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IMPROVED LUMPED-DIFFERENTIAL FORMULATION OF TRANSIENT CONJUGATED CONDUCTION-CONVECTION IN EXTERNAL FLOW

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Abstract: *Mathematica is employed in the symbolic computation implementation of a hybrid numerical-analytical solution for transient laminar forced convection over flat plates of non-negligible thickness, subjected to arbitrary time variations of applied wall heat flux from above. This conjugated conduction-convection problem is first simplified by employing the Coupled Integral Equations Approach (CIEA) to reformulate the heat conduction problem on the plate by averaging the related energy equation in the transversal direction. As a result, a partial differential formulation for the average wall temperature is obtained, while a third kind boundary condition is achieved for the fluid in the heat balance at the solid-fluid interface. An approximate solution is then proposed for the coupled partial differential equations by combining the classical integral method for the boundary layer equations and the method of lines implemented in the Mathematica routine* ***NDSolve****. The integral method is initially employed to yield polynomial approximations for both the steady velocity field and the transient temperature field within the fluid. Then, the governing partial differential equation for the ratio of thicknesses of the boundary layers is symbolically determined, from the solution of the related integral form of the energy equation in the transient state. Numerical solution for the thermal boundary layer thickness and for the average solid temperature is then obtained through the automatic use of the built in function* ***NDSolve****, yielding the time evolution and the longitudinal distribution of these parameters, for any specific prescribed wall heat flux time function. Finally, local heat transfer coefficients are readily determined from the wall temperature distributions, as well as the temperature values at any desired point within the fluid.*

Keywords*. forced convection, external convection, boundary layer, conjugated problem, conduction-convection*

1. Introduction

Forced convection over solid surfaces is usually studied by neglecting the participation of the wall in the heat transfer process through imposition of temperature or heat flux, as in the classical thermal boundary layer problem (Kays and Crawford, 1980, Schlichting, 1968, White, 1974), at the fluid-solid interface. The inclusion of wall conduction effects is of major relevance to the accurate prediction of heat transfer rates, but brings up a conjugated conduction-convection problem of a more involved nature when the full energy equations for both fluid and solid are to be solved simultaneously. Early work on approximate analytical solutions in both external and internal flows (Luikov *et al.*, 1971, Mori *et al.*, 1974) demonstrates the mathematical difficulties involved in handling this mixed parabolic-elliptic formulation. More recently, the advancement of purely numerical approaches has allowed for the computational handling of such classical problems in heat transfer, but has also confirmed the high computational costs for the accurate solution of coupled conduction-convection problems, governed by an increased number of parameters. However, simpler models have been proposed in the literature concerned with duct flows (Shah and London, 1978) that radially or transversally lump the temperature distribution at the duct wall, but retain the axial conduction information along the wall, thereby reducing the number of parameters to be explicitly considered. The simplified model is expected to be particularly useful in reducing computational costs and analytical involvement, aspects explored by previous works in channel flow, under different solution methodologies (Faghri and Sparrow, 1980, Zariffeh *et al.*, 1982, Wijeysundera, 1986, Guedes *et al.*,1989, Guedes *et al.*, 1991). The simple lumped formulation, although expected to be more adequate in the range of parameters that provides a not so significant radial or transversal temperature gradients within the wall, has been checked only briefly against numerical solutions that consider the two-dimensional effects (Pagliarini, 1988). In addition, for the duct flow situation, transient analysis is quite limited and only a few situations of periodic inlet and boundary conditions disturbances were previously considered, which lead to quasi-steady formulations, with or without longitudinal wall heat conduction (Guedes and Cotta, 1991, Guedes *et al.*, 1994, Sucec, 2002, Fourcher and Mansouri, 1998, Mansouri *et al.*, 2004). Similarly, the transient analysis of conjugated conduction-external convection is still quite restricted and a few contributions have been offered based on approximate analytical methodologies (Pozzi and Tognaccini, 2000, Lachi *et al.*, 2002, Lachi *et al.*, 2003, Lachi *et al.*, 2004-a).

The present work first brings a reformulation strategy to this class of transient conjugated conduction-external convection problems through application of the ideas in the coupled integral equations approach (CIEA) (Cotta and Mikhailov, 1997, Aparecido and Cotta, 1989 and Correa and Cotta, 1998), in order to allow for the simplification of the heat conduction formulation at the wall. Therefore, an improved lumping procedure is applied to the wall transversal direction, offering a simpler transient one-dimensional formulation for conduction along the wall, in terms of the transversally averaged temperature.

This so-called coupled integral equations approach (CIEA) is employed as a formulation simplification technique for the heat conduction problem, aimed at reducing the number of independent variables involved in the mathematical formulation, through an improved lumping procedure on those coordinates selected to be removed, in this case the transversal direction. The resulting lumped-differential formulation offers substantial enhancement over classical lumping schemes (Shah and London, 1978) in terms of accuracy, without introducing additional complexity in the corresponding final simplified differential equations to be handled. The approach is here demonstrated through a representative transient conjugated conduction-convection problem, for laminar air flow over a flat plate heated at the fluid-solid interface, and the enhancement characteristics are examined against the numerical solution of the fully differential formulation for the conduction problem.

Second, the approximate lumped-differential formulation for the solid-fluid interface facilitates the utilization of the classical Integral Method for thermal boundary layer analysis, (Kays and Crawford, 1980, Schlichting, 1968, White, 1974), in approximating the fluid temperature distribution and determining the transient thermal boundary layer thickness. In this approach, also quite popular in heat conduction analysis (Ozisik, 1980), the dependent variable is approximated by a prescribed functional form in one of the spatial variables, in general in a polynomial form, followed by the solution for the coefficients in such approximate formulation, as a function of the remaining independent variable (time or another space variable). In obtaining these coefficients, one attempts to satisfy boundary and initial conditions, as well as other asymptotic information of the original problem, in direct relation with the number of coefficients to be determined according to the proposed functional approximation. It is indeed a rather simple approximate analytic approach, with recognized practical importance, particularly in the analysis of non-similar problems in external convection.

