SIMULATION OF THE LADLE FURNACE PREHEATING PROCESS USING SPLIT AND GEOMETRIC SERIES

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Abstract. *The aim of this paper is to obtain exact solutions in a polynomial form to partial differential equations. The proposed method is based on the split of the unsteady heat equation. The respective formal solution is expressed in terms of geometric series of the differential operators involved. This method is applied to simulate the ladle furnace preheating process, considering that the thermal diffusivity of the wall does not vary with temperature. This assumption represents a weak restriction since this property can be regarded as constant along the temperaure interval considered. The advantage of this formulation is the analytical character of the solutions and the increasing of the processing speed which allows obtaining the results in real time.*

Keywords: *transient heat equation, split, ladle furnace, geometric series*

1. INTRODUCTION

Several analytical, numerical and hybrid methods can be found in literature to solve partial differential equations which mathematically describe problems in Engineering and Physics (Zwillinger, 1992). However, analytical solutions have some advantages from operational point of view: they are expressed in a closed form, the programs based on this kind of solutions require less processing time, since there is a reduction of the number of operations to be performed, and then the amount of memory required to execute the routines decreases significantly. Besides, the source codes based on closed-form solutions are short and easy to depurate. In general, these formulations are based on integral transforms (Cotta, 1993), separation of variables (Crank, 1975; Osizik,1993) and Lie symmetries methods in general (Polyanin, 2004; Zwillinger, 1992). The solutions obtained are generally expressed as infinite series, which have a large number of basis functions, even when truncated to perform the post-processing tasks. The problem related to the number of these basis functions is concerned about the completeness of the solution obtained, since a complete space of solutions can represent any boundary condition that can be applied, as well as restrictions based on the scenario that is being studied.

The aim of this paper is to obtain exact solutions in a polynomial form to partial differential equations. The proposed method is based on the split of the unsteady heat equation. The respective formal solution is expressed in terms of geometric series of the differential operators involved and it has sufficient number of arbitrary constants which allow that the boundary conditions of the practical problems to be satisfied. This method is applied to simulate the ladle furnace preheating process. The advantage of this formulation is the analytical form of the solutions and the increasing of the processing speed which allows obtaining the results in real time.

This article is outlined as follows. In section 2, the method is explained. In section 3, the application of the proposed formulation is described. Finally, in section 4, conclusion and recommendations for future work are drawn.

2. ANALYTICAL FORMULATION

Consider the problem described by the equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad x \le a, y \le b\,,\ t>0 \tag{1}$$

subjected to the boundary conditions

$$\frac{\partial T}{\partial x}+H_2 T=0 \quad \text{at } x=a\,,\ t>0 \tag{2}$$

$$\frac{\partial T}{\partial y} + H_4 T = 0 \quad \text{at } y = b, \tag{3}$$

where is the temperature, α is the thermal diffusivity, t is the time, x and y are the Cartesian coordinates.

In order to solve Eq. (1), the split method is applied. From Eq. (1), it is possible to write

$$LT = 0 \tag{4}$$

where L is a linear differential operator and which can written as $L = A - B$, then

$$AT = BT, \tag{5}$$

where

$$A = \frac{\partial}{\partial t}, \tag{6}$$

$$B = \alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{7}$$

and

$$A^{-1} = \int (\cdot)dt. \tag{8}$$

The application of A^{-1} on Eq. (5) gives

$$A^{-1}(AT) = A^{-1}(BT), \tag{9}$$

that is,

$$T = A^{-1}Bf + h_A, \tag{10}$$

where h_A is a solution that belongs to the null space of the linear operator A and which is represented by the set of solutions of the equation

$$Ah_A = 0. \tag{11}$$

Calculating T in Eq. (10) yields

$$T = \left[I - A^{-1}B\right]^{-1} h_A. \tag{12}$$

The inverse operator in Eq. (12) results

$$\frac{1}{I - A^{-1}B} = \sum_{k=0}^{\infty} \left(A^{-1}B\right)^k \tag{13}$$

using the geometric series, then

$$T = \sum_{k=0}^{\infty} \left(A^{-1}B\right)^k f_0. \tag{14}$$

In order to obtain a particular solution to Eq. (1) which describes the problem, it is necessary to choose a function f_0 that belongs to the null space of the operator A. For practical purposes, f_0 can be chosen as a function which belongs to the intersection of the null spaces of A and of $\left(A^{-1}B\right)^n$ to convert the given solution into a finite sum.

