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GENERALIZED FIBONACCI SEQUENCES AND FINS

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Abstract. In this work we studied the application of generalized Fibonacci sequences to resistances networks. We obtain the equivalent resistance for a polygon of non-identical resistors and for a plane fin, seen as a ladder network with an infinite number of differential elements, for different contour conditions.

***Keywords**. Fibonacci, resistance, fin.*

1. Fibonacci and Generalized Fibonacci Sequences

The study of aperiodically ordered structures has inspired a large amount of theoretical and experimental work, concerning natural and hand-made materials, like biological growth patterns, scale-rotational crystal growth (Boeyens, 2003), non-linear excitations in DNA (Cueda and Sanchez, 2004), defects in conducting polymers (Malhotra, 1988; Adame, Sanchez and Kiushar, 1995) and Josephson junction arrays (Lennholm and Hornquist, 1999). The aim is to simulate the behavior of these systems that relate non-linearity and disorder using aperiodic chains generated by specific inflation rules.

For example, the Fibonacci chain is generated from two basic units *A* and *B* using the rules: *A*→*AB* and *B*→*A*. Thus, beginning with a single unit chain, one generates after *n* successive applications of these rules a self-similar N unit aperiodic chain. Moreover, this chain is related to the Fibonacci sequence, defined by the recurrence relation:

$$\mathrm{F}_{n+2} = \mathrm{F}_{\mathrm{n}} + \mathrm{F}_{\mathrm{n+1}}$$

where $\mathrm{F}_0 = 0$ and $\mathrm{F}_1 = 1$, in such a way that it gives the number of units of the corresponding chain $(\mathrm{N} = \mathrm{F}_{\mathrm{n}})$. On the other hand, as the sequence proceeds, the ratio between two consecutive terms approaches the golden ratio (Dunlap, 1997; Huntley, 1970),

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618..$$

closely related to the self-similarity property in fractal structures (Boeyens, 2003; Dixon, 2002; Janner, 2001; Mandelbrot, 1988).

The Fibonacci sequence can be generalized in many ways (Paladino and Ferreira, 2000; Mouline and Rachidi 1995; Rachidi, Saidi and Zerouaoui, 2003). The simplest way is by introducing a coefficient in one of the terms of the equation (Paladino and Ferreira, 2000):

$$\mathrm{G}_{n+2} = \mathrm{G}_{\mathrm{n}} + \alpha \cdot \mathrm{G}_{\mathrm{n+1}}$$

in such a way that the general term of the sequence is given by the expression

$$\mathrm{G}_{\mathrm{n}} = \frac{1}{\sqrt{4+\alpha^2}} \left[\left(\frac{\alpha + \sqrt{4+\alpha^2}}{2} \right)^n - \left(\frac{\alpha - \sqrt{4+\alpha^2}}{2} \right)^n \right]$$

and the ratio between two consecutive terms approaches the value

$$\lim_{n\to\infty}\frac{G_{n+1}}{G_n}=\phi(\alpha)=\left(\frac{\alpha+\sqrt{4+\alpha^2}}{2}\right)$$

2. Resistance Networks and Fibonacci Sequences

Electric resistor networks are self-similar systems which behavior is related to the Fibonacci sequences (Paladino and Ferreira, 2000; March, 1993; Srinivasan, 1992; Basin, 1963). As an example be the ladder network of identical resistors (R=R') in Fig (1). The equivalent resistance R_n between the axis *X* and *Y* is given by Srinivasan (1992) and Basin (1963):

$$R_n=\frac{F_{2n}}{F_{2n-1}}R$$

in such a way that the equivalent resistance for the infinite network is $\phi\cdot R\approx 1.618\cdot R$.

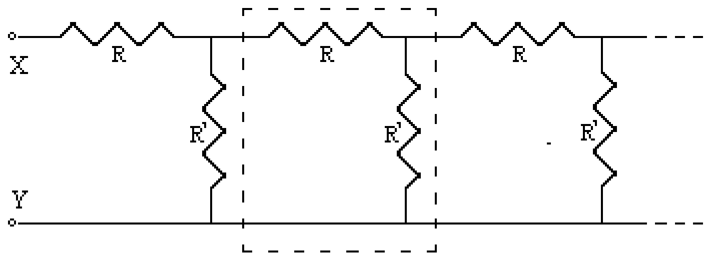

Figure 1. Ladder network of resistors.

