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A PROCEDURE FOR SIMULATING THE NONLINEAR CONDUCTION HEAT TRANSFER IN A BODY WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY.

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Abstract. *In this work a systematic procedure is proposed for simulating the conduction heat transfer process in a solid with a strong dependence of the thermal conductivity on the temperature. Such kind of inherently nonlinear partial differential equation, subjected to classical linear boundary conditions, will be solved with the aid of a Kirchoff Transform by means of a sequence of very simple linear problems. The proposed procedure provides the exact solution of the problem and induces finite dimensional approximations that can be used for computational simulations. Some typical cases are simulated by means of a finite difference scheme.*

Keywords: Temperature dependent thermal conductivity, Kirchoff transform, heat transfer, nonlinear equations system.

1. Introduction

Most of the conduction heat transfer phenomena are described under the assumption of temperature independent thermal conductivity. Such hypothesis is mathematically convenient because, in general, gives rise to linear partial differential equations. Nevertheless, the thermal conductivity is always a temperature dependent function. Many times, neglecting such dependence (assuming constant thermal conductivity), we have an inadequate mathematical description of the conduction heat transfer process.

The main objective of this work is to provide a reliable and systematic procedure for describing the conduction heat transfer in a rigid solid, with temperature dependent thermal conductivity, subjected to linear boundary conditions (Newton's law of cooling). This procedure is exact and employs only the tools utilized in problems in which the thermal conductivity is assumed to be a constant.

The first step consists of employing the Kirchoff transform in order to change the original problem into another one consisting of a linear partial differential equation subjected to a nonlinear boundary condition. The second step consists of regarding the new problem as the limit of a sequence of linear problems.

So, the problems with temperature dependent thermal conductivity will be regarded as the limit of a sequence whose elements are solution of linear problems.

2. Governing Equations

Let us consider a rigid, opaque and isotropic body at rest with domain represented by bounded open set Ω with boundary $\partial\Omega$. The steady-state heat transfer process inside this body is mathematically described by the following elliptic partial differential equation

$$div(kgradT)+\dot{q}=0 \quad in \quad \Omega \tag{1}$$

where T, $\dot{q}$ and k denote, respectively, the temperature field, the internal heat generation rate (per unit volume) and the thermal conductivity. In this work the thermal conductivity is assumed to be a function of the local temperature. In other words,

$$k=\hat{k}(T) \tag{2}$$

Assuming that body boundary and the environment exchanges energy according to Newton's law of cooling, the boundary conditions associated with equation (1) are given by

$$-(kgradT)\mathbf{n}=h(T-T_{\infty}) \quad on \quad \partial\Omega \tag{3}$$

in which $\mathbf{n}$ is the unit outward normal (defined on $\partial\Omega$), h is the convection heat transfer coefficient and T_{∞} is a temperature of reference

The resulting problem (nonlinear) is given by

$$\begin{aligned} div(kgradT)+\dot{q}=0 \quad in \quad \Omega \\ -(kgradT)\mathbf{n}=h(T-T_{\infty}) \quad on \quad \partial\Omega \end{aligned} \tag{4}$$

3. The Kirchoff Transform

Since the thermal conductivity is always positive-valued, the new variable ω (Kirchoff transform) defined by

$$\omega=\int_{T_0}^{T}\hat{k}(\xi)\,d\xi=\hat{f}(T) \tag{5}$$

being an invertible function of T. This definition allows us to write

$$grad\,\omega=k\;grad\,T \tag{6}$$

So, the original problem can be rewritten as follows

$$\begin{aligned} div(grad\omega)+\dot{q}=0 \quad in \quad \Omega \\ -(grad\omega)\mathbf{n}=h(\hat{f}^{-1}(\omega)-T_{\infty}) \quad on \quad \partial\Omega \end{aligned} \tag{7}$$

where $T\equiv\hat{f}^{-1}(\omega)$. Although above problem remains nonlinear, this nonlinearity takes place only on the boundary, not in the partial differential equation. It is to be noticed that, since the thermal conductivity is everywhere positive,

$$k>0 \quad \Rightarrow \quad \frac{d\omega}{dT}>0 \quad \text{and} \quad \frac{dT}{d\omega}>0 \quad \text{everywhere} \tag{8}$$

so, the temperature is a strictly increasing function of ω.

