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Filtering the Infinite Eigenvalues Originated from Linear Stability Analysis of Incompressible Flows

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Abstract. *Steady state, two-dimensional flows can become unstable to two and three-dimensional disturbances if the flow parameters exceed some critical values. In many practical examples, determining the parameters at which the flow becomes unstable is an essential ingredient of the full understanding of the situation. Linear hydrodynamic stability of a laminar flow may be determined by slightly disturbing the flow and tracking the fate of that disturbance. It may die away, persist as a disturbance of similar magnitude, or grow indefinitely leading to a different laminar flow state, to a transient flow, or even to a turbulent flow. Linear stability formulation leads to a generalize eigenvalue problem (GEVP) where the eigenvalues correspond to the rate of growth of the disturbances and the eigenfunction to the amplitude of the perturbation. Solving the GEVP has been highly challenging, because the incompressibility of the liquid creates singularities that lead to aphysical infinite eigenvalues that require intricate reformulation and heavy computation to get around. The complexity and high computational cost of solving the GEVP have probably discouraged the use of linear stability analysis of incompressible flows as a general engineering tool for design and optimization. In this work, a new procedure to eliminate the "infinite eigenvalues" from the GEVP that comes from linear stability analysis of incompressible flow is proposed. The procedure takes advantage of the structure of the matrices involved and avoids the computational effort of common mapping techniques used to compute the spectrum of incompressible flows. As an example, the proposed method is applied in the solution of linear stability analysis of plane Couette flow.*

keywords: *eigenvalues, stability analysis, matrix transformation, incompressible flows.*

1. Introduction

Thorough understanding of viscous flows in many situations requires not only the two-dimensional, steady state solution of the governing equations, but also the sensitivity of those flows to small upsets and to episodic perturbations, i.e. stability analysis. For example, when studying the flow that occurs in many manufacturing processes, where a steady state flow is crucial for uniform product quality, the stability limits of the flow determine bounds for the operability limits of the process.

In many situations, an asymptotic analysis with respect to infinitesimal disturbances is sufficient to predict the critical flow parameters at which a two-dimensional steady flow becomes unstable. There are many examples of such analysis in the literature. Ruschak, 1983, Christodoulou and Scriven, 1988, Coyle *et al.*, 1990 and Carvalho and Scriven, 1999 studied stability analysis of different coating flows. Ramanan and Homsy, 1994 studied the linear stability of the flow inside a lid-driven cavity, and Severtson and Aidun, 1995 analyzed the stability of stratified liquid layers in inclined channels. Linear stability analysis involves the linearization of the governing equations about the steady state flow. The perturbation variables are described by a linear system of coupled differential equations. The discretization of the system of linear differential equations that describe the amplitude of the perturbations and its rate of growth leads to a non-hermitian, generalized eigenvalue problem (GEVP) of the form

$$\mathbf{Jc} = \sigma \mathbf{Mc}.$$

where the eigenvalue σ is the growth rate of disturbances. $\mathbf{J}$ and $\mathbf{M}$ are usually referred to as the Jacobian and Mass matrices.

Finding the solution of the GEVP is extremely challenging. The level of discretization needed to describe the perturbed fields is usually high, leading to large sparse matrices. The large dimension of the problem rules out the calculation of the full spectrum. Only the leading eigenvalues, those with the largest real part, are calculated. Iterative methods have to be used to compute the relevant part of the spectrum. Moreover, the mass matrix $\mathbf{M}$, which is associated with the transient terms of the governing equations, is singular because the continuity equation for incompressible flows does not have a time dependent term. This singularity gives rise to "infinite eigenvalues". The presence of these very large eigenvalues represents an important difficulty in solving this class of problem, because most iterative methods favor the eigenvalues with the largest modulus, not those with the largest real part. Therefore, the aphysical "infinite eigenvalues" should be eliminated before the solution of the eigenproblem, otherwise they would be the first ones to be computed.

The most effective techniques to solve GEVP are the methods based on some form of preconditioning and Krylov subspace projection methods, such as Arnoldi's and Lanczos methods (see Saad, 1996). A simple way to eliminate the infinite eigenvalues is to use the shift-and-invert transformation, that maps the infinite eigenvalues to zero. However, iterative methods used to solve the transformed problem will favor the eigenvalues closest to the shift parameter, not the leading ones. Christodoulou and Scriven, 1988 used approximately exponential preconditioning by rational transformation to overcome these difficulties. The eigenvalues of the transformed problem are the exponentials of the original eigenvalues, and consequently this transformation maps leading eigenvalues of the original problem to ones of largest modulus, which are favored by the iterative procedures, like Arnoldi's algorithm. All the proposed preconditioning are computationally expensive and do not really eliminate the infinite eigenvalues from the problem. The dimension of the transformed eigenproblem is the same as the original one. The eigenvalues are only mapped to a part of the spectrum of the transformed eigenproblem that will not be favored by the iterative methods.

