BOUND VORTEX SINGULARITY FOR UNSTEADY COMPRESSIBLE TWO DIMENSIONAL MOTION APPLIED TO AN INDICIAL RESPONSE OF A THIN PROFILE

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Abstract. *This article presents numerical solutions for the aerodynamic lift coefficients for the indicial response of a thin profile in abrupt chance in the angle of attack in a compressible subsonic two dimensional flow field. Bound vortex singularity for a body fixed reference system is used in accordance to the convected wave equation singularity solution. Arbitrary motion of the profile can then be solved by superposition techniques of this indicial response. A field point is influenced by the continuous disturbances generated by the vortex with a delay relative to the time of action of the same vortex. Numerical panel methods techniques are used. Classical theory usually treats the profile, or the wing, as fixed body in a moving flow. This leads to a rather different function for the vortex singularity in comparison to a reference system fixed at the fluid with a moving body. Previous studies by the authors analyzed the behavior of these vortex functions which are now applied to the indicial response problem and compared to analytical solutions. Numerical steps are described and discussions about time and panel discretizations are presented. The benefits of using this method are highlighted and shown to be accurate enough to for use in solving routine unsteady aerodynamic phenomena.*

Keywords*: unsteady aerodynamics, vortex lattice, compressible flow*

1. INTRODUCTION

If the sound velocity of the flow is assumed infinite, the Mach number of the flow is zero and the flow is said to be incompressible. This means that when a vortex is introduced at the flow, all field points will be influenced by the vortex instantly. The vortex influence spreads as a wave, at sound velocity. When the flow is compressible, that is, for any Mach number greater then zero, the vortex will travel with finite velocity and the range of its influence will be limited. When working with compressible flows some modifications have to be made to the incompressible vortex lattice approach. For a fixed profile in a uniform flow, that is, using a coordinate system fixed at the profile, the vortex singularity presented in Souza et al. (2007) has to be used. In this article, the singularity will be tested for the calculation of the lift coefficient of a thin profile at compressible flow after a sudden pitch or sinking movement. This movement is equivalent to the indicial singularity function. By definition, an indicial function is the response to a disturbance which is applied abruptly at time zero and is held constant thereafter, that is, a disturbance given by a step function. An arbitrary movement can then be solved by superposition methods of a series of indicial responses (Bisplinghoff et al.,1955). There are many studies that uses indicial functions approach to obtain other motions making use of superposition (Leishman, 1997) (Sitaraman and Baeder, 2004).

2. INDUCED BOUND VORTEX AND POTENTIAL JUMP

In a compressible flow the piston theory may be applied to solve the series of impulses generated at each time interval. The theory can be applied to any profile or wing at any Mach number greater then zero. The theory proceeds as observed in Bisplinghoff et al. (1955). At a very first instant $t = 0^+$ each little element of wing surface starts off like a little piston with normal velocity $w_a = U\alpha$ into the gas at rest. Hence, over a profile with pitch angle α, freestream velocity U, sound velocity a_∞ and density ρ, the jump of pressure in the upper and lower surface, ΔP, is,

$$\Delta P = -2\rho \cdot a_\infty \cdot U\alpha \,. \tag{1}$$

So the pressure coefficient, ΔC_p, can be expressed as, where M is the Mach number,

$$\Delta C_p = \frac{\Delta P}{1/2 \cdot \rho \cdot U^2} = \frac{2\rho a_\infty \cdot U\alpha}{1/2 \cdot \rho \cdot U^2} = \frac{4a_\infty \alpha}{U} = \frac{4\alpha}{M}. \tag{2}$$

