INTEGRAL TRANSFORM APPROACH TO THE HYPERBOLIC BIOHEAT TRANSFER EQUATION

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Abstract. *Knowledge of the heat transfer rates in organic tissues is desirable due to its importance in guiding clinical procedures. A literature survey indicates that models based on the standard Pennes' bioheat transfer are heavily employed to predict important biological phenomena such as skin burn injuries, selective brain cooling for the treatment of ischemic patients and endometrial ablation of the uterus. However, this classical approach is based on a parabolic heat transfer model and over the years, empirical evidence suggests that the propagation of heat inside biological bodies occurs at a finite speed. Therefore, the aim of this contribution is to study the bioheat transfer problem by means of a hyperbolic heat transfer model. Initially, a generalized problem is presented and its solution is obtained by employing a finite integral transform methodology. Then, the relative merits of the proposed solution scheme are assessed by studying the transient temperature field related to a skin burn injury. On general grounds, it was found that the present methodology is capable of furnishing accurate predictions for the temperature distribution. It was also discovered that the temperature fields are strongly dependent on the thermal relaxation time.*

Keywords: *bioheat transfer, integral transform, burn injury, hyperbolic conduction.*

1. INTRODUCTION

The precise determination of the temperature field in biological bodies has attracted a great deal of interest over the years as a significant number of medical procedures rely on this information. For example, Torvi and Dale (1994) developed a multiple layer finite element model to predict the degree of burn injury in human skin when subjected to a flash fire exposure. The Henriques' criterion was employed and, consequently, the transient temperature distribution is integrated in order to produce a dimensionless number whose value indicates whether a first, second or even third degree burn results depending on the time of exposure. Baldwin *et al*. (2001) established a thermal model to analyze the endometrial ablation process which produces a burn injury in the outer layer of the uterus to minimize or eliminate the adverse effects of severe menorrhagia. Both situations were revisited by Presgrave *et al.*(2005b, 2006, 2007) where the advantages of furnishing eigenfunction expansion solutions to such problems are discussed.

A common trait to most problems dealing with the evaluation of the transient thermal response of organic tissues is that they are based on the assumptions made by Pennes (1948). One proposition is that the thermal effect of the blood flow within the organic tissue can be modeled by a heat sink term. Another major hypothesis is that heat propagates at an infinite speed in such situations. However, the experiments reported by Mitra *et al.*(1995) demonstrated that the heat diffusion in biological materials are more accurately described by a non-Fourier model rather the traditional parabolic one. They argued that while the microscopic description of the conduction process within such materials is not fully understood, a hyperbolic model could represent the macroscopic aspects of this situation in a more appropriate fashion. By examining the data collected in a set of four experiments, Mitra and his collaborators (1995) came to the conclusion that the thermal relaxation time in biological bodies could be as high as 16 s. Liu *et al.* (1999) used these findings to establish a more general bioheat transfer equation that accounts for a finite speed propagation of the thermal wave. This so-called "thermal wave model of bioheat transfer" was used to evaluate the temperature field in a human skin subjected to a constant surface temperature and to a constant flux heating. They found out that the value of the characteristic time possesses an substantial effect on the temperature levels and consequently, in the burn evaluation through the Henriques' criterion. In another contribution, Liu (2000) expanded this previous study by proposing a set of four situations commonly found in biomedical applications which were numerical solved by the finite difference method and, once again, substantial deviations were found between the results predicted by the thermal wave model and the classical Pennes' equation. More recently, Ozen *et al*. (2008) studied the effect of microwaves in biological tissues by considering a one dimensional transient model of a four-layered human tissue. The thermal wave bioheat transfer equation was solved by means of a finite difference approach and they found out that the Pennes model furnishes higher temperature values than the wave mode. Finally, in another recent contribution, Liu (2008) proposed a hybrid analytical numerical solution scheme to the bioheat wave model based on Laplace transform and its corresponding numerical inversion. This method was applied to three different modes of surface heating and a comparison between this procedure and previously reported works indicates that the severe numerical oscillations that are typical of hyperbolic problems are somewhat diminished.