In this work, realization on how incompressibility leads to infinite eigenvalues and analysis of the structure of the mass and jacobian matrices that arise from linear stability analysis of incompressible flows enabled us to eliminate the infinite eigenvalues. The procedure proposed transforms the generalized eigenproblem (GEVP) into a simple eigenproblem (EVP) whose dimension is smaller than the original one. Unlike the condensation procedure used by Ruschak, 1983 and Coyle *et al.*, 1990 for viscous free surface flows, and by Arora and Sureshkumar, 2002 for viscoelastic flows, the method proposed in this work is not limited to vanishing Reynolds number. Unlike the compressible flow formulation proposed by Sureshkumar, 2004, the proposed method does not include a penalty term in the mass conservation equation. The complete set of eigenvalues of the transformed problem is formed by all the finite eigenvalues of the original generalized eigenproblem. Therefore, the method presented really eliminates the infinite eigenvalue from the problem and also reduces the size of the matrices involved in the calculation. The proposed algorithm reduces not only the memory requirement but also the CPU time needed to compute the leading eigenvalues of incompressible viscous flows.

As an example, the proposed method is applied in the solution of linear stability analysis of plane Couette flow.

2. Linear Stability Analysis of Viscous Flow

2.1. Formulation

The velocity $\mathbf{v}$ and pressure p fields of two-dimensional, steady state, incompressible flow are governed by the continuity and momentum equations:

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$Re\ \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \boldsymbol{\tau}. \tag{2}$$

The Reynolds Number $Re \equiv \rho V L/\mu$ characterizes the ratio of inertial to viscous forces; V and L are suitable characteristic values of velocity and length, ρ is the liquid density and μ, the liquid viscosity. $\boldsymbol{\tau} \equiv \nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}$ is the viscous stress tensor for Newtonian fluid.

The goal of linear stability analysis is to determine if a two-dimensional, steady flow is stable with respect to infinitesimal disturbances. The stability of the flow can be judged by solving the time-dependent Navier-Stokes system for the long time behavior of infinitesimal perturbations to the base flow. Accordingly, the disturbed fields, i.e. velocity and pressure, are written as the sum of the base state and an infinitesimal perturbation

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v_0}(\mathbf{x}) + \epsilon\ \mathbf{v}'(\mathbf{x}) e^{\sigma t}, \tag{3}$$

$$p(\mathbf{x},t) = p_0(\mathbf{x}) + \epsilon\, p'(\mathbf{x})e^{\sigma t}. \tag{4}$$

$\mathbf{v_0}$ and p_0 are the velocity and pressure fields of the base flow, i.e. the two-dimensional, steady-state solution, which is known *a priori*. $\mathbf{v}'$ and p' are the fields that describe the amplitude of the perturbation and σ is the growth factor. If $\Re(\sigma) < 0$ ($\Re$ denotes the real part), the perturbation dies and the flow is said to be stable. If $\Re(\sigma) > 0$, the disturbance grows with time and the flow is said to be unstable .

The velocity $\mathbf{v}$ and pressure p of the disturbed flow are governed by the time-dependent Navier-Stokes system with the appropriate boundary conditions.

A system of linear differential equations that describe the perturbed fields is obtained after substituting the perturbed fields, e.g. eqs.(3,4), onto the transient Navier-Stokes system and neglecting terms of order $\mathcal{O}(\epsilon^2)$:

$$\nabla \cdot \mathbf{v}' = 0, \tag{5}$$

$$Re[\sigma \mathbf{v}' + \mathbf{v_0} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \mathbf{v_0}] = -\nabla p' + \nabla \cdot [\nabla \mathbf{v}' + \nabla \mathbf{v}'^T] = 0, \tag{6}$$

The unknowns of the problem are the perturbed fields $\mathbf{v}'$ and p' and the growth factor of the perturbation σ.