From the disturbance velocity potential formulation (Garrick 1957), where φ is the disturbance velocity potential,

$$\Delta P = -2\rho_\infty \left(\frac{\partial \varphi}{\partial t} + U \frac{\partial \varphi}{\partial x} \right) \Rightarrow \Delta C_P = -\frac{2}{U^2} \left(\frac{\partial \varphi}{\partial t} + U \frac{\partial \varphi}{\partial x} \right), \tag{3}$$

and it can be defined, from the Eq. (3), an impulsive unsteady contribution, $-\frac{2}{U^2}\frac{\partial \varphi}{\partial t}$, which is time dependent, and a steady circulatory contribution, $-\frac{2}{U}\frac{\partial \varphi}{\partial x}$, which is x dependent. The impulsive contribution can be associated with the piston theory, and therefore equation (2), to calculate the potential jump. The derivative is calculated by numeric approximation using finite elements,

$$\frac{\partial \varphi}{\partial t} = \frac{\delta \varphi}{dt}, \tag{4}$$

which for the initial state of the profile's movement,

$$\begin{aligned} \frac{4\alpha}{M} &= -\frac{2}{U^2}\frac{\delta \varphi}{dt}, \\ \delta \varphi &= 2\alpha \frac{U^2}{M} dt. \end{aligned} \tag{5}$$

For other instants of time, the boundary conditions changes. The induced velocities, w, generated by the vortex wake has, now, to be taken into account, that is,

$$\delta \varphi = 2\left(w + U\alpha \right) \frac{U}{M} dt. \tag{6}$$

The potential jumps are replaced by a pair of counter rotating vortices which do not necessarily influence the whole field of interest at each instant. The induced velocity of a bound vortex in a compressible flow, W_t, will be, as shown by Souza et al. (2007),

$$W_t = -\frac{\Gamma}{2\pi} \frac{\beta}{x} \frac{\sqrt{\left(a\beta^2\tau + M \cdot x \right)^2 - x^2}}{a\beta^2\tau + M \cdot x}. \tag{7}$$

Where Γ is the vortex intensity, β is a factor given by $\beta = \sqrt{1 - M^2}$, τ is the time necessary for the perturbation to reach a certain field point and x the distance from the field point to the vortex origin.

3. NUMERICAL METHOD

The steps for the numerical solution for a thin profile with a sudden change of the angle of attack, for fixed profile in a moving compressible flow, will now be given. First, the profile is divided into a chosen number of panels, p, of same length. The control points are placed at the center of each panel and are identified by the subscript k, where, $1 \le k \le p$. Variable Δt is the time step chosen between iterations. t is the number of iterations, where the first iteration is for the initial state 0^+. So control points are designated as z_k and, since the profile is fixed, it will remain the same in all calculations.

With the indicial movement the profile changes its angle of attack abruptly inside a moving flow and is held constant at that position. A potential jump, $\delta\varphi$, is generated, given by Eq. (5). Then, the potential disturbance is replaced by a pair of counter-rotating vortices. This is done in every instant and every new potential jump. The last vortex attached to the last panel is free to move and a vortex wake grows as time increases. Each free vortex moves a distance equal to $U \cdot \Delta t$. For every new step, new boundary conditions are formed with the growing vortices and growing wake, and all vortices created by that instant must be taken into account. Between each panel, points determined by subscript j, at every instant t, a balance of bound vortices, Γ, has to be made,

$$\sum\nolimits_j^t = \Gamma_j^t, \quad for \quad j = 1$$
$$\sum\nolimits_j^t = \Gamma_j^t - \Gamma_{j+1}^t, \quad for \quad 2 \le j \le p \tag{8}$$

These are the vortices that will grow at the left hand of each panel. Nevertheless, the time of appearance and the range of influence of the created vortices have to be taken into account.

The pulse, that is, the potential jump or the vortices just formed at every instant of time will be the unknowns of a system of equations. For the Δt chosen, the range of influence of the vortices is checked to see if the control points are within the range. If the term in equation (7),

$$R = \sqrt{\left(a\beta^2\Delta t + M \cdot x\right)^2 - x^2}\ , \tag{9}$$

is negative, it means that the control point lies outside the range of the vortex field and its influence must not be taken into account in calculations.