Although less cited in scientific research along the last few years, due to the wider availability of computational resources for simulations in fluid mechanics and heat transfer, the interest on this type of approach remains, essentially due to its simplicity and fairly ample applicability. While the Integral Method has been widely employed in the solution of steady-state external convection problems, and is well-documented even in various textbooks such as the above cited, much less information is readily available in its use within transient situations caused by temporal fluctuations of either wall or fluid conditions. Nevertheless, a number of fairly recent contributions in this direction have favored the use of this also called *Karman-Polhausen* approach in the approximate analysis of both impulsively and periodically heated walls (Lachi *et al.*,1998, Polidori, *et al.*,1998, Lachi *et al.*, 2004-b). The preferred solution path has therefore been the combined use of the Integral Method with a discrete approach for the numerical solution of the resulting partial differential equation for the wall temperature time evolution and longitudinal variation. This approach has also been validated against direct numerical solutions of the original partial differential problem and differential-similarity solutions of the transient boundary layers. On the other hand, symbolic computation platforms have been allowing for the revision and extension of a number of analytical procedures, either classical or more recently advanced, which to a certain extent have lost importance or were even almost abandoned from engineering practice, in light of the progress achieved along the last few decades by direct numerical analysis through traditional algorithm languages.

Besides the various possibilities open up through the symbolic derivation of previously just too tedious analytical approaches, a large number of hybrid developments have been observed, in different classes of engineering problems, merging automatic analytical derivations and modern numerical procedures with embedded error control (Cotta and Mikhailov, 1997). In this sense, the present work involves the construction of a *Mathematica* notebook (Wolfram, 1999) that deals with the approximate solution of the boundary layer equations for transient convective heat transfer of a Newtonian fluid flowing over a flat plate of non-negligible thickness, in laminar incompressible regime. The transients are caused by an arbitrary time variation of the applied uniform wall heat flux at the solid-fluid interface. After the reformulation of the wall heat conduction problem, as above discussed, the solution proceeds to the utilization of the Integral Method to obtain approximate polynomial approximations for the steady velocity and transient temperature fields in the fluid. From the integral form of the thermal boundary layer equation, a partial differential equation for the thermal boundary layer thickness is established, feasible of being numerically handled by the *Mathematica* system.

For this purpose, we employ the built in function **NDSolve**, which employs a Method of Lines approach to numerically solve the two coupled PDEs, for the thermal boundary layer thickness and for the average wall temperature. From this point, results for the interface temperature and heat fluxes are provided, in terms of the time variation of the thermal boundary layer thickness, for any arbitrary prescribed interface heat flux time variation. For illustration of the proposed symbolic-numerical approach, a typical application dealing with air heating (Lachi *et al.*, 2006) and with the wall conjugation was here considered more closely, for different wall materials and thicknesses, including different functional forms for the applied wall heat flux at the fluid-solid interface, which may also include an unheated starting length. This configuration is of particular interest in the so-called pulse method for the experimental determination of heat transfer coefficients in the transient regime (Petit *et al.*, 1981, Remy *et al.*, 1995, Rebay *et al.*, 2002), which provide the main motivation for the present effort of reaching an approximate hybrid numerical-analytical solution for the conjugated problem.

2. Problem Formulation

We consider laminar flow of a Newtonian fluid over a flat plate, with steady-state incompressible flow but transient convective heat transfer due to an arbitrarily varying delivered heat flux, $\phi(x,t)$, applied at the solid-fluid interface. This situation corresponds to the flash experiment for the determination of transient heat transfer coefficients in external convection (Petit *et al.*, 1981, Remy *et al.*, 1995, Rebay *et al.*, 2002). The fluid flows with a free stream velocity U_∞, which arrives at the plate front edge at the temperature T_∞, as described in Fig. (1).

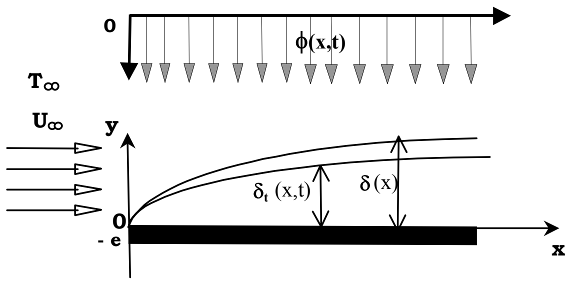

Figure 1. Description of the physical problem for transient conjugated external forced convection and wall heat conduction

The wall is considered to participate on the heat transfer problem, due to its thickness, *e*, length, *L*, and associated thermo-physical properties. The boundary layers equations are assumed to be valid for the flow and heat transfer problem within the fluid. The conjugated conduction-convection problem is mathematically described as:
Continuity:

$$\frac{\partial U(x,y)}{\partial x}+\frac{\partial V(x,y)}{\partial y}=0, \quad 0<y<\delta(x), 0<x<L \tag{1a}$$

Momentum in x-direction:

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\nu\frac{\partial^2 U}{\partial y^2} \quad , \quad 0<y<\delta(x), 0<x<L \tag{1b}$$

Momentum in y-direction:

$$\frac{\partial P(x,y)}{\partial y}=0 \tag{1c}$$

where U is the longitudinal velocity component, $m.s^{-1}$, V is the transversal velocity component, $m.s^{-1}$, ν is the kinematic viscosity, $m^2.s^{-1}$ and $\delta(x)$ is the velocity boundary layer thickness, *m.*