For instance, if we have

$$k=\begin{cases} k_1=\text{constant} & \text{if} \quad T<T_0 \\ k_2=\text{constant} & \text{if} \quad T\geq T_0 \end{cases} \tag{9}$$

it is easy to show that

$$T \equiv \hat{f}^{-1}(\omega) = |\omega| \left[\frac{1}{2k_2} - \frac{1}{2k_1} \right] + \omega \left[\frac{1}{2k_2} + \frac{1}{2k_1} \right] + T_0 \tag{10}$$

4. Constructing the Solution from a Sequence of Linear Problems

The solution of

$$\begin{aligned} div(grad\omega) + \dot{q} = 0 \quad in \quad \Omega \\ -(grad\omega)\mathbf{n} = h(\hat{f}^{-1}(\omega) - T_\infty) \quad on \quad \partial\Omega \end{aligned} \tag{11}$$

can be represented by the limit of the nondecreasing sequence $[\Phi_0, \Phi_1, \Phi_2, ...]$ whose elements are obtained from the solution of the linear problems below

$$\begin{aligned} div(grad\Phi_{i+1}) + \dot{q} = 0 \quad in \quad \Omega \\ -(grad\Phi_{i+1})\mathbf{n} = \alpha\Phi_{i+1} + \beta_i \quad on \quad \partial\Omega \\ \beta_i = h(\hat{f}^{-1}(\Phi_i) - T_\infty) - \alpha\Phi_i \end{aligned} \tag{12}$$

in which α is a, sufficiently large, constant and $\Phi_0 \equiv 0$.

It is remarkable that, for each i, the function Φ_{i+1} is the unknown and the function Φ_i is known. So, $\hat{f}^{-1}(\Phi_i)$ is always known in (12), being evaluated from the following equation

$$\Phi_i = \int_{T_0}^{\hat{f}^{-1}(\Phi_i)} \hat{k}(\zeta) d\zeta \tag{13}$$

For each spatial position, the root of the above equation is unique. This uniqueness is supported by equation (8).

The constant α must be large enough for ensuring that, at any point of Ω, $\Phi_{i+1} \geq \Phi_i$. In reference [2] such result is proven as well as it is provided an upper bound for the constant α [3]. For the problem considered in this work it is sufficient to choose α such that

$$\alpha \geq \frac{h}{k_{MIN}} \tag{14}$$

where k_{MIN} is the minimum value of the thermal conductivity.

5. Convergence

The limit of the sequence $[\Phi_0, \Phi_1, \Phi_2, ...]$, denoted here by Φ_∞ exists and is, in fact, a solution of the problem. To prove this assertion, let us begin showing that Φ_∞ is a solution of (7). In other words

$$\begin{aligned} div(grad\Phi_\infty) + \dot{q} = 0 \quad in \quad \Omega \\ -(grad\Phi_\infty)\mathbf{n} = h(\hat{f}^{-1}(\Phi_\infty) - T_\infty) \quad on \quad \partial\Omega \end{aligned} \tag{15}$$

Since β_∞ is given by

$$\beta_\infty = h\left(\hat{f}^{-1}(\Phi_\infty) - T_\infty\right) - \alpha\Phi_\infty \tag{16}$$

we have that (12) and (15) coincide. So, Φ_∞ is a solution. Now, taking into account that the sequence is nondecreasing and has an upper bound, we ensure the convergence, once that the solution of (15) belongs to the same space of the solutions of (12) for each i [4] e [5].

6. An Example

Let us consider the following problem (spherical body with uniform heat generation, surrounded by the same medium, with $h = 1$ and $T_\infty = 1$ - in some system of units)

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right)\right] + 1 = 0 \quad for \quad 0 \le r < 1$$
$$-k\frac{dT}{dr} = T \qquad at \quad r = 1 \tag{17}$$

in which it is assumed that T represents an absolute temperature and $k = 3T + 2$. The solution is easily reached and given by

$$T = -\frac{2}{3} + \left[\frac{(1-r^2)}{9} + 1\right]^{1/2} \tag{18}$$

Now, let us employ the proposed procedure. With the Kirchoff transform, the problem yields