2.2. Discretization by Galerkin's Method and Finite Element Basis Functions

The perturbation fields $\mathbf{v}', p'$ and its rate of growth σ can be found by applying Galerkin's weighted residual method to eqs. (5, 6). The weighting functions used for the momentum equation ϕ_j and continuity equations χ_j are piecewise Lagrangean biquadratic and linear discontinuous polynomial basis functions, respectively. The weighted residual equations of continuity and x component of the momentum conservation are

$$R_c^j = \int_\Omega \left(\frac{\partial u_h'}{\partial x} + \frac{\partial v_h'}{\partial y}\right) \chi_j \, d\Omega, \tag{7}$$

$$R_{mx}^j = \sigma \int_\Omega Re\; u_h'\; \phi_j \; d\Omega \; + \int_\Omega Re \left[u_0 \frac{\partial u_h'}{\partial x} + v_0 \frac{\partial u_h'}{\partial y} + u_h' \frac{\partial u_0}{\partial x} + v_h' \frac{\partial u_0}{\partial y} \right] \phi_j + \tag{8}$$

$$\left[-p_h' + 2\frac{\partial u_h'}{\partial x} \right] \frac{\partial \phi_j}{\partial x} + \left[\frac{\partial u_h'}{\partial y} + \frac{\partial v_h'}{\partial x} \right] \frac{\partial \phi_j}{\partial y} \; d\Omega - \int_\Gamma [\mathbf{n} \cdot (-p' + \boldsymbol{\tau}')]_x \phi_j \; d\Gamma,$$

the y component is similar to x one, Γ is the boundary of the two-dimensional domain Ω. Each perturbed field is approximated with a linear combination of the same basis functions.

$$\mathbf{u}_h' = \begin{bmatrix} u_h' \\ v_h' \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n U_k \phi_k \\ \sum_{k=1}^n V_k \phi_k \end{bmatrix}, \quad p_h' = \sum_{k=1}^m P_k \; \chi_k.$$

Once all the variables are represented in terms of the basis functions, the system of partial differential equations reduces to simultaneous algebraic equations for the coefficient of the basis functions of all fields and the growth rate σ. The number of algebraic equations is $N = 2n + m$, where n is the number of basis functions used to expand each component of the velocity perturbation and m is the number of basis functions used to expand the pressure disturbance. In vector form, the set of algebraic equations leads to a generalized eigenvalue problem:

$$\mathbf{Jc} = \sigma \mathbf{Mc}. \tag{9}$$

where $\mathbf{c}$ is the column vector of coefficients of the finite element basis function with which the perturbation of velocity and pressure are represented. $\mathbf{M}$, which multiplies the growth rate σ, is called the mass matrix and $\mathbf{J}$ is the jacobian matrix. It follows that the discretization of the differential equations that describe the perturbation fields give rise to a generalized, non-Hermitian eigenproblem.

3. Elimination of the "infinite eigenvalues"

As shown in the previous section, the linear stability analysis led to a generalized eigenproblem (9). The mass matrix $\mathbf{M}$ is diagonal by blocks and singular, because the continuity equation for incompressible liquids does not have a time derivative term. Consequently, the number of eigenvalues of (9) is smaller than the dimension of the problem $N = 2n + m$. The missing eigenvalues are commonly referred to as "infinite eigenvalues", because if the mass matrix is slightly perturbed to remove the singularity, e.g. $\mathbf{M}^* = \mathbf{M} + \epsilon \mathbf{I}$, large eigenvalues appear in the spectrum, and they grow unbounded as $\epsilon \to 0$. Truncation errors in the numerical methods used to calculate the spectrum of (9) are equivalent to perturbations on the mass matrix and lead to the appearance of very large eigenvalues (the infinite eigenvalues). Christodoulou and Scriven, 1988 discusses that the number of infinite eigenvalues is equal the number of algebraic constrains (equations with no time derivative) in the discrete eigenproblem, i.e. the number of rows identically equal to zero in the mass matrix. In viscous flows of incompressible liquids, they represent the number of continuity residuals (number of degrees of freedom associated with the pressure field) plus the number of essential boundary conditions on the velocity field.