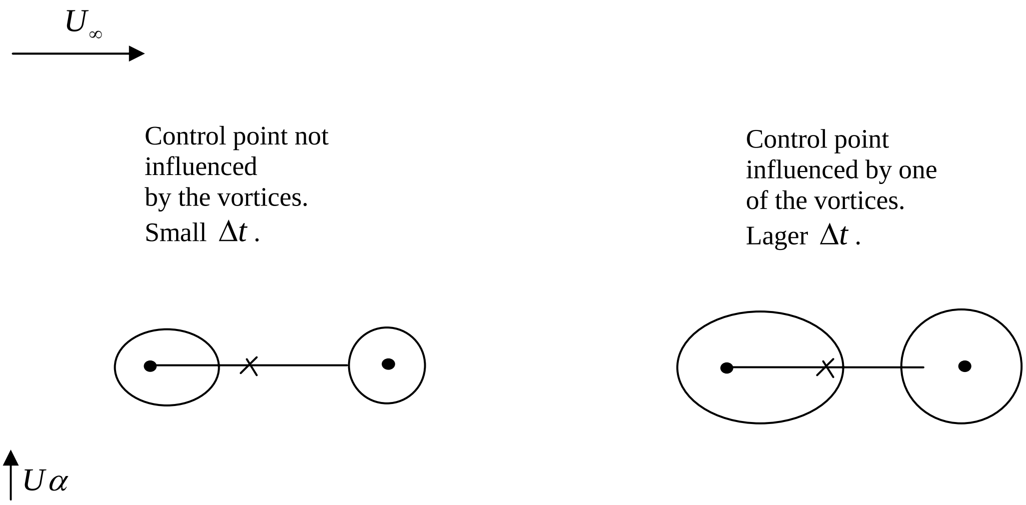

Figure 1. Compressible vortex in a one panel profile after time elapse of Δt

So the complex induced velocity for each vortex, $W_{k,j}^{*,t}$, is calculated by,

$$W_{k,j}^{*,t} = -\frac{\Gamma}{2\pi}\frac{\beta}{\left(z_k - z_j\right)}\frac{\sqrt{\left(a\beta^2 t + M \cdot \left(z_k - z_j\right)\right)^2 - \left(z_k - z_j\right)^2}}{a\beta^2 t + M \cdot \left(z_k - z_j\right)}\ , \tag{10}$$

meaning the induced velocity at point z_k by the vortex at point z_j that grew up to time t. The vortex intensity can be put as, Γ_j^t, where t is the time travel of the vortex and j its position at time t. Whenever the point at the center of the vortex lies before the control point, $j < k$, the sound velocity is overlapped by the velocity of the flow which has to be added when calculating the range of the vortex. The opposite occurs when $j > k$, when the stream velocity is subtracted from the sound velocity. This means that the vortex will travel at a speed greater, lesser than, or equal to, the speed of sound when,
for bound vortices,

$$j < k \Rightarrow a = a_\infty + U_\infty,$$
$$j > k \Rightarrow a = a_\infty - U_\infty, \tag{11}$$

and for free vortices, $a = a_\infty$.

The piston theory will guarantee that the boundary conditions are satisfied for the first impulse where no vortices are created yet. For a number of panels p, the sequence of events will be as shown in Fig (2), where Γ_{liv} stands for free vortex and the subscripts j has been neglected for means of clarity.