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\omega}{dr}\right)\right] + 1 = 0 \quad for \quad 0 \le r < 1$$
$$-\frac{d\omega}{dr} = \hat{f}^{-1}(\omega) = -\frac{2}{3} + \left[\frac{4}{9} + \frac{2\omega}{3}\right]^{1/2} \qquad at \quad r = 1 \tag{19}$$

The linear procedure for reaching the elements of the sequence is represented as follows

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi_{i+1}}{dr}\right)\right] + 1 = 0 \quad for \quad 0 \le r < 1$$
$$-\frac{d\Phi_{i+1}}{dr} = \alpha\Phi_{i+1} + \beta_i \qquad at \quad r = 1 \tag{20}$$
$$\beta_i = -\frac{2}{3} + \left[\frac{4}{9} + \frac{2\Phi_i}{3}\right]^{1/2} - \alpha\Phi_i$$

or, simply as

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi_{i+1}}{dr}\right)\right] + 1 = 0 \quad for \quad 0 \le r < 1$$
$$-\frac{d\Phi_{i+1}}{dr} = \alpha\Phi_{i+1} - \alpha\Phi_i - \frac{2}{3} + \left[\frac{4}{9} + \frac{2\Phi_i}{3}\right]^{1/2} \qquad at \quad r = 1 \tag{21}$$

Since the problem makes sense only for $T > 0$, we have that $k_{MIN} > 2$. So, we can work with any $\alpha \ge 1/2$. We shall use $\alpha = 3$!

The general solution of equation

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_{i+1}}{dr}\right)\right]+1=0 \quad for \quad 0\le r<1 \tag{22}$$

is

$$\Phi_{i+1}=-\frac{r^2}{6}+C_{i+1} \quad for \quad 0\le r<1 \tag{23}$$

where the constant C_{i+1}, for $i>0$, is obtained from the boundary condition. In other words,

$$\frac{1}{3}=3\left[-\frac{1}{6}+C_{i+1}\right]-3\left[-\frac{1}{6}+C_i\right]-\frac{2}{3}+\left[\frac{4}{9}+\frac{2}{3}\left(-\frac{1}{6}+C_i\right)\right]^{1/2} \tag{24}$$

Then,

$$C_{i+1}=C_i+\frac{1}{3}\left[1-\sqrt{\frac{1}{3}+\frac{2}{3}C_i}\right] \tag{25}$$

The constant C_1 is obtained from

$$\frac{1}{3}=3\left[-\frac{1}{6}+C_1\right] \quad\Rightarrow\quad C_1=\frac{5}{18} \tag{26}$$

If we employ $\alpha=20$, then

$$C_{i+1}=C_i+\frac{1}{20}\left[1-\sqrt{\frac{1}{3}+\frac{2}{3}C_i}\right] \tag{27}$$

and the constant C_1 is obtained from

$$\frac{1}{3}=20\left[-\frac{1}{6}+C_1\right] \quad\Rightarrow\quad C_1=\frac{11}{60} \tag{28}$$

Table 1 presents a comparison between the values obtained for C_i with two distinct values of α

	$\alpha=3$	$\alpha=20$
$i=$	$C_i=$	$C_i=$
1	0.371083678318	0.199585905322
2	0.450399853146	0.215439507012
3	0.518403223696	0.230908373418
4	0.577078085519	0.246005928926
5	0.627951577215	0.260744856563
10	0.800503112959	0.329459272507
20	0.940194171974	0.445715284244
50	0.998272514569	0.679342555325
100	0.999995202718	0.866108252119
500	0.999999985099	0.999842551009
1000	0.999999985099	0.999999949824

Table 1 – A comparison between results obtained with $\alpha=3$ and $\alpha=20$.