An exam of this brief literature review suggests that there is a body of sound empirical evidence supporting the thermal wave model for the heat transfer in biological bodies. However, the solution of this equation is most usually done by purely numerical schemes. On the other hand, eigenfunction expansion techniques were previously employed to accurately predict hyperbolic heat transfer processes (Frankel *et al.*, 1995). Certainly, an interesting aspect of this procedure is that the inversion process, when compared to the Laplace transform, is always analytical. In order to further explore this attractive feature, the main contribution of this work is to established a solution to a sufficiently general thermal wave bioheat problem by utilizing the ideas of the Generalized Integral Transform Technique (Cotta, 1993). In sequence, the merits of this solution procedure are studied within the context of an specific situation related to a skin burn injury.

2. ANALYSIS

In this section, we develop a solution to a generalized bioheat transfer equation that accounts for a finite speed in the heat propagation. Therefore, our starting point is the well-known Pennes' equation (Pennes, 1948), which can be written as:

$$\rho C \frac{\partial T(\vec{x},t)}{\partial t} = -\nabla \bullet \vec{q} - \omega(x)\rho_b C_b \left(T(\vec{x},t) - T_b\right) + g(\vec{x},t) \tag{1}$$

In order to fully express Eq. (1) in terms of the tissue temperature, *T(x,t)*, a relation between the heat flux and the temperature field must be established. Here, instead of the classical Fourier approach, we adopt the Vernotte-Cattaneo relation and, consequently:

$$\vec{q}(\vec{x},t+t_r) = \vec{q}(\vec{x},t) + t_r \frac{\partial \vec{q}(\vec{x},t)}{\partial t} = -k(\vec{x})\nabla T(\vec{x},t) \tag{2}$$

Inserting Eq. (2) in (1) and after some manipulations, the so-called "thermal wave model of bioheat transfer" (Liu *et al.*, 1999) is determined as:

$$t_r \rho C \frac{\partial^2 T(\vec{x},t)}{\partial t^2} + (\rho C + t_r \varpi(\vec{x})\rho_b C_b)\frac{\partial T(\vec{x},t)}{\partial t} = \nabla \bullet \left[k(\vec{x})\nabla T(\vec{x},t)\right] - \omega(x)\rho_b C_b \left(T(\vec{x},t) - T_b\right) +$$

$$g(\vec{x},t) + t_r \frac{\partial g(\vec{x},t)}{\partial t} \tag{3}$$

where t_r is the thermal relaxation time. Clearly, when t_r is set to zero, the standard Pennes's formulation is recovered. While in most ordinary homogeneous substances, t_r is indeed extremely low, its value for biological systems is predicted to be around 15 to 30 seconds (Mitra *et al.*, 1995). Moreover, Liu *et al.* (1999) points out that the Vernotte-Cattaneo relation when applied to the heat balance equation in an organic tissue, better explains the energy exchanges that are present within non-homogeneous structures such those of the human skin.

Equation (3) is rewritten in a more condensed way as:

$$t_r w(\vec{x}) \frac{\partial^2 T(\vec{x},t)}{\partial t^2} + \left[w(\vec{x}) + t_r d(\vec{x})\right]\frac{\partial T(\vec{x},t)}{\partial t} = \nabla \bullet \left[k(\vec{x})\nabla T(\vec{x},t\right] - d(\vec{x})T(\vec{x},t) + P(\vec{x},t) \tag{4}$$

where $d(\vec{x}) = \rho_b C_b \varpi(\vec{x})$, $P(\vec{x},t) = g(\vec{x},t) + t_r \dfrac{\partial g(\vec{x},t)}{\partial t} + d(\vec{x})T_b$ (5,6)

In fact, relation (4) could be interpreted as a hyperbolic version of the Class I problem described by Mikhailov and Ozisik (1984). Furthermore, as a great number of relevant problems in bioheat transfer occur in a direction perpendicular to the skin surface (Liu, 2008), in this research we pay attention to the one dimensional transient version of Eq (4). Consequently, this contribution is aimed at establishing a solution to the following bioheat problem:

$$t_r w(x) \frac{\partial^2 T(x,t)}{\partial t^2} + \left[w(x) + t_r d(x)\right]\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[k(x)\frac{\partial T(x,t)}{\partial x}\right] - d(x)T(x,t) + P(x,t)\,, \; x_0 < x < x_1, t > 0 \tag{7}$$

$$T(x,0) = f(x), \frac{\partial T(x,0)}{\partial t} = 0, \quad x_0 \le x \le x_1 \tag{8,9}$$

$$\alpha_0 T(x_0,t) - \beta_0 k(x_0) \frac{\partial T(x_0,t)}{\partial x} = \phi_0(t), \; \alpha_1 T(x_1,t) + \beta_1 k(x_1) \frac{\partial T(x_1,t)}{\partial x} = \phi_1(t), \; t > 0 \tag{10,11}$$