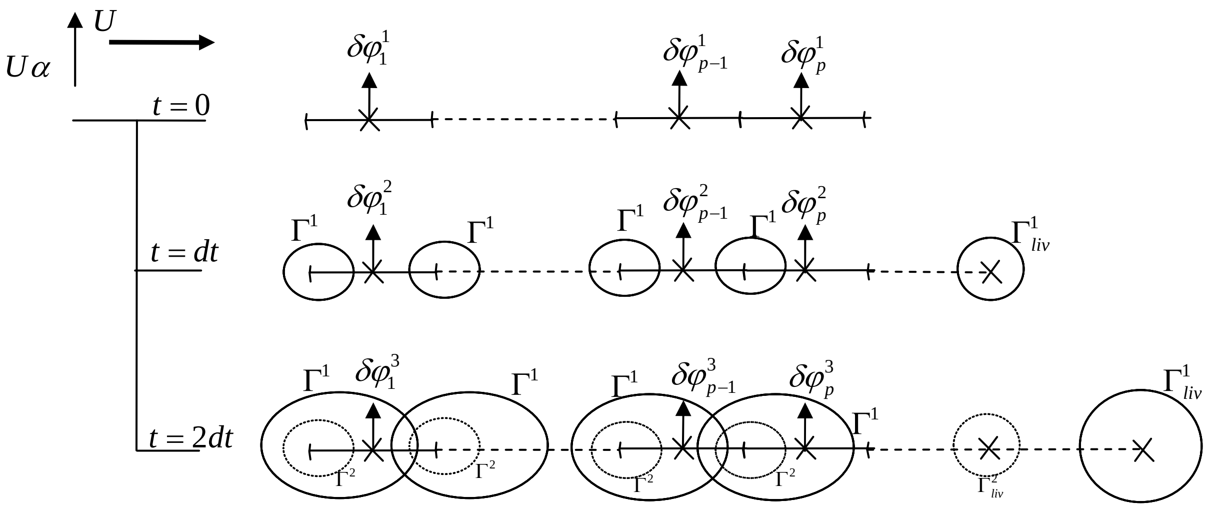

Figure 2. Sequence of events for a profile divided in p panels

The relation between the three dimensional steady Vortex Lattice Method and the analogy with the unsteady two dimensional flow will also establish a space-time relation between both 3D and 2D flows (Hernandes, 2003). While the center of the panel is taken at the three dimensional flow, this point is equivalent to half of the time interval of each iteration. This means that the boundary conditions should be taken at half the time interval, $t + 0.5 \cdot \Delta t$.

So finally, it is possible to define a system of equations that will solve the potential jump in every instant,

$$[A] \cdot [\delta\varphi]^t = [B]^t , \tag{12}$$

where,

$$[A] = \left[\begin{array}{cc|c} a_{11} + \dfrac{1}{2 \cdot a \cdot \Delta t} & a_{12} & a_{1n} \\ a_{21} & a_{22} + \dfrac{1}{2 \cdot a \cdot \Delta t} & a_{11} \\ \hline a_{n1} & a_{n2} & a_{nn} + \dfrac{1}{2 \cdot a \cdot \Delta t} \end{array} \right] , \tag{13}$$

and,

$$\begin{aligned} & a_{i,j} = \mathrm{Re}\left(\left(W^*_{k,j,dip} - W^*_{k,j+1,dip} \right) \cdot i \cdot t_j \right), \\ & W^*_{k,j,dip} = \begin{cases} 0, & if \;\; R < 0 \\ \dfrac{i}{2\pi} \dfrac{\beta}{\left(z_k - z_j\right)} \dfrac{\sqrt{\left(a\beta^2 t + M \cdot \left(z_k - z_j\right)\right)^2 - \left(z_k - z_j\right)^2}}{a\beta^2 \tau + M \cdot \left(z_k - z_j\right)}, & if \;\; R > 0 \end{cases} . \end{aligned} \tag{14}$$

Subscripts dip stands for dipole, liv for free, and lig for bound. The term from the piston theory appears only in the diagonal of the matrix, which is the panel influenced by itself.

$$[B]^t = \left[\begin{array}{c} U\alpha + \mathrm{Re}\left(\sum_{j=1}^{t} W^{*,t}_{v.liv.1,j} \cdot \delta\varphi_p^{t-j+1} \cdot i \cdot t_j \right) + \mathrm{Re}\left(\sum_{1}^{t} \sum_{j=1}^{p} W^{*,t}_{v.lig.1,j} \cdot \left(\delta\varphi_j^t - \delta\varphi_{j-1}^t \right) \cdot i \cdot t_j \right) \\ U\alpha + \mathrm{Re}\left(\sum_{j=1}^{t} W^{*,t}_{v.liv.2,j} \cdot \delta\varphi_p^{t-j+1} \cdot i \cdot t_j \right) + \mathrm{Re}\left(\sum_{1}^{t} \sum_{j=1}^{p} W^{*,t}_{v.lig.2,j} \cdot \left(\delta\varphi_j^t - \delta\varphi_{j-1}^t \right) \cdot i \cdot t_j \right) \\ \hline U\alpha + \mathrm{Re}\left(\sum_{j=1}^{t} W^{*,t}_{v.liv.p,j} \cdot \delta\varphi_p^{t-j+1} \cdot i \cdot t_j \right) + \mathrm{Re}\left(\sum_{1}^{t} \sum_{j=1}^{p} W^{*,t}_{v.lig.p,j} \cdot \left(\delta\varphi_j^t - \delta\varphi_{j-1}^t \right) \cdot i \cdot t_j \right) \end{array} \right] , \tag{15}$$