The functions from the null space of $\left(A^{-1}B\right)^n$ define the sum

$$T = \sum_{k=0}^{n} \left(A^{-1}B\right)^k f_0 \tag{15}$$

which is always an exact solution, since all the terms for $k \geq n$ are automatically zero. Hence, there is no problem concerned about convergence on the proposed formulation.

A simple choice to f_0 is

$$f_0 = x^2 + y^2 . \tag{16}$$

Applying $\left(A^{-1}B\right)$ over f_0 until it becomes zero, the series defined by Eq. (14) is converted into a finite sum given by Eq. (15), and consequently it is enough to find the operator power which defines the number of terms of the resulting sum, so

$$f_1 = \left(A^{-1}B\right)f_0 = 4\alpha t , \tag{17}$$

and the application of $\left(A^{-1}B\right)$ over f_1 yields

$$f_2 = \left(A^{-1}B\right)f_1 = 0 . \tag{18}$$

Therefore, Eq. (17) belongs to the null space of A and of $\left(A^{-1}B\right)^2$ and produces the following exact solution

$$T(x, y, t) = x^2 + y^2 + 4\alpha t . \tag{19}$$

Since f_0 can be prescribed as a linear combination of polynomials containing undetermined parameters, the proposed method yields polynomial solutions with several arbitrary constants, and hence, it becomes possible to apply all the boundary conditions in linear form to the expression. Sometimes it is necessary to convert boundary conditions of the third kind into conditions of the first kind over the interfaces. This conversion can be performed using differential constraints (Polyanin, 2004). Differential constraints consist of auxiliary equations which can be employed to restrict the space solution of the original one. They can be built including, in the original boundary conditions, source functions that become zero on the respective interfaces. Then, to convert the boundary conditions into differential constraints which are valid through all domain it is necessary to rewrite Eq. (2) and Eq. (3) as

$$\frac{\partial T}{\partial x} + H_2 T = r(x, y, t) \tag{20}$$

and

$$\frac{\partial T}{\partial y} + H_4 T = s(x, y, t) , \tag{21}$$

respectively, where $r(a, y, t) = 0$ and $s(x, b, t) = 0$.

The basic sketch of the procedure consists of the following steps:

1. Isolating the first terms from Eq. (20) and Eq. (21);
2. Deriving them to obtain the second order derivatives which appear in the original equation;
3. Performing the substitution of the corresponding expressions into Eq. (1);
4. Solving the resulting equation.

Deriving Eq. (20) twice with respect to x and also deriving Eq. (21) twice with respect to y yield

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial r}{\partial x} - H_2 (r - H_2 T) \tag{22}$$

and

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial r}{\partial y} - H_4 (r - H_4 T) \,. \tag{23}$$

It is convenient to specify the functions $r(x, y, t)$ and $s(x, y, t)$ before performing the substitution of Eq. (22) and Eq. (23) into Eq. (1). This can be done imposing that the mixed derivatives of the differential constraints are equal, that is

$$\frac{\partial^2 T}{\partial y \partial x} = \frac{\partial r}{\partial y} - H_2 (s - H_4 T) \tag{24}$$

and

$$\frac{\partial^2 T}{\partial x \partial y} = \frac{\partial s}{\partial x} - H_4 (r - H_2 T) \tag{25}$$

which results

$$\frac{\partial r}{\partial y} + H_4 r = \frac{\partial s}{\partial x} + H_2 s \,. \tag{26}$$

The solution of Eq. (26) can be obtained by the application of split which results the following system of partial differential equations

$$\frac{\partial r}{\partial y} + H_4 r = 0 \tag{27}$$

$$\frac{\partial s}{\partial x} + H_2 s = 0 \tag{28}$$

whose solutions are

$$r(x, y, t) = c_1(x, t) e^{-H_4 y} \tag{29}$$

and

$$r(x, y, t) = c_2(y, t) e^{-H_2 x} \,. \tag{30}$$

This way, Eq. (22) and Eq. (23) and the derivatives of Eq. (29) and Eq. (30) are applied to Eq. (1) resulting

$$\frac{\partial c_1}{\partial x} e^{-H_4 y} - H_2 c_1 e^{-H_4 y} + H_2^2 T + \frac{\partial c_2}{\partial y} e^{-H_2 x} - H_4 c_2 e^{-H_2 x} + H_4^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \,. \tag{31}$$