Clearly, a great deal of interesting cases can be derived from Eqs. (7) - (11) by prescribing appropriate values to functions *w(x)*, *k(x)*, *d(x)*, *f(x)*, $\phi_0(t)$, $\phi_1(t)$ and to coefficients α_0, α_1, β_0 and β_1. Also, it is worth mentioning that in Eq. (7) the classical Pennes formulation is slightly modified in order to incorporate a perfusion effect that depends upon the direction of the heat flow, which is represented in the generic term $d(x)\,T(x,t)$.

Now, we seek a solution to the above formulation based on eigenfunction expansion techniques. Following the ideas of the Generalized Integral Transform Technique (Cotta, 1993), the temperature field is expressed as:

$$T(x,t) = \sum_{i=1}^{\infty} A_i(t) \psi_i(x) \tag{12}$$

where $\psi_i(x)$ is the solution of the following eigenproblem:

$$\frac{d}{dx}\left[k(x)\frac{d\psi_i(x)}{dx}\right] + \left(\mu_i^2\, w(x) - d(x)\right)\psi_i(x) = 0 \tag{13}$$

$$\alpha_0 \psi_i(x_0) - \beta_0 k(x_0) \frac{d\psi_i(x_0)}{dx} = 0 \;, \; \alpha_1 \psi_i(x_1) + \beta_1 k(x_1) \frac{d\psi_i(x_1)}{dx} = 0 \tag{14,15}$$

Utilizing the orthogonality property of this standard Sturm-Liouville system, it is a simple matter to show that:

$$A_i(t) = \frac{1}{N_i} \int_{x_0}^{x_1} w(x) \psi_i(x) T(x,t) dx \tag{16}$$

and the transform-inverse pair is readily obtained as:

$$\overline{T_i}(t) = \frac{1}{N_i^{1/2}} \int_{x_0}^{x_1} w(x) \psi_i(x) T(x,t) dx \;, \; T(x,t) = \sum_{i=1}^{\infty} \frac{1}{N_i^{1/2}} \psi_i(x) \overline{T_i}(t) \tag{17,18}$$

The next step is to rewrite the original problem in terms of the transformed temperature field $\overline{T_i}(t)$. This task is accomplished through a series of mathematical operations which are briefly discussed now. Initially, the governing partial differential equation is operated on with $1/N_i^{1/2} \int_{x_0}^{x_1} \psi_i(x)\, dx$. Secondly, the Sturm-Liouville equation is multiplied by $1/N_i^{1/2} \int_{x_0}^{x_1} \theta(x,t)\, dx$ .These results are added and the boundary conditions of both the original and auxiliary problems are employed to furnish:

$$t_r \frac{d^2 \overline{T_i}(t)}{dt^2} + \frac{d\overline{T_i}(t)}{dt} + \mu_i^2 \overline{T_i}(t) + t_r \frac{d}{dt}\left[\frac{1}{N_i^{1/2}} \int_{x_0}^{x_1} d(x) \psi_i(x) T(x,t) dx\right] = \overline{g_i}(t) \tag{19}$$

where $\overline{g_i}(t) = \Omega_{1i}\phi_1(t) + \Omega_{0i}\phi_0(t) + \overline{P_i}(t)$ (20)

and the coefficients Ω_{0i} and Ω_{1i} together with the transformed source term $\overline{P_i}(t)$ are given by

$$\Omega_{1i} = \frac{1}{N_i^{1/2}} \left[\frac{\psi_i(x_1) - k(x_1)\dfrac{d\psi_i(x_1)}{dx}}{\alpha_1 + \beta_1} \right], \ \Omega_{0i} = \frac{1}{N_i^{1/2}} \left[\frac{\psi_i(x_0) + k(x_0)\dfrac{d\psi_i(x_0)}{dx}}{\alpha_0 + \beta_0} \right] \tag{21,22}$$

$$\overline{P_i}(t) = \frac{1}{N_i^{1/2}} \int_{x_0}^{x_1} \psi_i(x) P(x,t) dx + t_r \frac{1}{N_i^{1/2}} \int_{x_0}^{x_1} \psi_i(x) \frac{\partial P(x,t)}{\partial t} dx \tag{23}$$