If $j = 1$, that is, the first point of the first panel, the third term in the sum in Eq. (15) will be,

$$\mathrm{Re}\left(\sum_{1}^{t} \sum_{j=1}^{p} W_{v.lig.k,1}^{*,t} \cdot \left(\delta\varphi_1^t \right) \cdot i \cdot t_1 \right). \tag{16}$$

Where,

$$W_{v.liv.k,j}^{*,t} = \begin{cases} 0, & if\ R < 0 \\ \dfrac{i}{2\pi} \dfrac{\beta}{\left(z_k - z_{liv,j} \right)} \dfrac{\sqrt{\left(a\beta^2 t + M \cdot \left(z_k - z_{liv,j} \right) \right)^2 - \left(z_k - z_{liv,j} \right)^2}}{a\beta^2 \tau + M \cdot \left(z_k - z_{liv,j} \right)}, & if\ R > 0 \end{cases}, \tag{17}$$

and

$$z_{liv,j} = z_{p+1} + j \cdot U \cdot \Delta t , \tag{18}$$

$$W_{v.lig.k,j}^{*,t} = \begin{cases} 0, & if\ R < 0 \\ \dfrac{i}{2\pi} \dfrac{\beta}{\left(z_k - z_j \right)} \dfrac{\sqrt{\left(a\beta^2 t + M \cdot \left(z_k - z_j \right) \right)^2 - \left(z_k - z_j \right)^2}}{a\beta^2 \tau + M \cdot \left(z_k - z_j \right)}, & if\ R > 0 \end{cases} \tag{19}$$

and also,

$$\left[\delta\varphi \right]^t = \left[A \right]^{-1} \cdot \left[B \right]^t . \tag{20}$$

The solution is the potential jump for any time t considered. This is done from instant 2 up to any other chosen instant t. For the first instant, the potential jump is calculated by Eq. (5),

$$\delta\varphi_k^1 = 2 \frac{\alpha U^2}{M} \Delta t . \tag{21}$$

After all potential jumps have been solved, the aerodynamic coefficients can be calculated.

4. AERODYNAMIC LIFT COEFFICIENT

Numerical integration the non-circulatory contribution of the lift coefficient for a given instant t results in,

$$C_{L\alpha,n}^t = \sum_{k=1}^{p} \left(\Delta x \frac{2}{\alpha U^2} \frac{\partial \varphi_k^t}{\Delta t} \right), \tag{22}$$

where $\Delta x = c/p$.

The steady contribution of the lift, subscript s, is obtained by the total circulation around the profile for a given instant, that is,

$$L_s^t = \rho \cdot U \cdot \Gamma^t . \tag{23}$$

So the dimensionless coefficient, $C_{L\alpha,s}^t$, will then be,

$$C_{L\alpha,s}^t = \frac{L_s^t}{1/2 \cdot \rho U^2 \alpha} = \frac{\rho \cdot U \cdot \Gamma^t}{1/2 \cdot \rho U^2 \alpha}, \qquad C_{L\alpha,s}^t = \frac{2}{U\alpha} \cdot \Gamma^t. \tag{24}$$

To find the total circulation over the profile for any given instant, first, the circulation over each individual panel is calculated for all instants prior to one given. The circulation of each individual panel is given by the difference between the potential jumps over the panel,

$$\Gamma_k^t = \Delta\delta\varphi_k^t = \begin{cases} \delta\varphi_k^t, & k=1 \\ \delta\varphi_k^t - \delta\varphi_{k-1}^t, & 1<k\le p \end{cases} \tag{25}$$

$$\Gamma^t = \underbrace{\sum_{t=1}^{t} \underbrace{\sum_{k=1}^{p} \underbrace{\Gamma_k^t}_{\text{circulation over panel } k \text{ at instant } t}}_{\text{circulation over the profile at instant } t}}_{\text{circulation over the profile up to instant } t} . \tag{26}$$