This inhomogeneous first order equation can be readily solved for $T(x, y, t)$. Therefore, $T(a, y, t)$ is given by

$$T(a,y,t)=\left\{\int\alpha\left[e^{-t\alpha H_2^2-t\alpha H_4^2-H_4 y}\left(F_4(t)-H_2F_1(a,t)\right)+e^{-t\alpha H_2^2-t\alpha H_4^2-H_2 a}\left(\frac{\partial F_2(y,t)}{\partial y}-F_2(y,t)H_4\right)\right]dt+F_3(y)\right\}\cdot e^{\left(H_2^2+H_4^2\right)\alpha t}, \quad (32)$$

where $F_1(a,t)$, $F_2(y,t)$, $F_3(y)$ and $F_4(t)$ are arbitrary functions that are specified by the application of the initial condition.

Similarly, the expression to $T(x,b,t)$ is given by

$$T(x,b,t)=\left\{\int\alpha\left[e^{-t\alpha H_2^2-t\alpha H_4^2-H_4 b}\left(F_4(t)-H_2F_1(x,t)\right)+e^{-t\alpha H_2^2-t\alpha H_4^2-H_2 x}\left(\frac{\partial F_2(b,t)}{\partial y}-F_2(b,t)H_4\right)\right]dt+F_3(b)\right\}\cdot e^{\left(H_2^2+H_4^2\right)\alpha t}, \quad (33)$$

where $F_1(x,t)$, $F_2(b,t)$, $F_3(b)$ and $F_4(t)$ are arbitrary functions that are specified by the application of the initial condition. The expressions represented by Eq. (32) and Eq. (33) correspond to the boundary conditions of the first kind that should be applied at $x=a$ and at $y=b$.

It is important to emphasize that despite the existence of other formulations to find exact solutions to the heat equation, the proposed method is computationally advantageous. In Ozisik (1993), problems in rectangular domains are solved by separation of variables. This method produces eigenvalue problems involving ordinary differential equations whose solutions are previously known (trigonometric and hyperbolic functions, in case of Cartesian geometry). However, the final solutions are expressed as infinite series which are linear combination of the solutions of the corresponding eigenvalue problems. In general, numerical methods are applied to obtain these eigenvalues, besides, the series solution contains multiple summations in a quantity that corresponds to the number of spatial variables of the original problem. These features increase the computational cost of simulations based on these formulations.

The next section shows how this procedure can be applied to simulate the ladle furnace preheating process.

3. APPLICATION

3.1 Description of the problem

Consider the problem in cylindrical coordinates which describes the ladle furnace preheating process in the steel formulation.

The mathematical formulation of this problem can be described as

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad t>0 \quad (34)$$

subjected to the boundary conditions

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0 \quad (35)$$

$$T(l,t)=f_l(t) \quad (36)$$

where $T(r,t)$ represents the temperature, α is the thermal diffusivity, t corresponds to the time and r is the radial coordinate.

The mathematical model describes a process in which the ladle is being heated inside by a flame before receiving the liquid steel that comes from a main furnace. The boundary condition given by Eq. (35) represents that the outer surface is isolated during the process. Despite the surface being not really isolated, Eq. (35) is a good approximation for a thick wall composed by materials which have low thermal diffusivity. The second boundary condition states that the inner wall is at the same temperature of a flame that is being injected inside and it depends on time. In fact, despite the existence of a thermal resistance between the flame and the wall, the monomolecular layer which covers the inner wall and the flame reach a thermal equilibrium in a time scale which can be neglected comparing to the total time required to complete the preheating process. After the process, the ladle receives the liquid steel whose temperature also depends on the time.

3.2 Closed form solution

The same process that was shown in section 2 is applied. The split is used to solve Eq. (34). The differential operators A and B are defined as follows:

$$A = \frac{\partial}{\partial t}, \quad (37)$$

$$B = \alpha\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) \quad (38)$$

respectively. Hence, the solution to $T(r,t)$ is given by

$$T(r,t) = \sum_{k=0}^{\infty}\left[\int \alpha\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)dt\right]^k f_0, \quad (39)$$

In order to find a particular solution to the heat equation whose computational code allows to perform all the required tasks in real time, it is convenient to choose functions f_0 which belong to the intersection of the null spaces of the operators A and $\left(A^{-1}B\right)^n$. Consequently, the solution expressed as a series is converted into a finite sum. It is important to choose f_0 in a form which contains sufficient arbitrary elements that permit the solution to satisfy the boundary conditions, as well as to reproduce the time evolution of the temperature distribution described by the initial condition.