The presence of these very large eigenvalues represent an important difficulty in solving this class of problem, because most numerical methods to calculate the eigenvalues of a large eigenproblem favors the eigenvalues with the largest modulus. Therefore, these eigenvalues have to be eliminated before the solution of the eigenproblem. A common simple approach is to use transformations that map these infinite eigenvalues to zero, as the shift-and-invert transformation. In this case, the eigenproblem (9) is rewritten as

$$[(\mathbf{J}-\lambda\mathbf{M})-(\sigma-\lambda)\mathbf{M}]\,\mathbf{c}'=0.$$

$$\mathbf{A}\mathbf{c}'=\mu\mathbf{c}' \quad ; \qquad \mathbf{A}\equiv(\mathbf{J}-\lambda\mathbf{M})^{-1}\mathbf{M} \quad ; \qquad \mu\equiv 1/(\sigma-\lambda). \tag{10}$$

The shift-and-invert procedure transforms the generalized eigenproblem (9) to a simple eigenproblem (10). The eigenvalues σ of the GEVP can be calculated in terms of the eigenvalues μ of the simple EVP and of the shift parameter λ. The infinite eigenvalues are mapped to zero. However, the numerical methods to solve the simple eigenproblem will favors the largest eigenvalues μ, that correspond to the eigenvalues of the original problem σ closest to the shift λ.

Other more appropriate transformations can be use, such as bilinear and exponential transformations. Christodoulou and Scriven, 1988 and Ramanan and Homsy, 1994 used exponential transformation to map the leading eigenvalues to ones of largest modulus. They both found the method to be very robust, however the computational cost was extremely large, since the methods involves inverting a linear combination of the jacobian and mass matrices several times.

In the next section, a new method to filter the infinite eigenvalues of the GEVP (9) is proposed, taking advantage of the structure of the mass and jacobian matrices that come from linear stability analysis of flows of incompressible liquids. The analysis also shows that the number of infinite eigenvalues is actually larger than that proposed by Christodoulou and Scriven, 1988, it is equal to twice the number of residual equations associated with the mass conservation equations (twice the number of degrees of freedom associated with the pressure field) plus the number of residuals associated with essential boundary conditions on velocity.

3.1. Proposed Transformation

Following the ordering scheme explained before, both the mass and jacobian matrices are divided into blocks according to the equivalent residual equations and corresponding degrees of freedom.

$$\mathbf{M}=\begin{array}{cc}\left(\begin{array}{c|c|c}\mathbf{M_{11}} & \mathbf{0} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{M_{22}} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right) & \begin{array}{c} n\\ n\\ m\end{array}\\ \begin{array}{ccc} n & n & m\end{array} & \end{array} \qquad \mathbf{J}=\begin{array}{cc}\left(\begin{array}{c|c|c}\mathbf{J_{11}} & \mathbf{J_{12}} & \mathbf{J_{13}}\\ \hline \mathbf{J_{21}} & \mathbf{J_{22}} & \mathbf{J_{23}}\\ \hline \mathbf{J_{31}} & \mathbf{J_{32}} & \mathbf{0}\end{array}\right) & \begin{array}{c} n\\ n\\ m\end{array}\\ \begin{array}{ccc} n & n & m\end{array} & \end{array} \tag{11}$$

m and n indicate the dimension of each block, for example $[\mathbf{M_{11}}]=n\times n$.

The eigenvalues σ of the GEVP (9) are the roots of the characteristic polynomial

$$p(\sigma)\equiv det(\mathcal{A})\equiv det(-\sigma\mathbf{M}+\mathbf{J}). \tag{12}$$

As mentioned before, the algebraic equations associated with the Dirichlet's boundary conditions do not have a time derivative, and the perturbed velocity field at these boundaries are identically zero. Therefore, eliminating the rows and columns related to these equations and unknowns will not change the roots of the characteristic polynomial $p(\sigma)$. This is the first step in the procedure of eliminating the "infinite eigenvalues". The size of the new matrix $\mathbf{A}$ is $2n+m-b$, where b is the number of rows/columns related to the Dirichlet's boundary conditions that were eliminated from the eigenproblem. After eliminating the rows and columns associated with essential boundary conditions, it is also convenient to divide the new matrix $\mathbf{A}\equiv-\sigma\mathbf{M}+\mathbf{J}$ into a different structure of blocks, as shown below.

$$\mathbf{A}=\begin{array}{cc}\left(\begin{array}{c|c|c}\mathbf{A_{11}}(\sigma) & \mathbf{A_{12}}(\sigma) & \mathbf{A_{13}}\\ \hline \mathbf{A_{21}}(\sigma) & \mathbf{A_{22}}(\sigma) & \mathbf{A_{23}}\\ \hline \mathbf{A_{31}} & \mathbf{A_{32}} & \mathbf{0}\end{array}\right) & \begin{array}{c} m\\ 2n-m-b\\ m\end{array}\\ \begin{array}{ccc} m & 2n-m-b & m\end{array} & \end{array}$$

The blocks $\mathbf{A_{13}}$, $\mathbf{A_{23}}$, $\mathbf{A_{31}}$ and $\mathbf{A_{32}}$ do not have any contribution from the mass matrix and consequently do not depend on the growth factor σ.