So finally, the lift coefficient for any given instant is given by the sum of the non-steady and steady contributions,

$$C_{L\alpha,n}^t = \frac{2}{\alpha U}\left(\frac{1}{U\cdot\Delta t\cdot p}\sum_{k=1}^{p}\partial\varphi_k^t + \sum_{1}^{t}\sum_{k=1}^{p}\Gamma_k^t\right). \tag{27}$$

5. RESULTS AND CONCLUDING REMARKS

For calculations of arbitrary motions by superposition of indicial responses, precision is needed at the beginning of the movement. So, solutions presented here for the aerodynamics lift coefficient will have a maximum length of ten traveled chords, for that is the range of interest in the unsteady phenomena. Calculations of the lift coefficient was chosen to analyze the behavior of the bound vortex singularity function.

Beyond roughly one semi-chord length from the starting point, Wagner's function for compressible flow, $\phi_c(s)$, for subsonic airfoils are predicted by Fourier-integral superposition. It can be asserted (Bisplinghoff et al., 1955) that the result will be, for an impulse plunging of magnitude h,

$$L_c(k) = 2\pi\frac{\rho_\infty}{2}U^2(2b)(h)\phi_c(s), \tag{28}$$

and

$$\phi_c(s) = \frac{2}{\pi}\int_0^\infty \frac{F_c(k)}{k}\sin ksdk = 1+\frac{2}{\pi}\int_0^\infty \frac{G_c(k)}{k}\cos ksdk . \tag{29}$$

Equations (28) and (29) were integrated numerically for Mach numbers 0.5, 0.6 and 0.7 (Bisplinghoff et al., 1955). It is desirable to have approximate exponential representations of $\phi_c(s)$, which are provided by (Bisplinghoff et al., 1955),

$$\phi_c(s) = b_0 + b_1\cdot e^{-\beta_1\cdot s} + b_2\cdot e^{-\beta_2\cdot s} + b_3\cdot e^{-\beta_3\cdot s} . \tag{30}$$

Table (1) lists the various constants associated with the function for pitching motion.

Table 1 – Exponential representation of indicial functions for various Mach numbers.

Indicial Function	M	b_0	b_1	b_2	b_3	β_1	β_2	β_3
$\phi_c(s)$	0	1	-0.165	-0.335	0	0.0455	0.300	-
	0.5	1.155	-0.406	-0.249	0.773	0.0754	0.372	1.890
	0.6	1.250	-0.452	-0.630	0.893	0.0646	0.481	0.958
	0.7	1.400	-0.5096	-0.567	0.5866	0.0536	0.357	0.902

The numerical solution here presented is directly associated with the chosen discretization of the profile, Δx, and the time step, Δt. Since the profile is equally divided, the number of panels will determine the size of each panel, $\Delta x = c/p$. Hernandes (2003) shows in his work that the time step must be chosen as to satisfy the condition, $c/p \ge \Delta t$

or $\Delta x/\Delta t \geq 1$, that is, to obtain a good numerical result the time step must be equal or less to the size of the panel. Nevertheless, when calculating the aerodynamic coefficient, Hernandes (2003) uses the unsteady vortex singularity for a moving profile in a fixed coordinate system (free vortex). When using the unsteady vortex singularity for a fixed profile in a moving flow, as shown is this work (bound vortex), the results in Fig. (3) indicates that convergence of the solution happens in the opposite way. That is, solutions' accuracy seems to increases when the ratio $\Delta x/\Delta \mathrm{t}$ is set up to $\Delta x/\Delta \mathrm{t} \approx 0.67$. In fact, poor solution is obtained for $(c/p = 0.1) > (\Delta t = 0.05)$ (Fig. 4), and convergence is obtained as time step is increased (Fig. 4). This means that less computational time will be needed to calculate the aerodynamic coefficients. This phenomenon is due to the behavior of the two different vortex functions used in the two different works. The optimal time step was set as approximately 1.5, which is approximately the time the range of influence of vortex needs to travel by two consecutive control points.