The expression to f_0 is written as the linear combination defined by

$$T(r,t) = \sum_{k=0}^{5} C_k p_{2k}(r,t), \quad (40)$$

where the functions $p_{2k}(r,t)$ correspond to the solutions obtained to $f_0 = r^2$, $f_0 = r^4$, $f_0 = r^6$, $f_0 = r^8$ and $f_0 = r^{10}$. The coefficients C_k are shown in Tab. 1. They were obtained by a curve fitting procedure applied to data which correspond to the boundary condition at $r = l$. The values to the temperature are given in Tab. 2.

Table 1. Numerical values of the coefficients C_k.

Coefficient	Value	Coefficient	Value
C_0	102,5	C_3	8116
C_1	198,6	C_4	235,4
C_2	2830	C_5	28,74

Table 2. Values for the temperature

Temperature	Values (°C)
T_0	298
T_1	398
T_2	523
T_3	723
T_4	823
T_5	903
T_6	973

The data fitting generates a solution which represents the experimental values at $r = l$ and presents a mean square deviation about $1°C$ comparing with the fitted functions. This deviation is of the same order of own uncertainty of the experimental data. Although the processing time required to perform the curve fitting being of approximately 1s, it is greater than the time needed to calculate the solution. In most of the scenarios in the steel industry the initial profile is previously adjusted. Consequently, the processing time to this application corresponds to the time required to obtain the finite sum. Figure 1 shows the temperature distribution through the preheating process. The mean square deviation between experimental data provided by Aços Finos Piratini (Charqueadas, Brazil) and the results obtained by this formulation is 0,16% approximately.

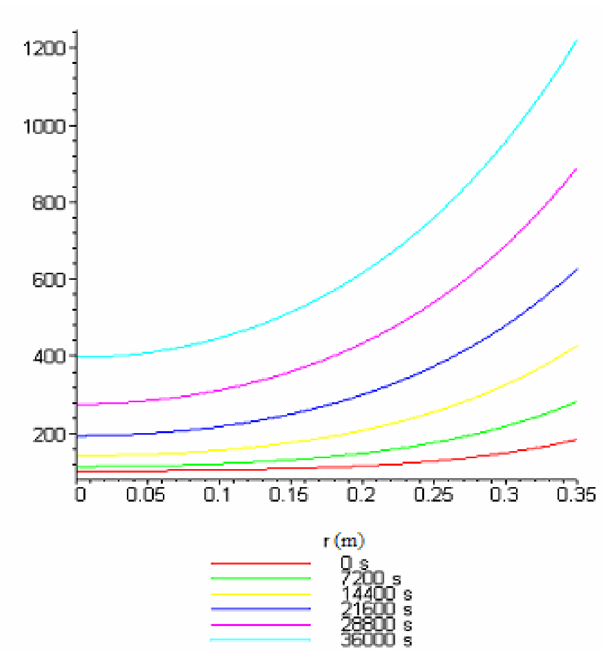

Figure 1. Temperature Distribution (°C) during the preheating process.

4. CONCLUSION

The processing time required to obtain the solution can be neglected (0,25 second), it corresponds to 5% of time needed to obtain the same solution employing the finite difference method (Alves, 2002). Both formulations were executed in a Pentium 4, 2.6GHz , 1Gb RAM. It is convenient to stress that the functions which define $p_{2k}(r,t)$ are exact solutions. Hence, Eq. (40) does not represent a truncated series, but it is an exact solution to the heat equation, being independent of the numerical values of the parameters involved. However, it is not an exact solution for the whole boundary problem since the boundary conditions given by Eq. (35) and Eq. (36) are not exactly fulfilled. The numerical results obtained agree with experimental data provided by the steel company Aços Finos Piratini. It is important to remark that the main advantage of this formulation is the analytical character of the solutions and the increasing of the processing speed which allows obtaining the results in real time. Since operators A and B were not specified in the description of the general formulation (see section 2), the proposed method can be applied to solve other problems in transport phenomena, such as simulation of pollutant dispersion and viscous flows.

5. REFERENCES

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