Polygons of resistors like the one in Fig. (2) have a similar convergence when the number *n* of sides becomes large. For a small number of sides, the equivalent resistance can be obtained by considering the symmetry of the figure and it is possible to identify recursion relations that relate these systems with Fibonacci sequences. For instance, the equivalent resistance of an n-sided polygon of identical resistors (R'=R) is given by (March, 1993):

$$R_n=\frac{F_{n-1}+F_{n+1}}{2(F_{n-1}+F_{n+1})+F_{n-2}+F_{n-4}}R \qquad n \text{ is even}$$

$$R_n=\frac{F_n}{F_{n-1}+F_{n+1}}R \qquad n \text{ is odd}$$

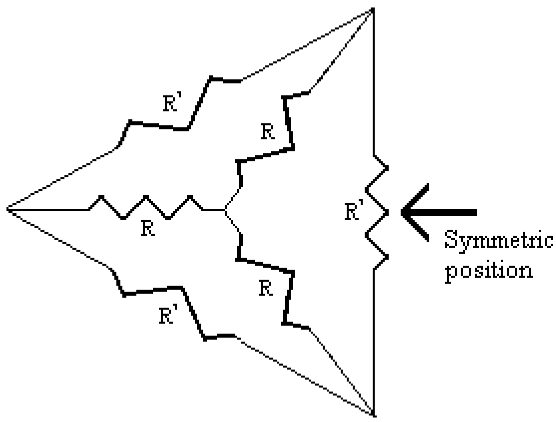

Figure 2. Resistors polygon.

Now, the generalized Fibonacci sequence is related to the description of networks with non-identical resistors. Paladino and Ferreira (2000) obtained the formula for the ladder network described in Fig (1), with $R=\alpha\cdot r$ and $R'=r/\alpha$:

$$R_n = \frac{G_n}{G_{n-1}} r$$

Our first objective of this work is to calculate the equivalent resistance in an n-side polygon with non-identical resistors (see Fig. (2), for $R' = \alpha^2 \cdot R$). This network has different symmetries for even and odd number of sides through at the limit $n \to \infty$ the result must be the same for both.

Polygons with an odd number of sides have a symmetry that allows the elimination of a resistor (in Fig (2), indicated by the arrow). Fig. (3) shows some networks after the elimination of the resistors and it indicates the self-similarity structure that relates the (2n+3)-sided polygon with the (2n+1)-sided polygon.