The flow problem solution is considered known at this point, by any of the known approximate analytical or numerical solution techniques, and the associated fluid and wall energy equations are given by:

$$\frac{\partial T_f(x,y,t)}{\partial t}+U\frac{\partial T_f(x,y,t)}{\partial x}+V\frac{\partial T_f(x,y,t)}{\partial y}=\alpha_f\frac{\partial^2 T_f(x,y,t)}{\partial y^2} \quad , \quad 0<y<\delta_t(x), 0<x<L, t>0 \tag{2a}$$

$$\frac{\partial T_s(x,y,t)}{\partial t} = \alpha_s\left(\frac{\partial^2 T_s(x,y,t)}{\partial x^2} + \frac{\partial^2 T_s(x,y,t)}{\partial y^2}\right) \quad , \quad -e < y < 0, 0 < x < L, t > 0 \tag{2b}$$

with initial conditions

$$T_f(x,y,0) = T_\infty, \quad 0 < y < \delta_t(x), 0 < x < L \tag{2c}$$

$$T_s(x,y,0) = T_\infty, \quad -e < y < 0, 0 < x < L \tag{2d}$$

and boundary and interface conditions:

$$T_f(x,\delta_t,t) = T_\infty \quad , \quad 0 < x < L, t > 0 \tag{2e}$$

$$T_f(x,0,t) = T_s(x,0,t) \quad , \quad 0 < x < L, t > 0 \tag{2f}$$

$$-k_f \left.\frac{\partial T_f}{\partial y}\right|_{y=0} = -k_s \left.\frac{\partial T_s}{\partial y}\right|_{y=0} + \phi(x,t) \quad , \quad 0 < x < L, t > 0 \tag{2g}$$

$$-k_s \left.\frac{\partial T_s}{\partial y}\right|_{y=-e} = 0 \quad , \quad 0 < x < L, t > 0 \tag{2h}$$

$$T_f(0,y,t) = T_\infty \quad , \quad 0 < y < \delta_t(x), t > 0 \tag{2i}$$

$$\left.\frac{\partial T_s}{\partial x}\right|_{x=0} = \left.\frac{\partial T_s}{\partial x}\right|_{x=L} = 0 \quad , \quad 0 < y < \delta_t(x), t > 0 \tag{2j}$$

where T_f is the fluid temperature, °*C*, T_s is the wall temperature, °*C*, $\delta t(x)$ is the thermal boundary layer thickness, *m*, α_f is the thermal diffusivity (fluid), $m^2.s^{-1}$, α_s is the thermal diffusivity (solid), $m^2.s^{-1}$, k_f is the thermal conductivity (fluid), $W.\ m^{-1}.K^{-1}$ and k_s is the thermal conductivity (solid), $W.\ m^{-1}.K^{-1}$.

3. Coupled Integral Equations Approach (C.I.E.A.)

The coupled integral equations approach (C.I.E.A.) is a very straightforward reformulation tool employed in the simplification of convection-diffusion problems via averaging processes in one or more of the involved space variables. In this sense, simpler formulations of the original partial differential systems are obtained, through a reduction of the number of independent variables in the multidimensional situations, by integration (averaging) of the full partial differential equations in one or more space variables, but retaining some information in the direction integrated out, provided by the related boundary conditions. Different levels of approximation in such mixed lumped-differential formulations can be used, starting from the plain and classical lumped system analysis, towards improved formulations, obtained through Hermite-type approximations for integrals (Cotta and Mikhailov, 1997). Such approach has been already exploited in different heat and fluid flow problems (Cotta and Mikhailov, 1997, Aparecido and Cotta, 1989, Correa and Cotta, 1998).

The Hermite formulae of approximating an integral, based on the values of the integrand and its derivatives at the integration limits, are given in the form (Cotta and Mikhailov, 1997):

$$\int_{x_{i-1}}^{x_i} y(x)\mathrm{d}x \cong \sum_{\nu=0}^{\alpha} C_\nu\, y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} D_\nu\, y_i^{(\nu)} \tag{3a}$$

where $y(x)$ and its derivatives $y^{(n)}(x)$ are defined for all $x \in (x_{i-1}, x_i)$. Furthermore, it is assumed that the numerical values of $y^{(\nu)}(x_{i-1}) \equiv y_{i-1}^{(\nu)}$ for $\nu = 0,1,2,...,\alpha$ and $y^{(\nu)}(x_i) \equiv y_i^{(\nu)}$ for $\nu=0,1,2,...,\beta$, are available at the end points of the interval.

In such a manner, the integral of $y(x)$ is expressed as a linear combination of $y\left(x_{i-1}\right)$, $y\left(x_i\right)$ and their derivatives, $y^{(\nu)}\left(x_{i-1}\right)$ up to order $\nu=\alpha$, and $y^{(\nu)}\left(x_i\right)$ up to order $\nu=\beta$. This is called the $H_{\alpha,\beta}$ approximation. The resulting expression for the $H_{\alpha,\beta}$ - approximation is given by:

$$\int_{x_{i-1}}^{x_i} y(x)\mathrm{d}x = \sum_{\nu=0}^{\alpha} C_\nu(\alpha,\beta) h_i^{\nu+1} y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} C_\nu(\beta,\alpha)(-1)^\nu h_i^{\nu+1} y_i^{(\nu)} + O\left(h_i^{\alpha+\beta+3}\right) \tag{3b}$$

where,

$$h_i = x_i - x_{i-1} \quad ; \quad C_\nu\left(\alpha,\beta\right) = \frac{\left(\alpha+1\right)!\,\left(\alpha+\beta+1-\nu\right)!}{\left(\nu+1\right)!\,\left(\alpha-\nu\right)!\,\left(\alpha+\beta+2\right)!} \tag{3c,d}$$

In the present work, we consider just the two approximations, $H_{0,0}$ and $H_{1,1}$, given by:

$$H_{0,0} \rightarrow \int_0^h y(x)\mathrm{d}x \cong \frac{h}{2}\left(y(0)+y(h)\right) \tag{4a}$$

$$H_{1,1} \rightarrow \int_0^h y(x)\mathrm{d}x \cong \frac{h}{2}\left(y(0)+y(h)\right) + \frac{h^2}{12}\left(y'(0)-y'(h)\right) \tag{4b}$$

which correspond, respectively, to the well-known trapezoidal and corrected trapezoidal integration rules.