Now, the last term on the left hand side of Eq. (19) needs to be expressed in terms of the transformed temperature field $\overline{T_i}(t)$. Accordingly, Eq. (18) is inserted in relation (19) to yield:

$$t_r \frac{d^2\overline{T_i}(t)}{dt^2} + \frac{d\overline{T_i}(t)}{dt} + \mu_i^2 \overline{T_i}(t) + t_r \sum_{j=1}^{\infty} A_{ij} \frac{d\overline{T_i}(t)}{dt} = \overline{g_i}(t), \ A_{ij} = \frac{1}{N_i^{1/2}} \frac{1}{N_j^{1/2}} \int_{x_0}^{x_1} d(x) \psi_i(x) \psi_j(x) dx \tag{24,25}$$

Finally, the initial conditions for problem (24) are now set by rewriting the initial conditions represented in Eqs. (8) and (9) in terms of $\overline{T_i}(t)$. By employing Eq. (17) in relations (8) and (9), we obtain:

$$\overline{T_i}(0) = \frac{1}{N_i^{1/2}} \int_{x_0}^{x_1} w(x) \psi_i(x) f(x) dx = \bar{f}_i \ , \ \frac{d\overline{T_i}(0)}{dt} = 0 \tag{26,27}$$

Relation (24) together with its initial conditions, Eqs. (26) and (27) form a second order, linear, non-homogeneous infinite system of coupled ordinary of differential equations. In order to address its solution, we transform the second order problem to a first order one in such a way that:

$$\frac{d\overline{T_i}(t)}{dt} = \overline{U}_i(t) \ , \ t_r \frac{d\overline{U}_i(t)}{dt} + \sum_{j=1}^{\infty} B_{ij} \overline{U}_i(t) + \mu_i^2 \overline{T_i}(t) = \overline{g_i}(t) \tag{28,29}$$

$$\overline{T_i}(0) = \bar{f}_i, \quad \overline{U}_i(0) = 0, \ B_{ij} = \delta_{ij} - t_r A_{ij} \tag{30,31,32}$$

From a practical point of view, the infinite summation in Eq. (18) should be truncated to a sufficiently large number, N. This results in the following 2N X 2N explicit system:

$$\begin{Bmatrix} \overline{T_i}'(t) \\ \overline{U_i}'(t) \end{Bmatrix}_{2N\,x\,1} = \begin{bmatrix} 0 & \delta_{ij} \\ -\delta_{ij}\mu_j^2/t_r & -B_{ij}/t_r \end{bmatrix}_{2N\,x\,2N} \begin{Bmatrix} \overline{T_i}(t) \\ \overline{U}_i(t) \end{Bmatrix}_{2N\,x\,1} + \begin{Bmatrix} 0 \\ \overline{g}_i(t) \end{Bmatrix}_{2N\,x\,1}, \ \begin{Bmatrix} \overline{T_i}(0) \\ \overline{U}_i(0) \end{Bmatrix}_{2N\,x\,1} = \begin{Bmatrix} \bar{f}_i \\ 0 \end{Bmatrix}_{2N\,x\,1} \tag{33,34}$$

As pointed out by Cotta (1993), system (33) – (34) does possess a working analytical solution if the eigenvalues of the 2N X 2N matrix are distinct yielding to a 2N set of linearly independent eigenvectors. On the other hand, well established numerical routines for solving a system of first order ordinary differential equations based on the Gear method (Cotta, 1998), could also be employed to determine the transformed temperature field, especially in the cases where the non-homogeneous terms are present. In either way, once the solution of system (33) – (34) is available, the inverse relation represented by Eq. (18) is used to computed the original transient temperature distribution. It should also be noted that a robust algorithm, such as the “Sign-Count Method” (Mikhailov and Ozisik, 1984) must also be employed in order to determine the eigenvalues and related eigenquantities to the eigenproblem described in Eqs. (13) – (15) with a good numerical precision.