It can be proven that $C_\infty=1$. Hence, the limit of the sequence is given by

$$\Phi_\infty=-\frac{r^2}{6}+1 \quad for \quad 0\le r<1 \tag{29}$$

The solution ω is exactly the limit of the sequence. Taking into account that

$$\hat{f}^{-1}(\omega) = T = -\frac{2}{3} + \left[\frac{4}{9} + \frac{2\omega}{3}\right]^{1/2} \quad (30)$$

we have the following result (coincident with the exact solution, previously obtained)

$$T = -\frac{2}{3} + \left[\frac{4}{9} + \frac{2}{3}\left(-\frac{r^2}{6} + 1\right)\right]^{1/2} \quad (31)$$

7. A computational numerical simulation example

A computational support has been developed parallel to the mathematical modeling in order to provide numerical results of the proposed method. All the programs were made in MATLAB. Now, a computational simulation for a typical problem will be shown. The physical characteristics of the problem are described by the following parameters: convection heat transfer coefficient $h = 5$, heat generation $\dot{q} = 10$, reference temperature $T_\infty = 20$. The conduction is processed in a rectangular flat plate. The spatial domain is described in a rectangular cartesian coordinates system. The plate is represented by a rectangular elements mesh for Finite Differences Method application, and the linear equations system is solved by the Gauss-Seidel iterative method. The temperature dependence of the thermal conductivity is given by:

$$k = \begin{cases} k_1 = 45 \rightarrow T < 21 \\ k_2 = 10 \rightarrow T \geq 21 \end{cases}$$

In this simulation we assume $\alpha = 8$. The following figures show the iterative evolution of the simulation.