The CIEA is here employed in approximating the formulation for the conduction problem within the solid. According to this approach (Cotta and Mikhailov, 1997), the average transversal temperature is to be approximated by a Hermite formula for integrals, here by taking the $H_{1,1}$ approximation, the well known corrected trapezoidal rule. In addition, the average transversal wall heat flux shall be approximated by the simplest $H_{0,0}$ approximation, the trapezoidal rule. This $H_{1,1}$ / $H_{0,0}$ combined solution does not change the nature of the classical lumped formulation, but only slightly modifies the equation coefficients, and is expected to be more accurate than the classical lumped system analysis in the applicable range of the governing parameters.

The transversally averaged wall temperature, T_{av}, is thus approximated as:

$$T_{av}(x,t) \equiv \frac{1}{e}\int_{-e}^{0} T_s(x,y,t)dy \approx \frac{1}{2}\left[T_s(x,0,t)+T_s(x,-e,t)\right] - \frac{e}{12}\left.\frac{\partial T_s}{\partial y}\right|_{y=0} \tag{5}$$

The average heat flux is approximated as:

$$\int_{-e}^{0} \frac{\partial T_s(x,y,t)}{\partial y}dy \equiv \left[T_s(x,0,t)-T_s(x,-e,t)\right] \approx \frac{e}{2}\left.\frac{\partial T_s}{\partial y}\right|_{y=0} \tag{6}$$

An expression for the temperature at $y=-e$, to be eliminated, is thus obtained:

$$T_s(x,-e,t) = 2T_{av}(x,t) - T_s(x,0,t) + \frac{e}{6}\left.\frac{\partial T_s}{\partial y}\right|_{y=0} \tag{7a}$$

This expression is substituted into the average heat flux expression, eq.(6), providing:

$$\left[T_s(x,0,t) - \left(2T_{av}(x,t) - T_s(x,0,t) + \frac{e}{6}\left.\frac{\partial T_s}{\partial y}\right|_{y=0}\right)\right] = \frac{e}{2}\left.\frac{\partial T_s}{\partial y}\right|_{y=0} \tag{7b}$$

Then, the interface condition, eq.(2f), is recalled, yielding:

$$\left.\frac{\partial T_s}{\partial y}\right|_{y=0} = \frac{3}{e}\left[T_f(x,0,t) - T_{av}(x,t)\right] \tag{7c}$$

and the interface condition, eq.(2g), is now reformulated as:

$$-k_f \left.\frac{\partial T_f}{\partial y}\right|_{y=0} = \phi(x,t) - \frac{3k_s}{e}\left[T_f(x,0,t) - T_{av}(x,t)\right] \tag{8}$$

Clearly, according to the expression above, the boundary condition for the fluid at y=0 was reformulated as a third kind boundary condition that includes the participation of the wall through its averaged temperature. When the interface temperature, $T_f(x,0,t)$, and the average solid temperature, $T_{av}(x,t)$, have the same value, the wall does not participate and the conventional second kind boundary condition for an imposed heat flux is recovered.

The energy equation for the solid is now reformulated by taking the average on the transversal direction, operating with $\frac{1}{e}\int_{-e}^{0} \_\_\_ dy$, to yield:

$$\begin{aligned}\frac{\partial T_{av}(x,t)}{\partial t} &= \alpha_s \frac{\partial^2 T_{av}(x,t)}{\partial x^2} + \frac{\alpha_s}{e}\int_{-e}^{0} \frac{\partial^2 T_s(x,y,t)}{\partial y^2} dy \\ &= \alpha_s \frac{\partial^2 T_{av}(x,t)}{\partial x^2} + \frac{\alpha_s}{e}\left[\left.\frac{\partial T_s(x,y,t)}{\partial y}\right|_{y=0} - \left.\frac{\partial T_s(x,y,t)}{\partial y}\right|_{y=-e}\right]\end{aligned} \tag{9}$$

We can then eliminate the derivative at y =0 and at y =-e by applying the interface conditions, (2g, 2h):

$$\frac{\partial T_{av}(x,t)}{\partial t} = \alpha_s \frac{\partial^2 T_{av}(x,t)}{\partial x^2} + \frac{\alpha_s}{ek_s}\left[k_f \left.\frac{\partial T_f(x,y,t)}{\partial y}\right|_{y=0} + \phi(x,t)\right] \tag{10}$$

or, by recalling the reformulated fluid boundary condition:

$$\frac{\partial T_{av}(x,t)}{\partial t} = \alpha_s \frac{\partial^2 T_{av}(x,t)}{\partial x^2} - \frac{3\alpha_s}{e^2}\left[T_{av}(x,t) - T_f(x,0,t)\right] \tag{11a}$$

This equation is followed by the also averaged initial and boundary conditions as:

$$T_{av}(x,0) = T_\infty \tag{11b}$$

$$\left.\frac{\partial T_{av}(x,t)}{\partial x}\right|_{x=0} = 0 \quad ; \quad \left.\frac{\partial T_{av}(x,t)}{\partial x}\right|_{x=L} = 0 \tag{11c,d}$$

Again, the difference between the average solid temperature and the fluid interface temperature is responsible by the coupling of the two processes along the longitudinal coordinate "x". The problem may now solved in the transversal coordinate for the fluid, by employing the conventional integral method, for instance, and thus reducing the problem to a pair of coupled partial differential equations for $T_f(x,0,t)$ and $T_{av}(x,t)$.