The blocks $\mathbf{A_{23}}$ and $\mathbf{A_{32}}$ can be eliminated using the following transformation:

$$\widetilde{\mathbf{A}} = \left(\begin{array}{c|c|c} \mathbf{A_{11}}(\sigma) & \widetilde{\mathbf{A}}_{\mathbf{12}}(\sigma) & \mathbf{A_{13}} \\ \hline \widetilde{\mathbf{A}}_{\mathbf{21}}(\sigma) & \widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma) & \mathbf{0} \\ \hline \mathbf{A_{31}} & \mathbf{0} & \mathbf{0} \end{array}\right) = \mathbf{T_l}\ \mathbf{A}\ \mathbf{T_r}, \tag{13}$$

where

$$\mathbf{T_l} = \left(\begin{array}{c|c|c} \mathbf{I}_{[m]} & \mathbf{0} & \mathbf{0} \\ \hline -\mathbf{A_{23}}\mathbf{A_{13}^{-1}} & \mathbf{I}_{[2n-m-b]} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_{[m]} \end{array}\right) \quad , \quad \mathbf{T_r} = \left(\begin{array}{c|c|c} \mathbf{I}_{[m]} & -\mathbf{A_{31}^{-1}}\mathbf{A_{32}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_{[2n-m-b]} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_{[m]} \end{array}\right). \tag{14}$$

The characteristic polynomial $p_1(\sigma)$ of the transformed matrix $\widetilde{\mathbf{A}}$ is

$$p_1(\sigma) = det(\widetilde{\mathbf{A}}) = det(\mathbf{T_l}) \cdot det(\mathbf{A}) \cdot det(\mathbf{T_r}) = det(\mathbf{A}) = p(\sigma). \tag{15}$$

Because the blocks used in the definition of $\mathbf{T_l}$ and $\mathbf{T_r}$ do not contain the eigenvalue σ, the determinants of these matrices are independent of the eigenvalue. Moreover, because each transformation matrix is a triangular matrix with diagonal entries equal to one, their determinants are equal to one. Therefore, the characteristic polynomial of the transformed matrix $\widetilde{\mathbf{A}}$, $p_1(\sigma)$ is exactly the characteristic polynomial of the original matrix $\mathbf{A}$, $p(\sigma)$; both polynomials have the same roots. The multiplication of $\mathbf{A}$ by $\mathbf{T_l}$ and $\mathbf{T_r}$ does not change the spectrum of the original problem.

The determinant of the transformed matrix may be calculated using the cofactor expansion of the matrix:

$$p_1(\sigma) = det(\widetilde{\mathbf{A}}) = det(\left(\begin{array}{c|c|c} \widetilde{\mathbf{A}}_{\mathbf{11}}(\sigma) & \widetilde{\mathbf{A}}_{\mathbf{12}}(\sigma) & \mathbf{A_{13}} \\ \hline \widetilde{\mathbf{A}}_{\mathbf{21}}(\sigma) & \widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma) & \mathbf{0} \\ \hline \mathbf{A_{31}} & \mathbf{0} & \mathbf{0} \end{array}\right)) = det(\mathbf{A_{13}}) \cdot det(\left(\begin{array}{c|c} \widetilde{\mathbf{A}}_{\mathbf{21}}(\sigma) & \widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma) \\ \hline \mathbf{A_{31}} & \mathbf{0} \end{array}\right)) =$$

$$-det(\mathbf{A_{13}}) \cdot det(\mathbf{A_{31}}) \cdot det(\widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma)) = \kappa \cdot det(\widetilde{\mathbf{A}}_{\mathbf{22}}(\sigma)) = \kappa \cdot p_2(\sigma).$$

Consequently, the characteristic polynomial of the original matrix, $p(\sigma)$, is proportional to the characteristic polynomial of the $(2n-m-b) \times (2n-m-b)$ matrix $\widetilde{\mathbf{A}}_{\mathbf{22}}$; both polynomial have the same roots. Because the matrix $\widetilde{\mathbf{A}}_{\mathbf{22}}$ is non-singular, the number of roots of its characteristic polynomial is $2n-m-b$, the number of finite eigenvalues of the original problem. The number of infinite eigenvalues is twice the number of degrees of freedom associated with the mass conservation equation plus the infinite eigenvalues that come from essential boundary conditions, i.e. $2m+b$.

