction with period $2L$.

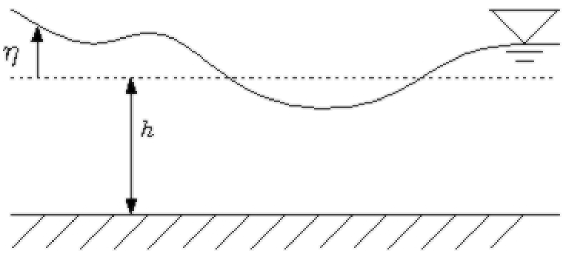

Figure 1. Problem domain.

The solution method is as follows. We compute the following truncated Fourier harmonic solution for a given initial condition, which satisfy identically Laplace's equation, periodicity, and the bottom boundary conditions:

$$\eta_j = \sum_{-N/2+1}^{N/2} \frac{a_n(t)}{2} e^{inkx_j}, \quad a_{-n} = a_n^*, \quad k = \frac{\pi}{L} \tag{10}$$

$$\psi_j = \sum_{-N/2+1\,N/2} -\frac{b_n(t)}{2}\frac{\cosh nk(h+\eta_j)}{\cosh nkh} e^{inkx_j}, \quad b_{-n} = b_n^* \tag{11}$$

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=\eta_j} = \sum_{-N/2+1}^{N/2} -ink\frac{b_n(t)}{2}\frac{\sinh nk(h+\eta_j)}{\cosh nkh} e^{inkx_j}, \tag{12}$$

where the asterisk denotes complex conjugate, and discretized position $x = j\Delta x$. We then compute the x-derivatives by Fourier collocation:

$$\frac{\partial\psi}{\partial x} = F^{-1}\left\{inkF\left\{\psi\right\}\right\}, \tag{13}$$

$$\frac{\partial\eta}{\partial x} = F^{-1}\left\{inkF\left\{\eta\right\}\right\}, \tag{14}$$

where F and F^{-1} are discrete Fourier transform and its inverse, computed by an FFT algorithm. Let us define and compute the $N \times N$ matrix

$$A_{jn} = \frac{1}{2}\frac{\cosh nk(h+\eta_j)}{\cosh nkh} e^{inkx_j}, \tag{15}$$

and notice that:

$$\psi_j = A_{jn} b_n. \tag{16}$$

The inverse of A_{jn} is defined as $\tilde{A}_{jn}$, and hence we can evaluate:

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \tilde{A}_{jn} b_n. \tag{17}$$

With all the right-hand-sides of the evolution equations evaluated, we step in time using a standard fourth order Runge-Kutta time integration scheme.

3. RESULTS

We have applied this scheme to a configuration consisting of two infinite (periodic) solitary wave trains with exactly the same amplitude colliding head-on. The initial condition used for the solitary wave trains were constructed from the nearly exact solution by Tanaka (1986), for a wave of height $\eta_{max} = 0.269$ m, at a mean depth of $h = 1$ m. The periodic domain size was 63 m, and was discretized with a number of points $N = 70$ (at every N points or 63 m, the domain repeats itself). The number of points used in the solution is of minor relevance since the derivatives are calculated in Fourier space, and therefore has a quite good convengency rate when compared to finite differences or finite element discretization schemes. For this particular example, there was no need for filtering or de-aliasing.

The acuracy of the model was checked by observing the total mass and total energy (both per unit length) of the system, which is required to be conserved, except for numerical errors, as the system is not dissipative. Figure **??** shows a plot of the evolution of the mass and energy for 5000 s of simulation, both normalized by their value at $t = 0$ s. Notice that numerical errors build up along the simulation. To avoid misinterpretation of the results we only looked at the first 1000 seconds of the simulation.