Higher order formulations could be achieved but then the nature of the formulation would somehow change. For instance, by introducing the $H_{1,1}$ approximation also for the average heat flux, the formulation would then incorporate a partial differential equation for the temperature at $y = -e$, which is not entirely eliminated, coupled to the average wall and interface temperatures. At the present contribution we have preferred to obtain a simpler formulation for the conjugated problem as above described.

4. Solution Methodology

The heat transfer problem within the fluid can be represented by the time-dependent semi-integral form of the energy equation, written as:

$$\frac{\partial}{\partial t}\int_0^{\delta_t} T_f\, dy + \frac{\partial}{\partial x}\int_0^{\delta_t} U(T_f - T_\infty)\, dy = -\alpha_f \left.\frac{\partial T_f}{\partial y}\right|_{y=0} \tag{12}$$

The solution methodology applied to eq.(12) is here illustrated by the 3th-order polynomial Karman-Pohlhausen approach for the velocity profile and the 2th-order polynomial for the temperature field. Higher order polynomials for the temperature field were here avoided not to introduce the time derivative of the interface temperature in the coefficients determination, which would require one more coupling differential equation. The present orders for the velocity and temperature fields polynomial approximations were selected from the accuracy analysis of the different combinations in the steady-state situation. Thus, the velocity profile is modeled by:

$$U(x,y) = U_\infty \left[\frac{3}{2}\frac{y}{\delta(x)} - \frac{1}{2}\left(\frac{y}{\delta(x)}\right)^3 \right] \tag{13}$$

With the related boundary conditions given by Eqs. (2e, 2f, and 8), the temperature profile results in:

$$T_f(x,y,t) = T_\infty + \frac{\left(y-\delta_t(x,t)\right)^2 \left(e\,\phi(x,t) + 3k_s (T_{av}(x,t) - T_\infty)\right)}{\delta_t(x,t)\left(2e\,k_f + 3k_s\,\delta_t(x,t)\right)} \tag{14a}$$

and for the interface temperature

$$T_f(x,0,t) = T_\infty + \frac{\delta_t(x,t)\left(e\,\phi(x,t) + 3k_s (T_{av}(x,t) - T_\infty)\right)}{\left(2e\,k_f + 3k_s\,\delta_t(x,t)\right)} \tag{14b}$$

Substitution of the polynomial approximations, eqs.(13, 14a), into the integral form of the boundary layer equation, eq.(12), yields the partial differential equation for the thermal boundary layer thickness, $\delta_t(x,t)$, as a function of the longitudinal coordinate and the time variable, coupled to the average wall temperature. All of the steps in the derivation of the solution methodology were accomplished by making use of symbolic computation, as made possible by the *Mathematica* system (Wolfram, 1999). As an illustration of the symbolic computation procedure, we reproduce below the obtained partial differential equation that governs the thermal boundary layer thickness, with the time derivative provided in explicit form:

$$\begin{aligned}
\delta t^{(0,1)}[x,t] == \Bigg(3\,(2\,e\,k_f + 3\,k_s\,\delta t[x,t])^2 \Bigg(& \frac{6\,\alpha_f\,(e\,\phi[x,t] + 3\,k_s\,(-T_\infty + T_{av}[x,t]))}{6\,e\,k_f + 9\,k_s\,\delta t[x,t]} - \\
& \frac{\delta t[x,t]^2\,(e\,\phi^{(0,1)}[x,t] + 3\,k_s\,T_{av}{}^{(0,1)}[x,t])}{6\,e\,k_f + 9\,k_s\,\delta t[x,t]} - \\
& (u_\infty\,\delta t[x,t]^2\,(3\,\delta t[x,t]\,(2\,e\,k_f + 3\,k_s\,\delta t[x,t])\,(-15\,\delta[x]^2 + \delta t[x,t]^2) \\
& \quad (e\,\phi[x,t] + 3\,k_s\,(-T_\infty + T_{av}[x,t]))\,\delta'[x] - 3\,k_s\,\delta[x]\,\delta t[x,t] \\
& \quad (-15\,\delta[x]^2 + \delta t[x,t]^2)\,(-e\,\phi[x,t] + 3\,k_s\,(T_\infty - T_{av}[x,t]))\,\delta t^{(1,0)}[x,t] + \\
& \quad 3\,\delta[x]\,(2\,e\,k_f + 3\,k_s\,\delta t[x,t])\,(-15\,\delta[x]^2 + \delta t[x,t]^2) \\
& \quad (-e\,\phi[x,t] + 3\,k_s\,(T_\infty - T_{av}[x,t]))\,\delta t^{(1,0)}[x,t] + \\
& \quad \delta[x]\,\delta t[x,t]\,(2\,e\,k_f + 3\,k_s\,\delta t[x,t])\,(-e\,\phi[x,t] + 3\,k_s\,(T_\infty - T_{av}[x,t])) \\
& \quad (-30\,\delta[x]\,\delta'[x] + 2\,\delta t[x,t]\,\delta t^{(1,0)}[x,t]) + \\
& \quad \delta[x]\,\delta t[x,t]\,(2\,e\,k_f + 3\,k_s\,\delta t[x,t])\,(15\,\delta[x]^2 - \delta t[x,t]^2) \\
& \quad (e\,\phi^{(1,0)}[x,t] + 3\,k_s\,T_{av}{}^{(1,0)}[x,t]))) / \\
& (120\,\delta[x]^4\,(2\,e\,k_f + 3\,k_s\,\delta t[x,t])^2)\Bigg)\Bigg) \Big/ (\delta t[x,t] \\
& (4\,e\,k_f + 3\,k_s\,\delta t[x,t])\,(e\,\phi[x,t] + 3\,k_s\,(-T_\infty + T_{av}[x,t])))
\end{aligned}$$