Because the infinite eigenvalues were eliminated by the transformations, the generalized eigenvalue problem (GEVP) can be transformed into a simple eigenvalue problem (EVP) ($\widetilde{\mathbf{M}}_{\mathbf{22}}$ is not singular):

$$\left(-\sigma\ \widetilde{\mathbf{M}}_{22} + \widetilde{\mathbf{J}}_{22}\right)\mathbf{c} = 0 \quad \Rightarrow \quad \underbrace{\widetilde{\mathbf{M}}_{22}^{-1}\widetilde{\mathbf{J}}_{22}}_{\mathbf{D}}\,\mathbf{c} = \sigma\mathbf{c} \tag{16}$$

The computational cost of the proposed matrix transformations corresponds to the cost of inverting two $m \times m$ matrices and one $(2n-m-b) \times (2n-m-b)$ matrix. The benefits is that the complete physically relevant spectrum of the original problem can be evaluated by solving a simple EVP that is approximately $1/3$ of the size of the original GEVP.

4. Example: Stability of Plane Couette Flow

4.1. Perturbed Equations and Solution Method

The method described in the previous section to eliminate the infinite eigenvalues of the spectrum is applied to study the stability of plane Couette flow.

The flow geometry and boundary conditions are shown in Fig.1: liquid flows between two parallel plates located at $y = \pm 1$ that are moving with velocity $U = \pm 1$.

The steady state solution is $\mathbf{v_0} = (y, 0, 0)$ and $p_0 = 0$. The base flow is perturbed as

$$\mathbf{v}(x, y, t) = \mathbf{v_0}(y) + \epsilon\ \mathbf{v}'(y)e^{i\alpha x + \sigma t} \text{ and } p(x, y, t) = p_0(y) + \epsilon\ p'(y)e^{i\alpha x + \sigma t}. \tag{17}$$

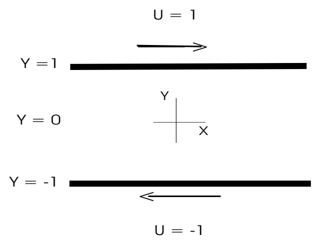

Figure 1: Configuration of plane Couette flow.

For simplicity, only two-dimensional perturbation was considered. α is the wavelength of the periodic perturbation along the flow direction. Substituting the perturbed fields into the transient conservations equations and neglecting the higher order terms ($\mathcal{O}(\epsilon^2)$), a system of differential equations on the perturbed variables, e.g. $\mathbf{v}' = (u', v')$ and p', is obtained:

$$\begin{aligned} & i\alpha u' + \frac{dv'}{dy} = 0, \\ & Re[(\sigma + i\alpha y)u' + v'] = -i\alpha p' + \frac{d^2u'}{dy^2} - \alpha^2 u', \\ & Re[(\sigma + i\alpha y)v'] = -\frac{dp'}{dy} + \frac{d^2v'}{dy^2} - \alpha^2 v'. \end{aligned} \tag{18}$$

This system of equation can be manipulated to eliminate the pressure field, and by using the definition of stream function, a fourth order operator on the amplitude of the perturbed stream function can be obtained (the Orr-Sommerfeld operator).

It is common practice to analyze the stability of Couette flow by working with the Orr-Sommerfeld operator. The fourth-order operator is usually discretized by spectral methods. An example of such procedure is presented by Dongarra *et al.*, 1996, that used Chebyshev-τ method to discretize the Orr-Sommerfeld equation and QZ method to solve the generalized eigenvalue problem. They reported that the singularity of the mass matrix $\mathbf{M}$ might account for the appearance of the spurious eigenvalues.

In this work, instead of using stream function formulation (Orr-Sommerfeld equation), the stability analysis is formulated in terms of primitive variables, i.e. velocity and pressure eqs.(18). At a fixed wave number α, the amplitude of the perturbation u', v' and p', and its growth rate σ are found by applying Galerkin's weighted residual method to eqs.(18), as explained in the previous section. In this example, the velocity perturbations are expanded using piecewise Lagrangian quadratic polynomials ϕ_j and the pressure perturbation are approximated using piecewise linear discontinuous polynomials χ_j.