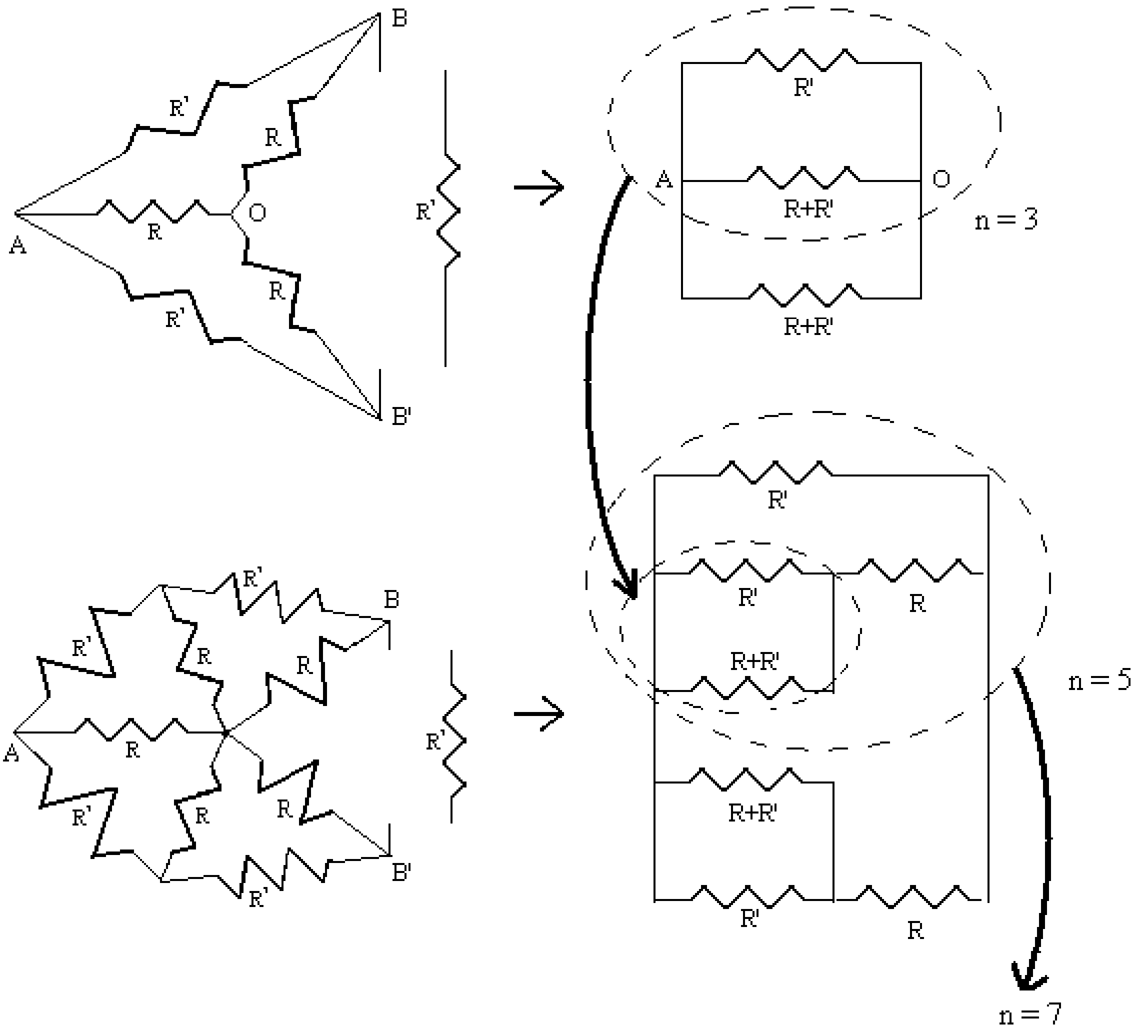

Figure 3. Similarity in polygons with an odd number of sides.

$$\frac{1}{R'_{2n+3}} = \frac{1}{R'_{2n+1} + R'} + \frac{1}{R} \tag{1}$$

where

$$\frac{1}{R'_n} = \frac{1}{R_n} + \frac{1}{R}$$

This relation allows us to obtain the equivalent resistance:

$$R_n = \frac{G_n}{G_{n-1} + G_{n+1}} r \qquad n \text{ is odd}$$

where $R = r/\alpha$ and $R' = \alpha \cdot r$. In case of a polygon with an even number of sides, it is necessary to replace the 2n-sided polygon by an equivalent (2n+1)-sided polygon and then identify the self-similarity structure that relates the networks. It is shown in Fig. (4). Equation (1) holds, so we obtain the formula for the equivalent resistance:

$$R_n = \frac{\alpha}{4+\alpha^2} \frac{G_{n+1} - G_{n-1}}{G_{n+1} + G_{n-1}} r \qquad n \text{ is even}$$