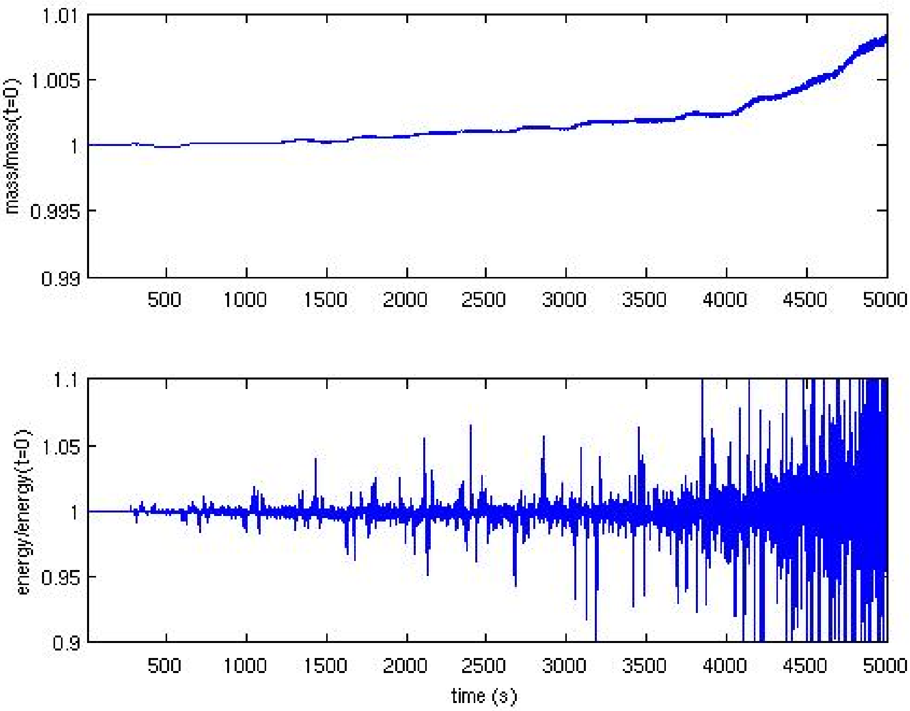

Figure 2. Evolution of mass (top) and energy (bottom) of the system.

We find a strong recurrence phenomenon as these waves interact. For the interactions between the first few waves in the wave train we obtain results very similar to those by Mirie and Su (1980): dispersive low amplitude tails develop behind the main waves. As more waves interact with each other, these dispersive tails grow and assume nearly standing wave patterns, which is expected. However, after a certain (usually large) number of solitary wave interactions we notice that the dispersive low amplitude waves decrease in amplitude losing energy to the main solitary waves until virtually all energy returns to the main waves and the process restarts as a striking recurrence phenomenon. Figure 3 (left) shows a sequence of 5 frames (200 seconds each) of surface elevation (shaded) against space and time for interactions of two trains of solitary wave of initial height-depth ratio equal to 0.269. The interactions between solitary waves appear as reflections at the left and right walls. The shading of the plot was intentionally made rough to show a qualitative picture of what is happening. The bright white thick lines are regions near the crest of the main solitary wave. The smaller amplitude dispersive tails can be seen as localized gray spots around the main waves. In this particular example the recurrence can be identified as these smaller waves decrease amplitude drastically on 3 occasions with a periodicity of around 362 seconds. Figure 4 (right) shows surface profiles (only one wave is shown) snapshots at several times, and it can be seen that the solitary wave practically reforms its original shape at times 362 and 724.5 seconds. Profiles of the other variable (ψ) are not shown but the results are very similar to the free surface, and the recurrence can be observed there as well.

Another way of looking at the recurrence is by ploting the evolution of the Fourier spectra of the variables. To illustrate this we ploted (Figure 5) the time evolution of the amplitude of the tenth harmonic of the free surface elevation.

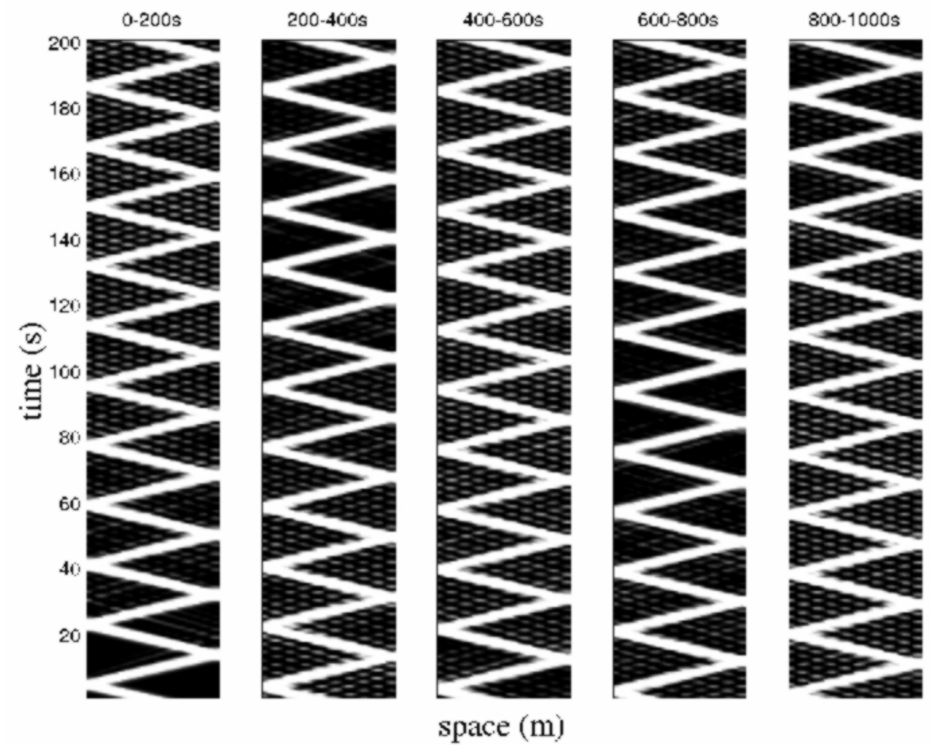

Figure 3. Shading with wave collisions/evolution.