At each row corresponding to the Dirichlet boundary condition applied at both walls, e.g. $u' = 0$ and $v' = 0$, all the entries of the mass and jacobian matrices are equal to zero, except the diagonal entry of the jacobian matrix, which is equal to one.

4.2. Filtering the infinite eigenvalues

In order to illustrate the structure of the matrices at each step of the process of filtering the infinite eigenvalues for the plane Couette flow, a case with only three finite elements is discussed first. For this discretization level, the total number of degrees of freedom of the problem is $N = 2n + m = 20$, as $n = 7$ and $m = 6$. The scheme used to number the elements, nodes and degrees of freedom of the problem is illustrated in Fig.2.

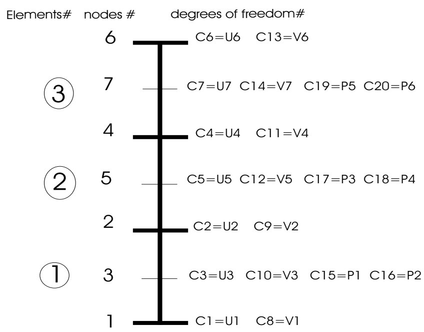

Figure 2: Numbering scheme for elements, nodes and degrees of freedom for a mesh with three finite elements.

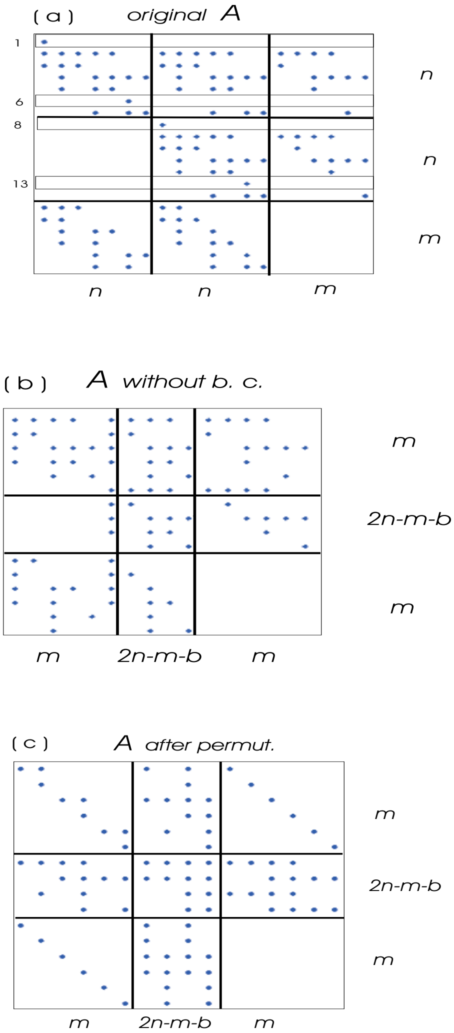

Figure 3: Example of the structure of the matrix $-\sigma\mathbf{M}+\mathbf{J}$ during the different steps of the proposed procedure. (a) Structure of the original matrix $\mathcal{A}$; (b) structure of the matrix after eliminating rows and columns associated with the Dirichlet boundary conditions $\mathbf{A}$; (c) Structure of the matrix after permutation of rows and columns to create diagonal sub-matrices that are easy to invert.

The structure of the non-zero entries of the matrix $\mathcal{A} = -\sigma\mathbf{M} + \mathbf{J}$ is shown in Fig.3(a). The only entries in rows $1, 6, 8$ and 13 (the ones associated with the Dirichlet boundary conditions) different than zero are the diagonal elements, that are equal to one. As explained before, the first step is to remove the rows and columns associated with the essential boundary conditions. The structure of the resulting matrix is shown in Fig.3(b). The next step is to eliminate the blocks $\mathbf{A_{32}}$ and $\mathbf{A_{23}}$ using the transformation defined in eq.(14). In order to construct the transformation matrices $\mathbf{T_l}$ and $\mathbf{T_r}$, the inverse of the blocks $\mathbf{A_{13}}$ and $\mathbf{A_{31}}$ need to be evaluated. In the particular case of a one-dimensional problem, like this one, it is possible use the structure of the matrix to minimize the time to compute the inverse of these sub-matrices. A permutation of the first $2n-b$ columns and the first $2n-b$ rows will lead to the matrix structure shown in Fig.3(c). The blocks $\mathbf{A_{13}}$ and $\mathbf{A_{31}}$ became diagonal matrices, and the evaluation of the inverse is extremely fast. The structure of the resulting transformed matrix $\widetilde{\mathbf{A}} = \mathbf{T_l}\ \mathbf{A}\ \mathbf{T_r}$ is shown in Fig.4. All the information on the spectrum of the original problem is contained in the $(2n-m-b)\times(2n-m-b)$ central block in the transformed matrix (4×4 in this particular case), indicated in Fig.4.