3. APPLICATION, RESULTS AND DISCUSSION

The solution scheme for the generalized hyperbolic bioheat transfer equation outlined in the previous section is now tested by studying a specific situation related to a burn injury in a human skin caused by surface heating. Apparently, this problem was first proposed by Liu *et al*. (1999) and was further studied by the same author in another contribution (Liu, 2000). Recently, it was revisited by Liu (2008) and it can briefly described as follows. A human skin of thickness *l* is initially in an equilibrium condition when its outer surface comes in perfect thermal contact with a hot metal plate of temperature T_H. The thermal conductivity, blood perfusion and metabolic heat of the tissue are taken as constants and its inner surface is supposed to be at a zero flux condition. Therefore, this one-dimensional transient bioheat problem can be mathematically stated as:

$$\rho C\, t_r \frac{\partial^2 T(x,t)}{\partial t^2} + (\rho C + t_r \varpi \rho_b C_b)\frac{\partial T(x,t)}{\partial t} = k\frac{\partial^2 T(x,t)}{\partial x^2} - \varpi\rho_b C_b (T(x,t) - T_b) + \dot{Q}_{met}, \quad 0 < x < l,\, t > 0 \tag{35}$$

$$T(x,0) = T_{init}(x), \quad \frac{\partial T(x,0)}{\partial t} = 0, \quad 0 \le x \le l\,,\ T(0,t) = T_H\,, \quad \frac{\partial T(l,t)}{\partial x} = 0, \qquad t > 0 \tag{36,37,38,39}$$

By employing the following dimensionless coordinate and time variables

$$\chi = \frac{x}{l}, \quad \tau = \frac{k}{\rho C}\frac{t}{l^2} \tag{40,41}$$

the above formulation is expressed in a more compact way as:

$$\tau_r \frac{\partial^2 T(\chi,\tau)}{\partial \tau^2} + (1 + \tau_r P_f)\frac{\partial T(\chi,\tau)}{\partial \tau} = \frac{\partial^2 T(\chi,\tau)}{\partial \chi^2} - P_f (T(\chi,\tau) - T_b) + G, \quad 0 < \chi < 1, \tau > 0 \tag{42}$$

$$T(\chi,0) = T_{init}(\chi), \quad \frac{\partial T(\chi,0)}{\partial \tau} = 0, \quad 0 \le \chi \le 1\,,\ T(0,\tau) = T_H\,, \quad \frac{\partial T(1,t)}{\partial \chi} = 0, \qquad \tau > 0 \tag{43,44,45,46}$$

where $\tau_r = \frac{k}{\rho C}\frac{t_r}{l^2}, \quad P_f = \frac{\varpi\rho_b C_b l^2}{k}, \quad G = \frac{\dot{Q}_{met} l^2}{k}$ (47,48,49)

The initial temperature distribution is supposed to follow a simple energy balance expressed by:

$$\frac{d^2 T_{init}(\chi)}{d\chi^2} - P_f (T_{init}(\chi) - T_b) + G = 0\,,\ T_{init}(0) = T_{sur}\,, \quad \frac{dT_{init}(1)}{d\chi} = 0 \tag{50,51,52}$$

where T_{sur} is the outer skin surface temperature before it comes into contact with the hot plate.