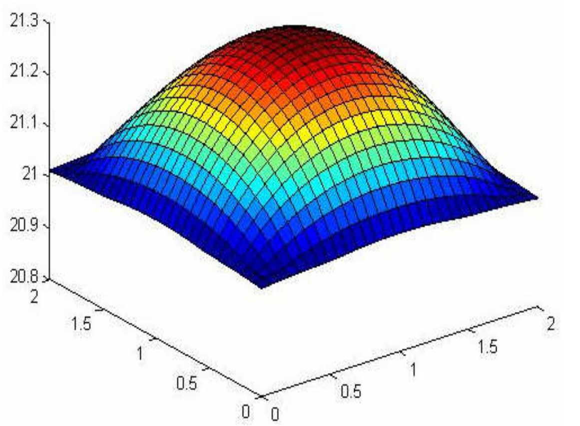

Figure 1. first iteration result

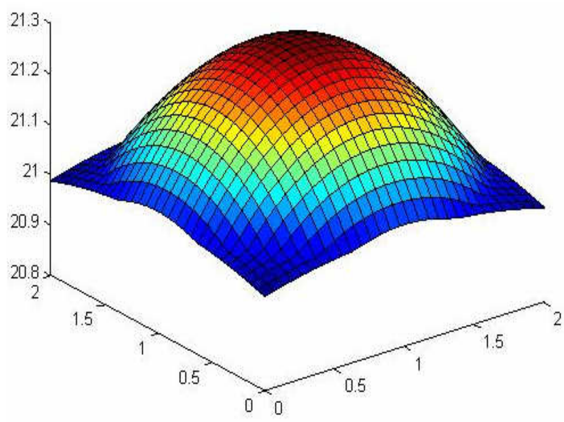

Figure 2. seventh iteration result

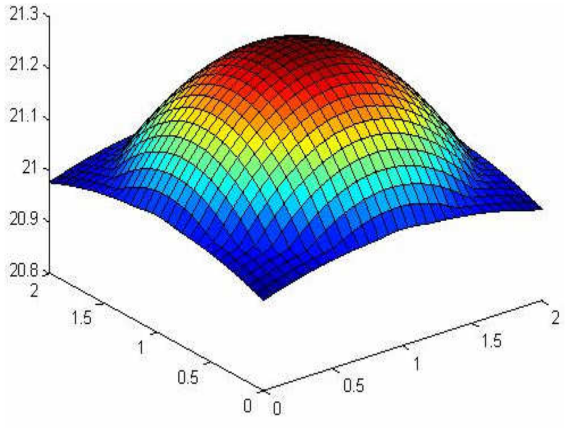

Figure 3. thirteenth iteration result

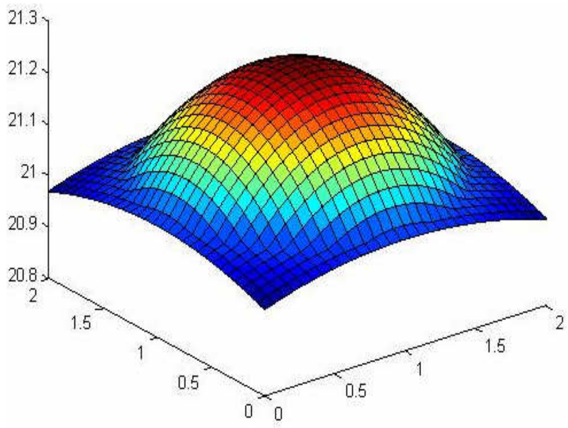

Figure 4. seventeenth iteration result

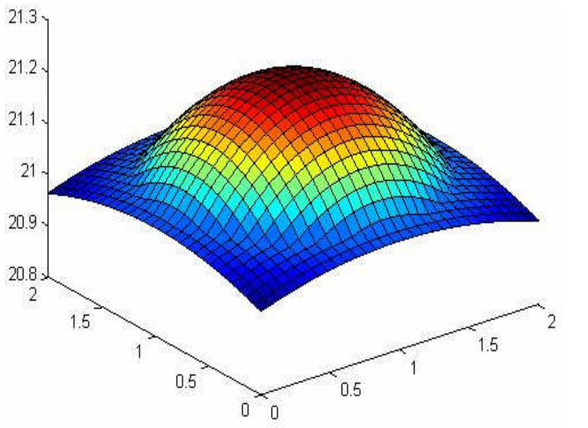

Figure 5. twenty third iteration result

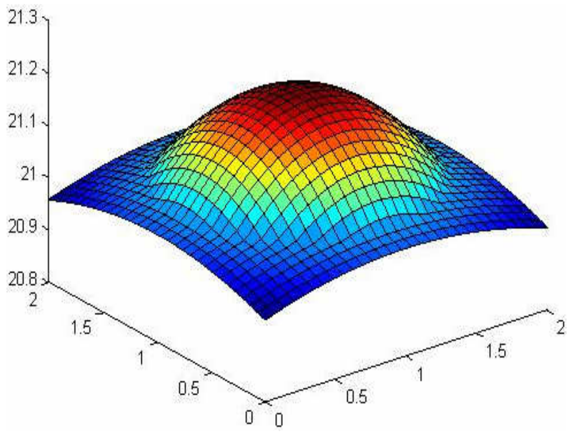

Figure 6. thirtieth and final result

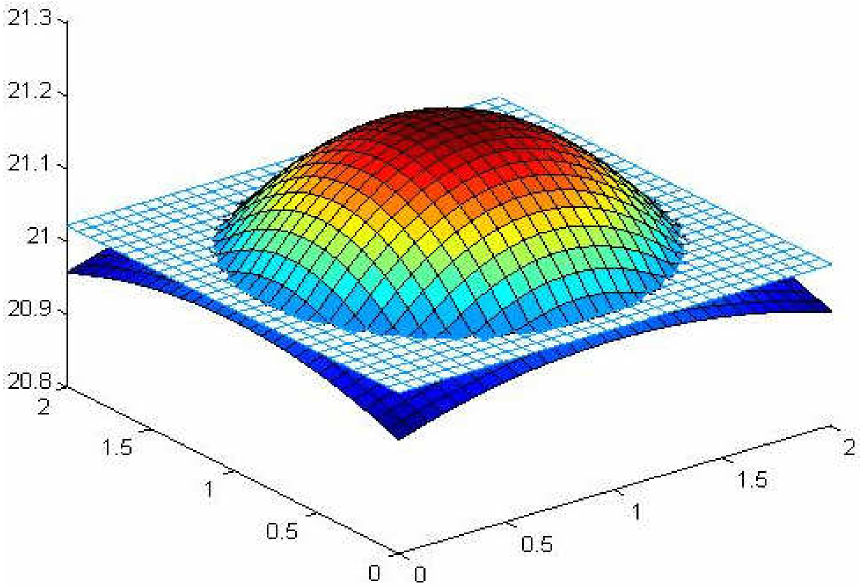

Figure 7. plane dividing the temperature field in two regions of constant thermal conductivity

8. Final remarks

The presented procedure proved to be a simple, but efficient subsidy for solving problems of conduction heat transfer with temperature dependent thermal conductivity, by means of classic tool employed on heat transfer linear problems, such as finite differences.

9. References

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