The energy equation for the transversally averaged solid temperature is then symbolically prepared, together with the corresponding boundary and initial conditions:

$$\frac{T_{av}{}^{(0,1)}[x,t]}{\alpha_s} == -\frac{3\left(-T_\infty - \frac{\delta t[x,t]\,(e\,\phi[x,t] - 3\,k_s\,(T_\infty - T_{av}[x,t]))}{2\,e\,k_f + 3\,k_s\,\delta t[x,t]} + T_{av}[x,t]\right)}{e^2} + T_{av}{}^{(2,0)}[x,t]$$

Non-zero values for the boundary conditions might be required, due to the singularity at the plate edge or at the starting length. Therefore, once the above coupled equations for the thermal boundary layer thickness and average solid temperature are numerically solved, we can readily compute the wall and fluid temperatures. Those equations were solved by making use of the routine **NDSolve** of the *Mathematica* system, and interpolated expressions for the thermal boundary layer thickness and for the average solid temperature are then automatically offered by the routine, allowing for their computation at any requested position *x* and time *t*.

5. Results and Discussion

Following the symbolic derivation through the *Mathematica* system (Wolfram, 1999), numerical computations are performed within this integrated platform. The presently constructed notebook is an extension of a previous one for transient external convection without the wall conjugation (Lachi *et al.*, 2006), where a thorough validation and numerical analysis of the **NDSolve** solution of the thermal boundary layer thickness is provided. Here we shall concentrate on the study of the wall conjugation effects and the errors involved in the approximate formulation that has been proposed. We have considered the analysis of four specific cases with wall participation for two different materials (Norcoat and PVC), provided by the pertinent data in Table 1, having air at ambient temperature as the cooling fluid and with a step change in time on the uniform heat flux applied at the interface. The common parameters in all five cases are given by T_∞=20 C, U_∞=1 m/s, L=0.1 m, ϕ=100 W/m^2.

Table 1 – Selected test cases and governing parameters
(Air, T_∞=20 C, U_∞=1 m/s, L=0.1 m, ϕ=100 W/m^2)

CASE	Material	e (m)	k (W/m C)	α (m/s^2)
1	Norcoat	0.007	0.12	1.67 10^{-4}
2	Norcoat	0.002	0.12	1.67 10^{-4}
3	Norcoat	0.012	0.12	1.67 10^{-4}
4	PVC	0.012	0.15	0.11 10^{-6}

For the validation of the CIEA reformulation, we have also implemented the two-dimensional solution of the wall heat conduction problem by employing the same routine **NDSolve**, and using the interface temperature obtained by the conjugated problem solution as the boundary condition. Then, the computation of the average temperature distribution from the two-dimensional solution provides a direct comparison with the same quantity in the one-dimensional lumped-differential formulation. Table 2 presents an illustration of the deviations between the two values of the transversally averaged wall temperature for the case of a Norcoat wall (case 1) with a 7 mm thickness. For different longitudinal positions and time values, the two dimensional and the improved lumped-differential formulations agree to at least three significant digits, fairly uniformly within the entire domain.

Table 2 – Comparison of average wall temperatures, $T_{av}(x,t)$, for
lumped-differential (1D) and two-dimensional (2D) formulations (case 1).

t [s]	**0.25**		**0.5**		**0.75**		**1.0**	
x [m]	**Tav 1D**	**Tav 2D**	**Tav 1D**	**Tav 2D**	**Tav 1D**	**Tav 2D**	**Tav 1D**	**Tav 2D**
0.01	22.159	22.172	23.122	23.144	23.563	23.575	23.765	23.770
0.02	22.549	22.556	23.878	23.903	24.536	24.553	24.855	24.864
0.04	22.860	22.856	24.752	24.776	25.882	25.903	26.513	26.527
0.05	22.925	22.918	24.985	25.007	26.312	26.332	27.105	27.119
0.06	22.967	22.958	25.148	25.168	26.634	26.654	27.578	27.594
0.08	23.018	23.007	25.357	25.375	27.076	27.095	28.274	28.291
0.1	23.045	23.029	25.469	25.478	27.323	27.329	28.684	28.684

Similar comparisons were constructed for the other two cases of the norcoat wall, and as an example we compare the three cases in Table 3 for the average temperatures at the time t=1.0 s. All three situations provided demonstrate an excellent agreement between the two formulations, but it can be noticed a slight loss of accuracy of the lumped-differential formulation when the thickness is increased to 12 mm. As for the classical lumped system analysis, the present lumped-differential formulation is expected to present a loss of precision as the temperature gradients in the transversal direction increase, but in fact the CIEA formulation is not as much influenced by the non-uniform distributions as is the classical analysis, since the temperature spatial variations are somehow accounted for by the Hermite formulae for integrals.

Table 3 – Effect of wall thickness on the average wall temperatures, $T_{av}(x,t)$, for lumped-differential (1D) and two-dimensional (2D) formulations (cases 1, 2 and 3), t=1.0 s.