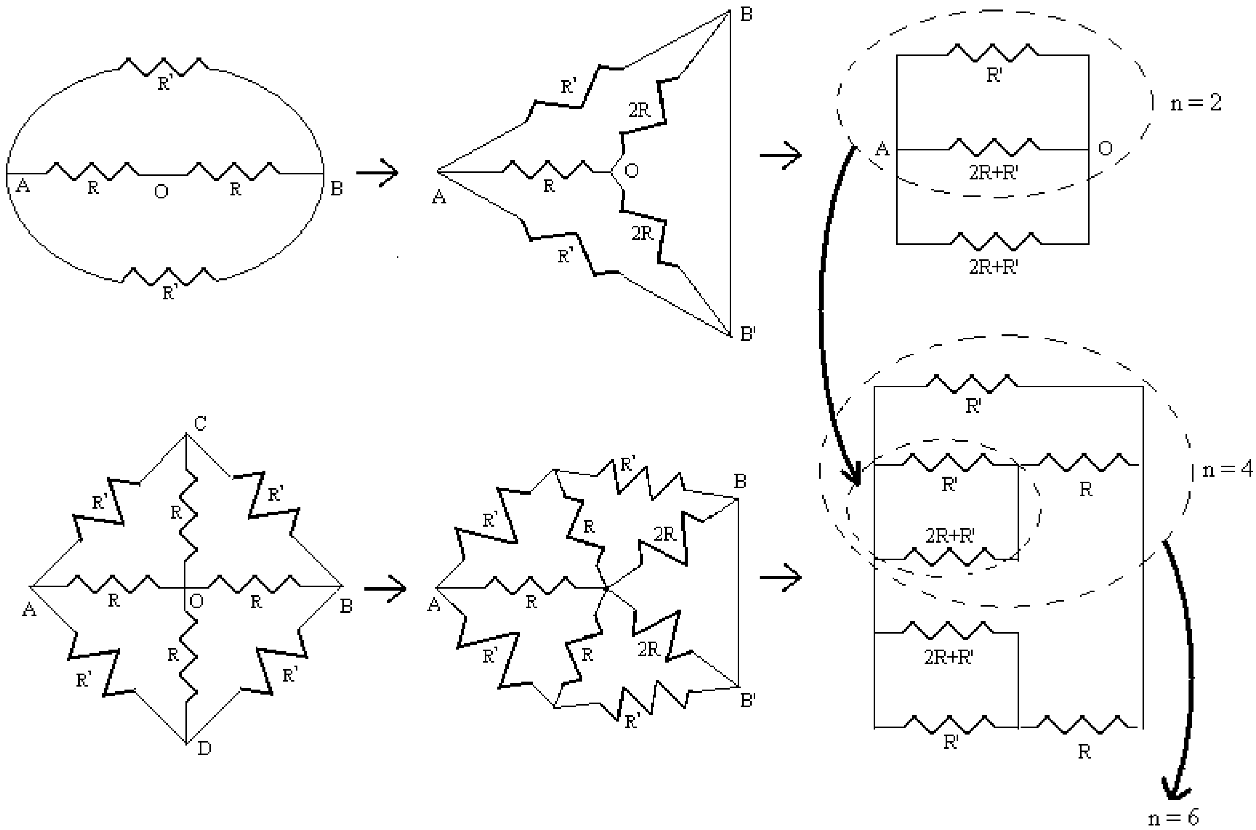

Figure 4. Similarity in polygons with an even number of sides.

3. Plane Fins and Generalized Fibonacci Sequences

Resistance networks may describe other energy transfer processes, such as thermal and optical systems. For instance, a plane fin defined by its length *L* and its transverse section with area *A* and perimeter *P* (see Fig. (5)) transfers heat according to the equation:

$$q = \frac{T_{base} - T_{\infty}}{R_{fin}}$$

where T_{base} is the temperature at the basis of the fin, T_{∞} is the free fluid temperature and R_{fin} is the equivalent thermal resistance of the fin. Although the process is two-dimensional, the high thermal conductivity *k* of the (metallic) fin and the low convection coefficient *h* of the surface allows the one-dimensional analysis described in Fig. (5) (Incropera and DeWitt, 2002). The thermal resistances $R = \Delta x(kA)^{-1}$, $R' = (hP\Delta x)^{-1}$ and $R'' = (hA)^{-1}$ represent respectively the conduction process along a differential section Δx, the convection process around this differential section and the convection process at the end of the fin.