The number of panels chosen has a strong influence in the precision of solutions. Figure (3) shows how an increase in dicretization will cause the numerical model to notice all pressure discontinuities and thus, affecting the accuracy of the lift curves.

From Fig. (5) it may be concluded that the influence of compressibility manifests itself through a more gradual approach of all indicial quantities to their final values. Once the impulsive starting effect, which is largely non-circulatory, has died out, it can be stated that the defect of lift from its final magnitude, at any particular time, increases with the Mach number.

Figure (3) shows that good approximation can be obtained with as little as 10 panels. This comes in handy when computational time and storage is a problem. Results for 10 panels are obtained with just a few seconds when working with a computer with processor AMD Turion 64, with 1.6 GHz, and 512 RAM. For twenty panels this grows up to approximately three hours. The longer the number of chords traveled the slower the calculations get. For all disturbances in each time step has to be stored.

Results and considerations of the bound vortex approach here presented inserts in the context of research in unsteady aerodynamics and the elementary solutions used in the compressible vortex lattice method. The results shown in this article are a further step to understanding potential flow methods in unsteady aerodynamics. Detailed numerical steps and overall comparison of unsteady aerodynamics lift coefficients are given. All final results appear to be fully accurate enough for use in solving routine aeroelastic problems.

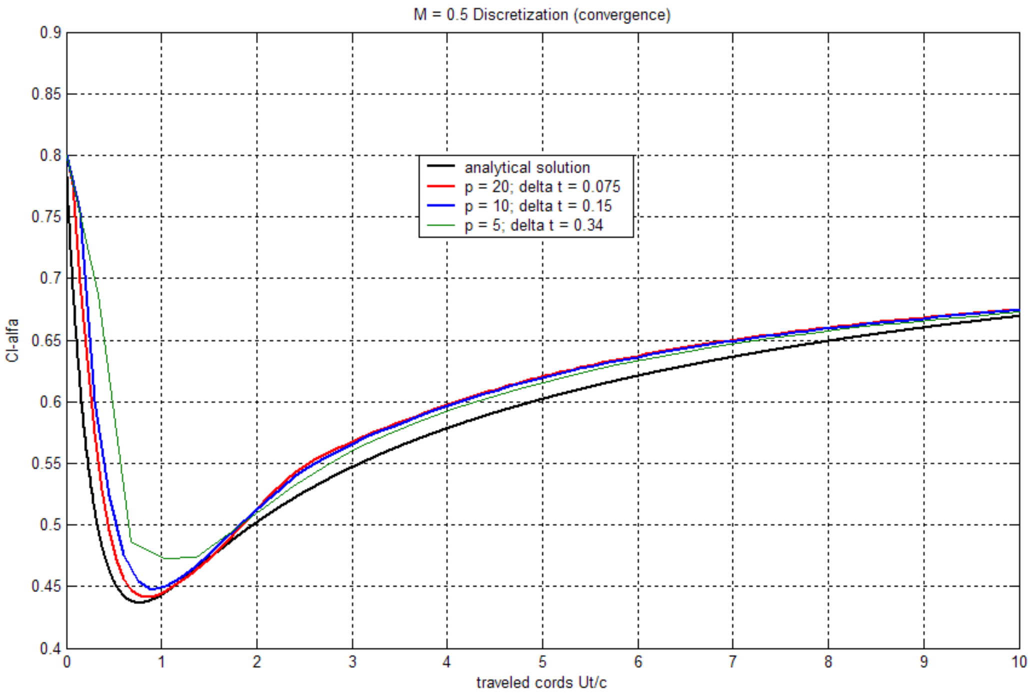

Figure 3. Panel discretization (convergence)