CASE	2 (e=2mm)		1 (e=7mm)		3 (e=12mm)	
x [m]	Tav 1D	Tav 2D	Tav 1D	Tav 2D	Tav 1D	Tav 2D
0.01	23.796	23.795	23.765	23.770	23.081	23.136
0.02	25.166	25.169	24.855	24.864	23.802	23.851
0.04	27.288	27.291	26.513	26.527	24.842	24.895
0.05	28.154	28.157	27.105	27.119	25.185	25.237
0.06	28.935	28.938	27.578	27.594	25.449	25.501
0.08	30.338	30.341	28.274	28.291	25.827	25.875
0.1	31.478	31.460	28.684	28.684	26.041	26.071

Figure 2 then illustrates the variation of the interface (solid) and the wall average (dashed) temperatures along the plate length, for five different time values (t=0.25, 0.5, 0.75, 1.0 and 1.25 s), from bottom to top. It can be noticed that the high heat transfer coefficients near the plate leading edge can bring the interface temperatures to lower values than the averaged value. Also, as time progresses, the differences between the local and the average values tend to diminish, as the steady-state solution is approached, further favoring the present approximation.

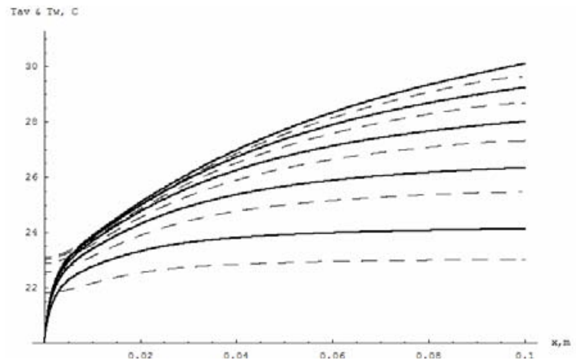

Figure 2 – Average wall (dashed) and interface (solid) temperatures distributions for different times (t=0.25, 0.5, 0.75, 1.0 and 1.25 s, from bottom to top), for the Norcoat wall (case 1)

Figure 3 provides the partition of the interface heat fluxes along the plate length, between the wall and the fluid. Four different time values t=0.25, 0.5, 0.75, 1.0 s are considered, and their respective results are represented within the graph in increasing order with the dash length. Along the plate length, the heat flux to the fluid decreases, following the heat transfer coefficient decrease, while the heat flux to the solid increases for the present uniform energy delivery to the interface. Along time, as the wall temperature transversal gradients tend to be smoother, as observed from Fig.(2), the wall heat fluxes at the interface decrease, while the heat fluxes to the fluid increase. In the early transient, and for larger values of the longitudinal coordinate, the energy partition can even favor the solid, as the heat flux to the wall crosses over the fluid curve.

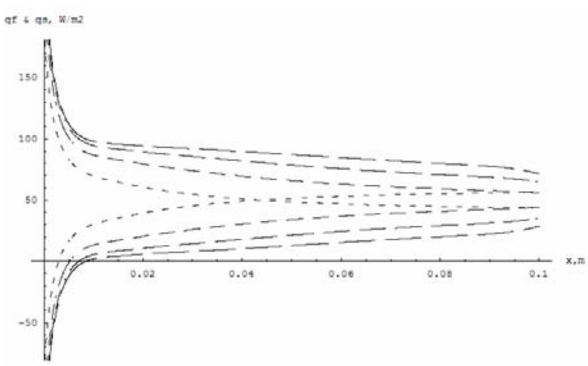

Figure 3 – Heat fluxes distributions at the interface for solid (lower curves) and fluid (upper curves) for different time values (t=0.25, 0.5, 0.75 and 1.0 sec), in order of dash length, for the Norcoat wall (case 1).

All the above results were obtained solely from the one-dimensional lumped-differential formulation for the wall, which can only provide information on the average and boundary quantities, temperatures and heat fluxes. However, if the knowledge of the local solid temperature is for any reason essential, the two-dimensional analysis can be approximately accomplished as above suggested for inspection of the average temperature behavior, employing the interface temperature as a boundary condition for the pure heat conduction problem. This allows one to also investigate the local behavior of the wall temperature, as illustrated in Fig. (4). For a fixed position x=0.05m, we show the transversal profiles of both solid and fluid temperatures for different time values (t=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 s). Clearly, one can observe the reduction with time of the wall temperature gradients at the interface, due to the progressively smoother temperature profiles across the solid, while the increase of the heat fluxes to the fluid are also quite evident from the steeper temperature curves all the way to the steady-state.

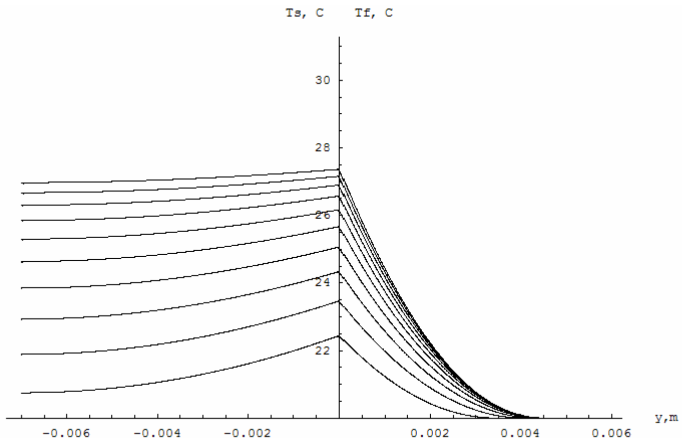

Figure 4 – Local wall (left) and fluid (right) temperatures distributions for different time values (t=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 s), at x=0.05 m, for the Norcoat wall (case 1).