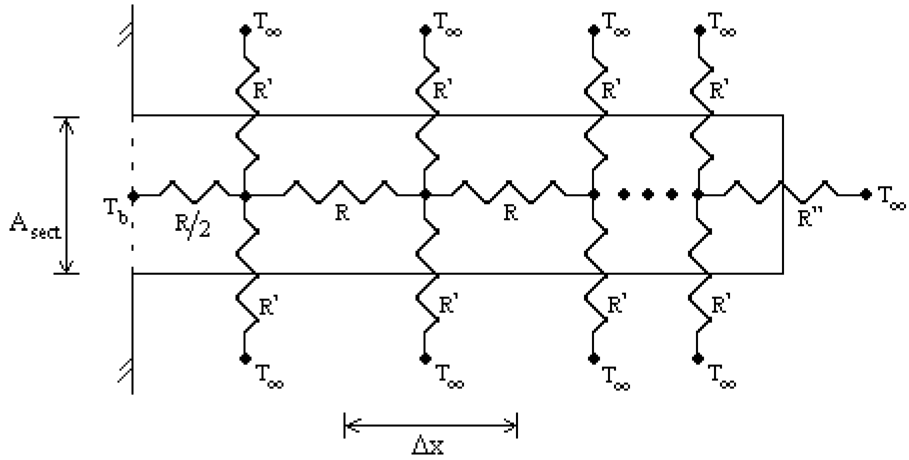

Figure 5. A plane fin, seen as a ladder network.

This system is a ladder network with an infinite number of differential elements. Here, we introduce three adimensional parameters: $\alpha = \Delta x\sqrt{hP/kA}$, $\beta = \sqrt{hA/kP}$ and $r = 1/\sqrt{hPkA}$, in such a way that R = αr, R'=r/α and R"=r/β.

In the case the end of the fin is isolated, R" = 0, and the network can be represented by a generalized Fibonacci sequence, with $G_0 = 0$ and $G_1 = 1$. Then, for a fin $L_n = n\Delta x$ long,

$$\frac{R_n}{r} = \frac{G_{2n+1}}{G_{2n}} - \frac{\alpha}{2}$$

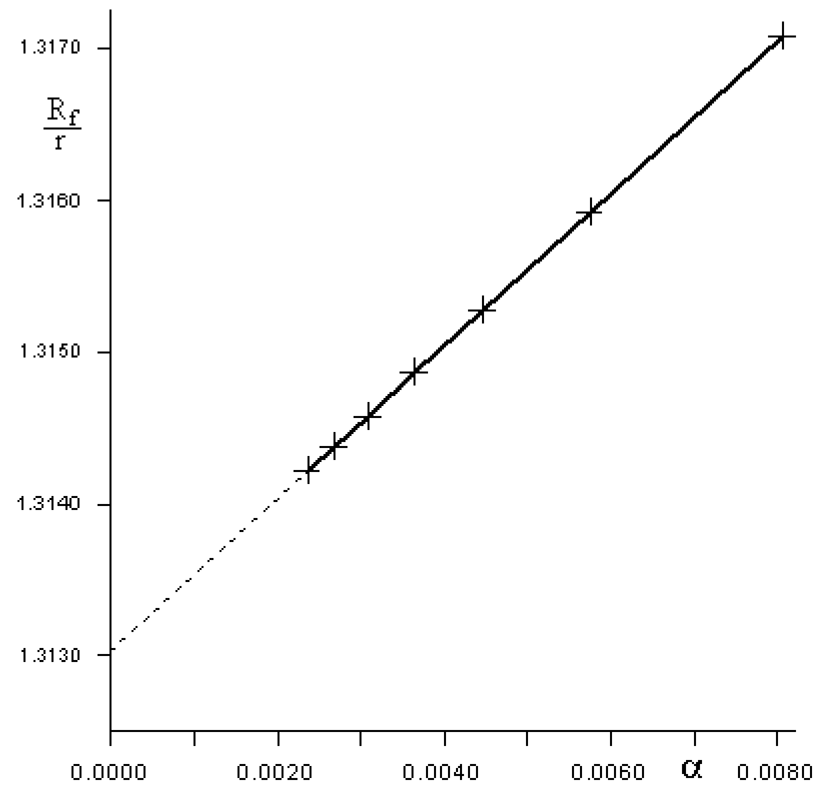

Figure 6. The generalized golden rule ϕ.

and we obtain the general term of the sequence:

$$\frac{G_{2n+1}}{G_{2n}} = \lim_{n\to\infty} \frac{\left(\alpha_o/n + \sqrt{4+(\alpha_o/n)^2}\right)^{2n+1} - \left(\alpha_o/n - \sqrt{4+(\alpha_o/n)^2}\right)^{2n+1}}{2\left(\left(\alpha_o/n + \sqrt{4+(\alpha_o/n)^2}\right)^{2n} - \left(\alpha_o/n - \sqrt{4+(\alpha_o/n)^2}\right)^{2n}\right)} \tag{2}$$