The relations (42) to (46) are, in fact, a special case of the general problem described in the previous section and, in principle, its solution scheme could be directly used to predict the transient temperature distribution. However, as pointed out by Frankel *et al.* (1995), solutions to hyperbolic problems based on eigenfunction expansions may need hundreds of terms in order to obtain a reasonable degree of accuracy. Since non-homogeneous terms are known to slow down the convergence rate of eigenfunction expansions, it seems convenient to use the split-up technique (Mikhailov and Ozisik, 1984) and therefore, the original temperature distribution is considered to be a sum of two contributions :

$$T(\chi,\tau) = T_S(\chi) + \theta(\chi,\tau) \tag{53}$$

where the problem for $T_S(\chi)$ is given by:

$$\frac{d^2 T_S(\chi)}{d\chi^2} - P_f (T_S(\chi) - T_b) + G = 0\,,\ T_S(0) = T_H\,, \quad \frac{dT_S(1)}{d\chi} = 0 \tag{54,55,56}$$

By inserting Eq. (54) in relations (42) – (46), the problem for $\theta(\chi,\tau)$ is found to be:

$$\tau_r \frac{\partial^2 \theta(\chi,\tau)}{\partial \tau^2} + (1 + \tau_r P_f)\frac{\partial \theta(\chi,\tau)}{\partial \tau} = \frac{\partial^2 \theta(\chi,\tau)}{\partial \chi^2} - P_f\,\theta(\chi,\tau), \quad 0 < \chi < 1, \tau > 0 \tag{57}$$

$$\theta(\chi,0) = T_{init}(\chi) - T_S(\chi), \quad \frac{\partial \theta(\chi,0)}{\partial \tau} = 0, \quad 0 \le \chi \le 1\,,\ \theta(0,\tau) = 0, \quad \frac{\partial \theta(1,t)}{\partial \chi} = 0, \qquad \tau > 0 \tag{58,59,60,61}$$

The eigenproblem for this situation is chosen as:

$$\frac{d^2\psi_i(\chi)}{d\chi^2}+\mu_i^2\psi_i(\chi)=0,\quad 0<\chi<1,\qquad \psi_i(0)=0,\quad \frac{d\psi_i(1)}{d\chi}=0 \tag{62,63,64}$$

where the eigenfunctions, norm and eigenvalues are easily determined as:

$$\psi_i(\chi)=sin(\mu_i\chi),\quad N_i=1/2,\quad \mu_i=(2i-1/2)\pi \tag{65,66,67}$$

Moreover, the integral-transform pair is immediately found to be:

$$\bar{\theta}_i(\tau)=\frac{1}{N_i^{1/2}}\int_0^1\psi_i(\chi)\,\bar{\theta}_i(\tau)\,d\chi\,,\ \theta(\chi,\tau)=\sum_{i=1}^{\infty}\frac{1}{N_i^{1/2}}\psi_i(\chi)\,\bar{\theta}_i(\tau) \tag{68,69}$$

and by applying the methodology outlined earlier, the differential equations that governs the transformed temperature field $\bar{\theta}_i(\tau)$ together with its initial conditions is given by:

$$\tau_r\frac{d^2\bar{\theta}_i(\tau)}{d\tau^2}+(1+\tau_r P_f)\frac{d\bar{\theta}_i(\tau)}{d\tau}+(\mu_i^2+P_f)\bar{\theta}_i(\tau)=0\,,\ \bar{\theta}_i(0)=\bar{f}_i=\sqrt{2}\,\frac{\mu_i(T_{sur}-T_H)}{\mu_i^2+P_f},\quad \frac{d\bar{\theta}_i(0)}{d\tau}=0 \tag{70,71,72}$$

An interesting aspect of this particular problem is that the solution to system (70)-(72) is analytically obtainable by means of the characteristic equation. Accordingly, three different possibilities are envisioned. Firstly, if the roots of the characteristic equation λ_{1i} and λ_{2i} are distinct, the solution for the transformed temperature is given by:

$$\bar{\theta}_i(\tau)=\frac{\bar{f}_i}{\lambda_{2i}-\lambda_{1i}}\left(\lambda_{2i}\exp(\lambda_{1i}\tau)-\lambda_{1i}\exp(\lambda_{2i}\tau)\right),\ \lambda_{1i,2i}=\frac{-(1+\tau_r P_f)\pm\sqrt{(1+\tau_r P_f)^2-4\tau_r(\mu_i^2+P_f)}}{2\tau_r} \tag{73,74}$$

and in the case where there is a complex conjugate root, the solution to $\bar{\theta}_i(\tau)$ is found to be:

$$\bar{\theta}_i(\tau)=\bar{f}_i\left(\cos(q_i\tau)-\frac{p_i}{q_i}sin(q_i\tau)\right)\exp(p_i\tau)\,,\ p_i=\frac{-(1+\tau_r P_f)}{2\tau_r},\quad q_i=\frac{\sqrt{4\tau_r(\mu_i^2+P_f)-(1+\tau_r P_f)^2}}{2\tau_r} \tag{75,76,77}$$

and finally, if two equal roots are found, we have:

$$\bar{\theta}_i(\tau)=\bar{f}_i(1-p_i)\exp(p_i\tau) \tag{78}$$

At this point, the inverse relation represented by Eq. (68) is recalled in order to determine $\theta(\chi,\tau)$. It should also be noted that the initial and final temperature distributions, $T_{init}(\chi)$ and $T_S(\chi)$ are easily found to be:

$$T_{init}(\chi)=T_b+\frac{G}{P_f}+\left(T_{sur}-T_b-\frac{G}{P_f}\right)\frac{\cosh\left[\sqrt{P_f}(1-\chi)\right]}{\cosh\sqrt{P_f}}, \tag{79}$$

$$T_S(\chi)=T_b+\frac{G}{P_f}+\left(T_H-T_b-\frac{G}{P_f}\right)\frac{\cosh\left[\sqrt{P_f}(1-\chi)\right]}{\cosh\sqrt{P_f}} \tag{80}$$

In order to assess the role of the thermal relaxation time in this problem, it seems interesting to compare the results predicted by Eq.(53) with the standard (parabolic) Pennes' model. It is a simple matter to show that the transient temperature distribution for this latter situation is given by (Presgrave, 2005):

$$T_{parab}(\chi,\tau)=T_S(\chi)+2(T_{sur}-T_H)\sum_{i=1}^{\infty}\frac{\mu_i}{\mu_i^2+P_f}sin(\mu_i\chi)\exp\left[-(\mu_i^2+P_f)\tau\right] \tag{81}$$

Numerical simulations were carried out for various situations of interest but only a few of our finding are reported in this contribution due to space limitations. The various numerical parameters needed for the simulations are: rate of blood perfusion (ϖ) = 0.000463 $m^3{}_b\, m^{-3}\, s^{-1}$, thermal conductivity – skin (k) = 0.2 or 0.5 $W\, m^{-1}\,{}^{\circ}C^{-1}$, specific heat – skin and blood (C, C_b) = 4200 $J\, kg^{-1}\,{}^{\circ}C^{-1}$, density – skin and blood (ρ, ρ_b) = 1000, 1080 $kg\, m^{-3}$, arterial blood temperature (T_b) = 37 $^{\circ}C$, thickness – skin (l) = 0.001208 m, hot plate temperature (T_H) = 44.5$^{\circ}C$, surface temperature - skin (T_{sur}) = 32.5 $^{\circ}C$, metabolic heat (Q_{met}) = 200 $W\, m^{-3}$.

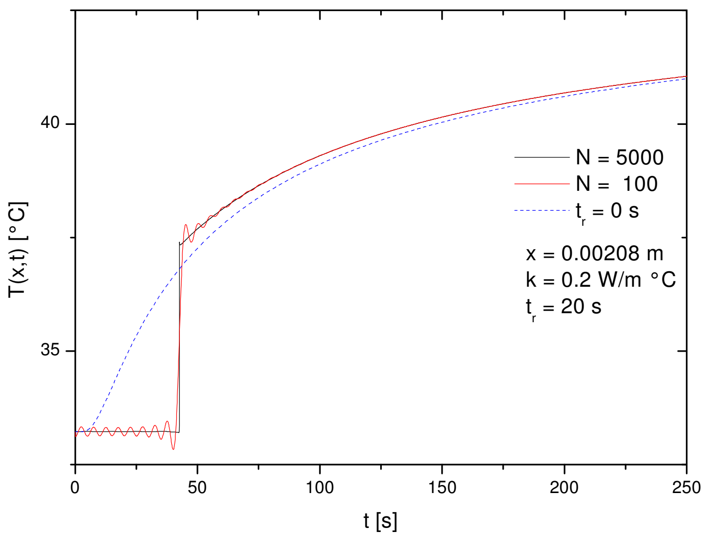

Figure 1. Convergence history of the proposed solution scheme.