The effect of varying the wall thickness is again analyzed through Fig. (5) and Fig.(6) for the transversal wall and fluid temperature distributions, in comparison with the above mentioned Fig. (4). Figures 5 and 6 are respectively for the transversal temperature profiles with a Norcoat wall of thickness e=2mm (case 2) and 12 mm (case 3), again at a fixed position x=0.05m and for the same time values above. It can be noticed that the steady-state is reached sooner for the thinner wall, figure 5, than for the other two situations. In addition, the wall temperature gradients are less pronounced in this case (Fig. (5)) and the interface temperature reaches a larger value. The thicker wall offers a larger thermal resistance and results in more pronounced temperature gradients within the solid and the increased thermal capacitance leads to a slower evolution to the steady-state solution.

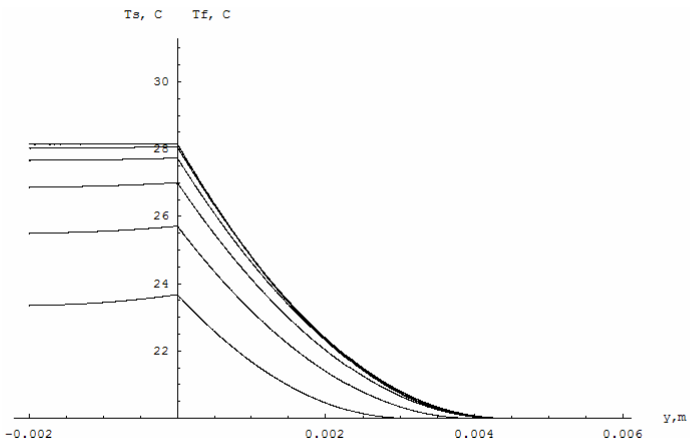

Figure 5 – Local wall (left) and fluid (right) temperatures distributions for different time values (t=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 s), at x=0.05 m, for the Norcoat wall (case 2).

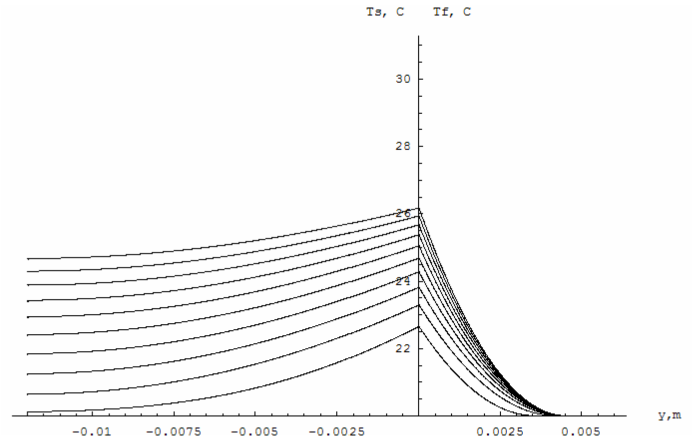

Figure 6 – Local wall (left) and fluid (right) temperatures distributions for different time values (t=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 s), at x=0.05 m, for the Norcoat wall (case 3).

The next figure (Fig 7), illustrates the transversal temperature profiles for a material with a considerably smaller thermal diffusivity, as represented by the PVC wall (case 4), in comparison with the above mentioned Norcoat wall (case 3), both with a 12 mm thickness. Clearly, observing the quite different temperature and time scales employed for this visualization, the temperature distributions are smoother in the Norcoat plate case, and in addition the transients are faster, as expected.

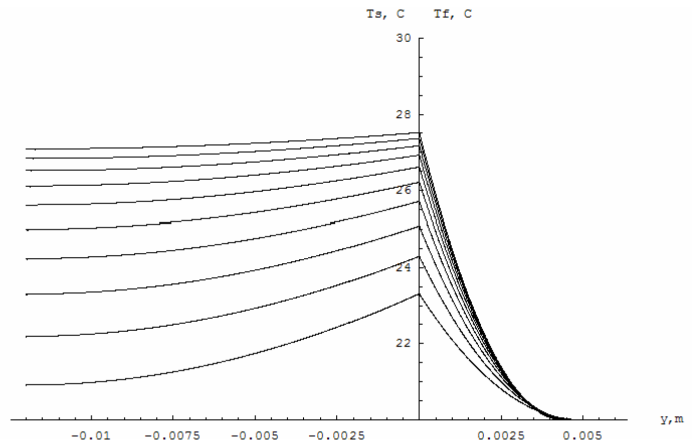

Figure 7– Local wall (left) and fluid (right) temperatures distributions for different time values (t=360. to 3600.s, intervals of 360 s), at x=0.05 m, for the PVC wall (case 4).

6. Conclusions

The problem of transient conjugated conduction-external convection over a flat plate of finite thickness is approximately solved, first by providing an improved lumped-differential formulation for the wall heat conduction problem, thus eliminating the transversal coordinate, and then approximating the fluid temperature by a polynomial according to the classical integral method for boundary layers. Symbolic computation is employed throughout the development of the solution, thus eliminating the cumbersome analysis that in general associated with analytic-type approaches. The resulting coupled partial differential equations for the thermal boundary layer thickness and transversally averaged wall temperature are then numerically solved along the longitudinal coordinate and the time variable, by making use of the routine **NDSolve** of the *Mathematica* system. The approach is first validated against the two-dimensional wall formulation and a few different physical situations are examined, for different materials and slab thicknesses.

The accuracy level achieved by the improved lumped-differential formulation for this class of problems, encourages the use of more accurate solution methodologies for the fluid, as well as the extension of this analysis to more involved situations, including different geometric configurations and boundary conditions variations with space and time.

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