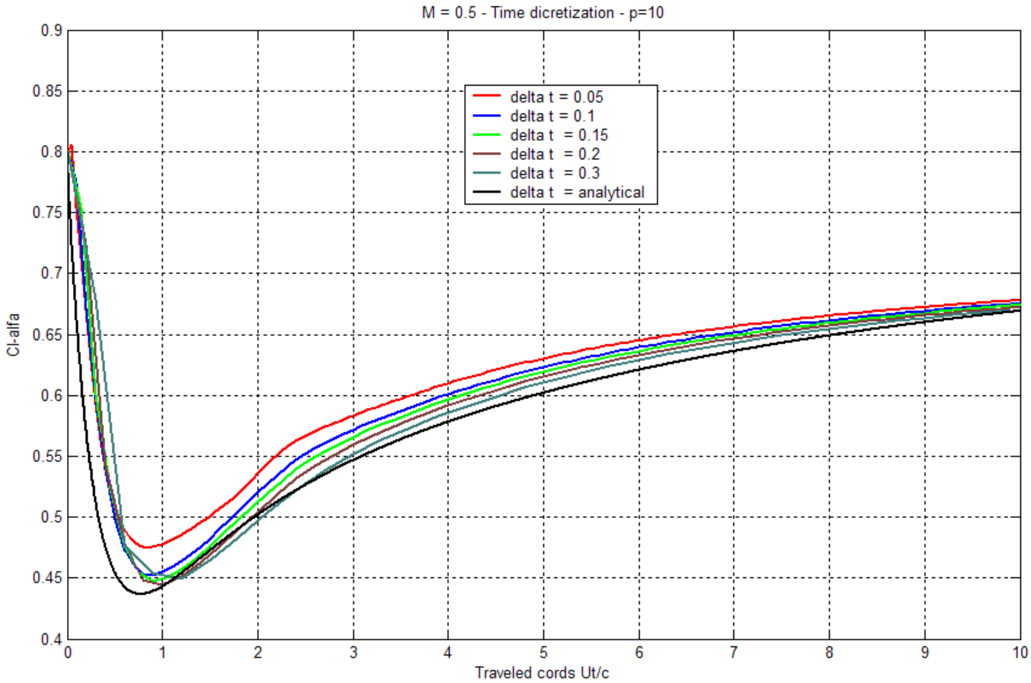

Figure 4. Time discretization

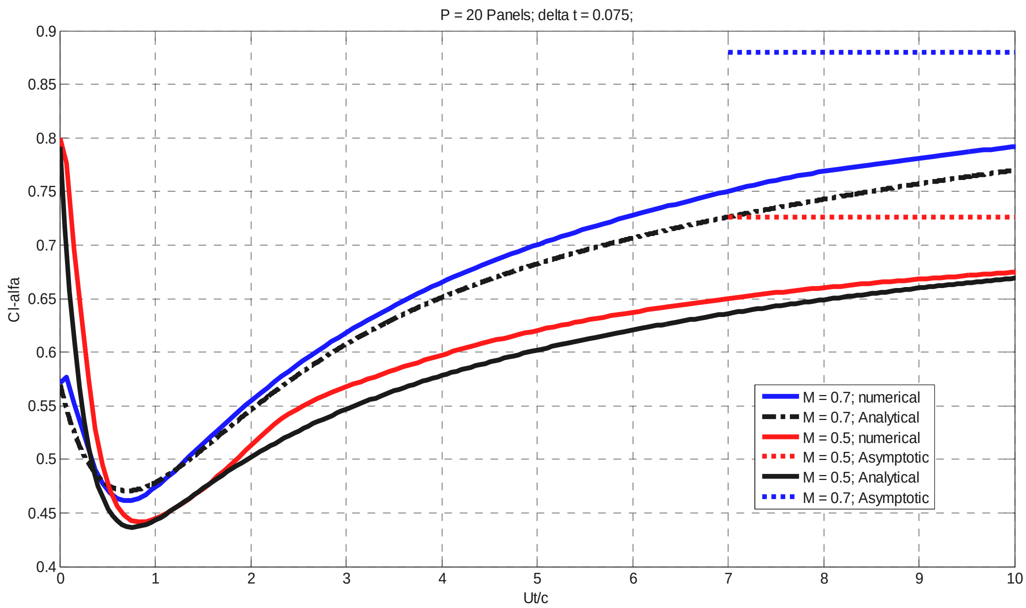

Figure 5. Mach variation

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