Figure 1 explores the unusual features of the convergence characteristics, which are typical of eigenfunction expansion techniques applied to the solution of hyperbolic problems. The 100 term expansion for $t_r = 20\ s$ yields poor results in the early stages of the transient process since it exhibits a pronounced oscillation in the vicinity of the temperature jump. However, once the thermal signal reaches the selected position, this result is quite satisfactory since it agrees closely with the 5000 term expansion. In fact, this trend can be anticipated by inspecting relations (73)-(78). It is a well known fact that in parabolic heat diffusion, the convergence rate is very rapid and this is due to the decaying exponential term that contains the eigenvalues, Eq. (81). On the other hand, no such terms are found in Eqs. (73)-(78) and therefore it is naturally expected that the rate of convergence of the hyperbolic solutions is much slower than the one suggested by Eq. (81). This figure also presents the fully converged ($N = 50$) parabolic solution to this physical situation, evaluated by Eq. (81). It is apparent from this figure that before the thermal wave reaches $x = 0.00208\ m$, the two solutions differ significantly from one another.

Figure 2 depicts the transient thermal response obtained at the interface of the dermis and the sub-cutaneous tissue and is expressed in terms of the temperature elevation, $\Delta T(x,t)$ which is defined as the difference between the local temperature, $T(x,t)$, to the initial steady state distribution, $T_{init}(x)$. An inspection of this results shows that the present solution closely agrees to the one reported by Liu (2008). Moreover, a comparison is made between the hyperbolic ($t_r = 20s$) and parabolic ($t_r = 0\ s$) model. Once again, there is a noticeable deviation between these two models which diminishes as the new state of thermal equilibrium is gradually attained. Consequently, as time evolves, the effect of the thermal wave becomes less pronounced and so, parabolic models should not be used to predict the early stages of the transient process in skin burn injuries. This observation is especially usefully in the context of flash fire accidents where the combustion process lasts for less than 5 s and the time necessary for a second degree burn to develop is usually less than 3 s (Torvi and Dale, 1994).

In conclusion, this research is aimed at producing an analytical-numerical solution procedure for the evaluation of the transient temperature field in the thermal wave bioheat transfer equation. This task was achieved by selecting an appropriate Sturm-Liouville system that lead to the development of an integral-transform pair. Once the transformed problem is solved by either an analytical or a numerical means, the original temperature field is determined with the aid of the inverse relation. This methodology was applied to a test-case related to a skin burn injury and the results obtained corroborate the ideas outlined in this contribution. Our present research effort lies at further testing this methodology by selecting and simulating other relevant problems in bioheat transfer and at implementing numerical schemes, such as the Kummer transform (Frankel *et al.*, 1985), in order to accelerate the convergence rate of the series solution basic to hyperbolic bioheat transfer problems.

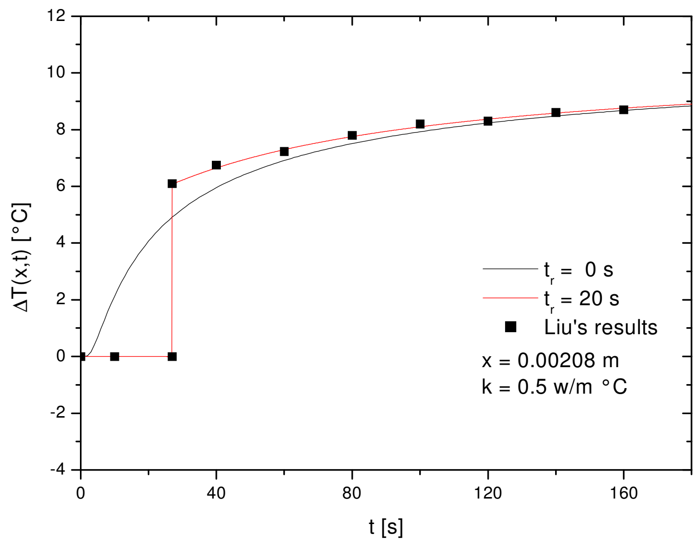

Figure 2. Comparison between present methodology and Liu's results.

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5. RESPONSIBILITY NOTICE

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