4. CONCLUSION

We solved the full equations of irrotational flow of an incompressible, inviscid fluid over a flat bed. The equations are expressed in terms of canonical variables of a continuous Hamiltonian system. We solve the equations in a periodic domain for identical colliding waves, which is equivalent to a one dimensional box with perfectly reflective walls. As the initial condition we use the solutions obtained by Tanaka (1980). The solution method involved a spectral collocation approach to compute spatial derivatives, otherwise the equations are evaluated in physical space. Time integration was performed with a fourth order Runge-Kutta scheme. Discrete Fourier transforms and time integration are hence the only approximations involved in the solution of the full problem. We do a few tests to check the accuracy of our model and solution approach.

We have arrived at a stunning recurrence phenomenon which poses a question of how the energy of the dispersive tail is pumped back into the solitary wave. No analytical satisfactory explanation has been found for this phenomenon.

5. REFERENCES

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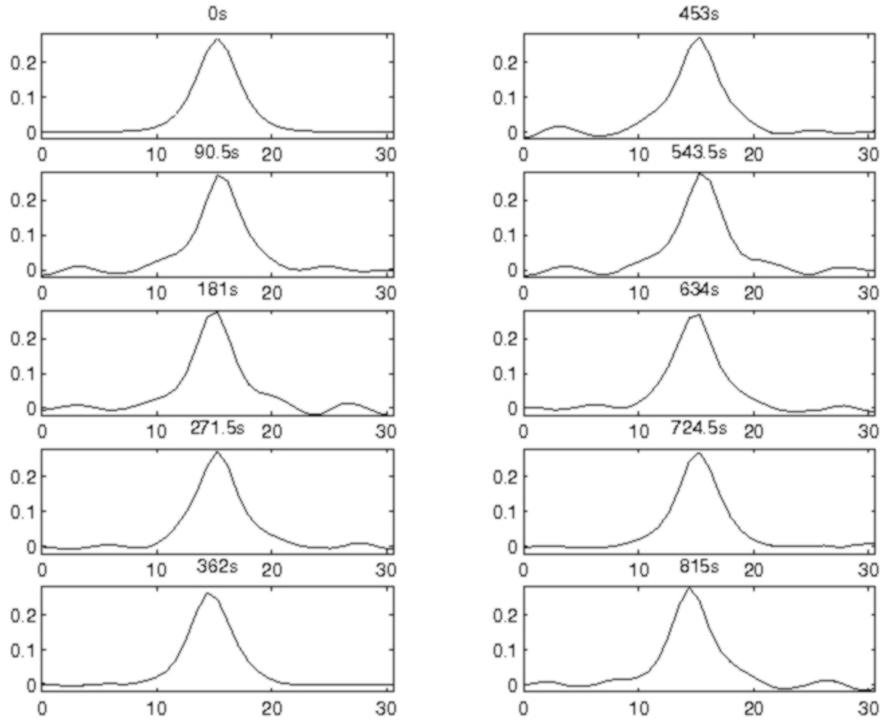

Figure 4. Surface elevations for one-half-period of the periodic domain.

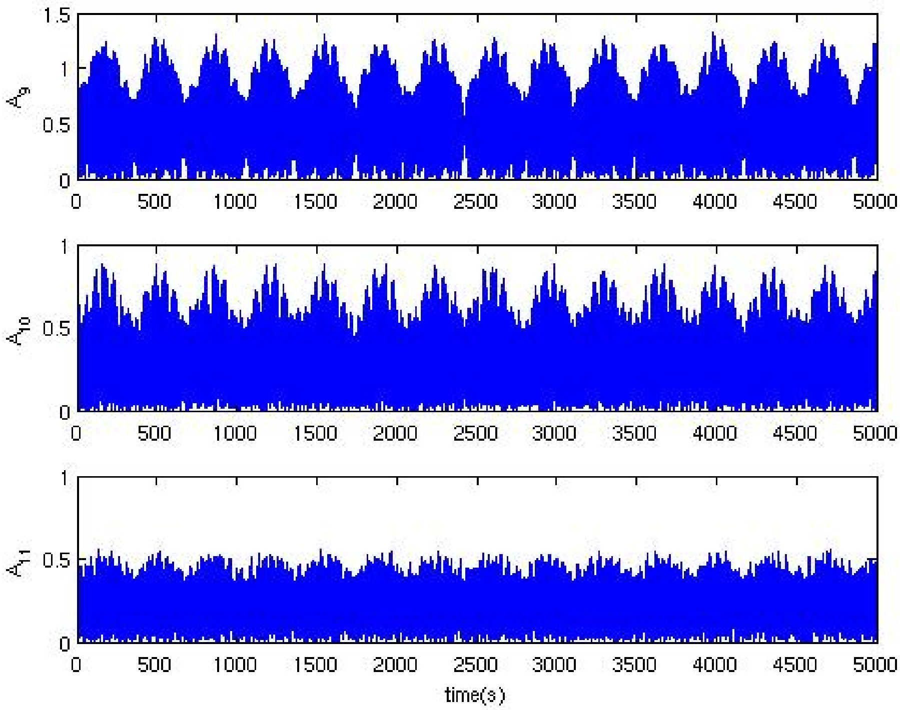

Figure 5. Amplitude evolution of the nineth, tenth, and eleventh harmonics of the free surface elevation.

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6. Responsibility notice

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