$$\alpha_o = L\sqrt{hP/kA}$$

in such a way that $R_n \to R_{fin}$ as $n \to \infty$. At this limit Δx and α_o go to zero while the ratio in Eq. (2) goes to $1{,}31303 \approx (\tanh(1))^{-1}$ (see Fig. (6)) and

$$R_{fin} \approx \frac{1{,}31303}{\sqrt{hPkA}}$$

the classic result (Incropera and DeWitt, 2002). We also considered the case R" ≠ 0. Then, a distinct generalized Fibonacci sequence, with $G_0 = 1$ and $G_1 = \beta^{-1}$, applies. We could not obtain the general term of the equation, but it is possible to calculate R_{fin} and show that it agrees with the classic result (Incropera and DeWitt, 2002).

$$R_{fin} \approx \frac{\cosh(\alpha_0) + \beta\,\mathrm{senh}(\alpha_o)}{\mathrm{senh}(\alpha_0) + \beta\cosh(\alpha_o)}\, r \tag{3}$$

For instance, fixing $\alpha_0 = 1.0$ and $\beta = 0.5$ we get $R_{fin} \approx 1.0944r$ (see Fig. (7), the same value obtained by Eq. (3).

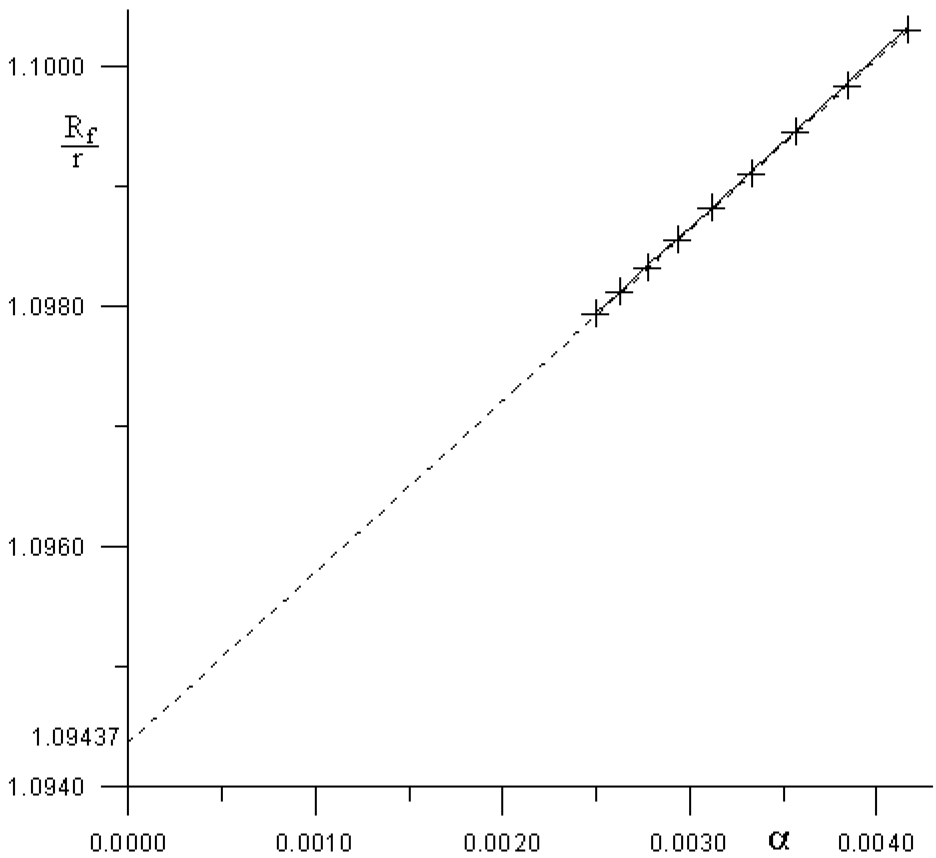

Figure 7. The equivalent resistance for the fin with R’’≠0.

4. Final Considerations

In this work, we identify an application for Fibonacci sequences, with differential coefficients and different initial conditions, G_0 and G_1. Moreover, we analyzed the problem of an n-sided polygon of resistors in an innovative way, by identifying self-similar elements in the networks. This approach may be applied to other thermal and optical systems, for which the resistance correspondence holds.

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