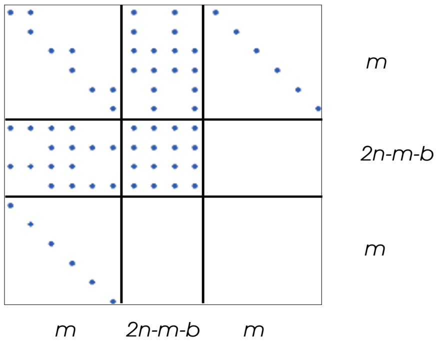

Figure 4: Structure of final transformed matrix $\widetilde{\mathbf{A}}$. The complete finite portion of the eigenspectrum of the original problem is contained in the center $(2n-m-b)\times(2n-m-b)$ block.

4.3. Results

To compare the spectrum of the plane Couette flow predicted with the method described in this work with the one presented by Bottaro *et al.*, 2003, the analysis was performed at $Re = 500$ and $\alpha = 1.5$.

In order to verify the independence of the predicted eigenvalues to the level of the discretization (number of elements), the spectrum of the original generalized eigenproblem — eq.(9) — was solved using QZ method. The leading (largest real part, not considering the infinite numbers) eigenvalues predicted with three different meshes are shown in Fig. 5, together with the results presented by Bottaro *et al.*, 2003. In the range of $-3.5 < \Re(\sigma) < 0$, doubling the number of elements from 100 to 200 did not affect the predicted spectrum. A mesh of 100 elements was considered to be fine enough to predict the leading eigenvalues of the problem. The results obtained with the formulation based on the primitive variables (velocity and pressure) and discretization by Galerkin's / Finite Element method agree well with the one obtained by solving the Orr-Sommerfeld operator using Chebyshev-τ method.

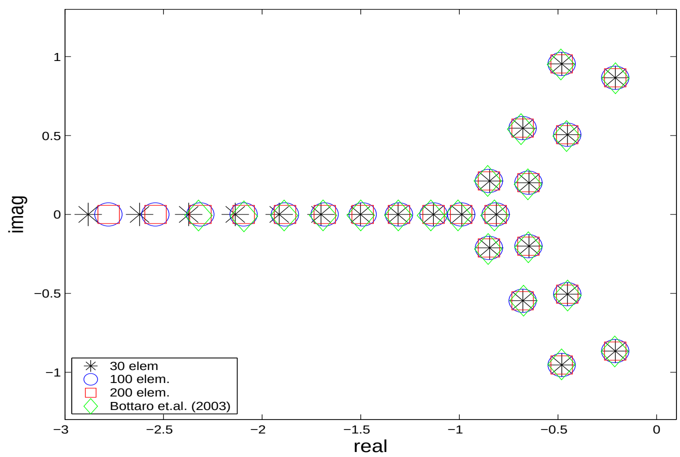

Figure 5: Leading part of the spectrum of plane Couette flow at $Re = 500$ and $\alpha = 1.5$ reported by Botaro and computed using the QZ method to solve the original GEVP with 30, 100 and 200 elements.

With 100 elements, the number of degrees of freedom of the problem (dimension of the original generalized eigenproblem) is $N = 602$, with $n = 201$, $m = 200$ and $b = 4$. After using the transformations presented here to eliminate the infinite eigenvalues of the problem, the reduced matrix, that contains all the information of the spectrum of the problem, has a dimension of $2n - m - b = 198$. Considering that typically $\mathcal{O}(n) \simeq \mathcal{O}(m)$, the dimension of the original GEVP is $N = 2n + m \simeq \mathcal{O}(3n)$. Because the number of essential boundary conditions is usually much smaller than the number of degrees of freedom associated with each velocity component, i.e. $b << n$, the dimension of the reduced EVP is $2n - m - b \simeq \mathcal{O}(n)$, approximately 1/3 of the size of the original problem.

The reduced simple EVP (16) was solved by the LAPACK routine ZGEEV (for non-Hermitian matrix). As expected, the eigenvalues of the simple EVP corresponded to the finite portion of the spectrum of the GEVP. The leading eigenvalues computed with both formulations are shown in Fig.6; they are exactly the same.

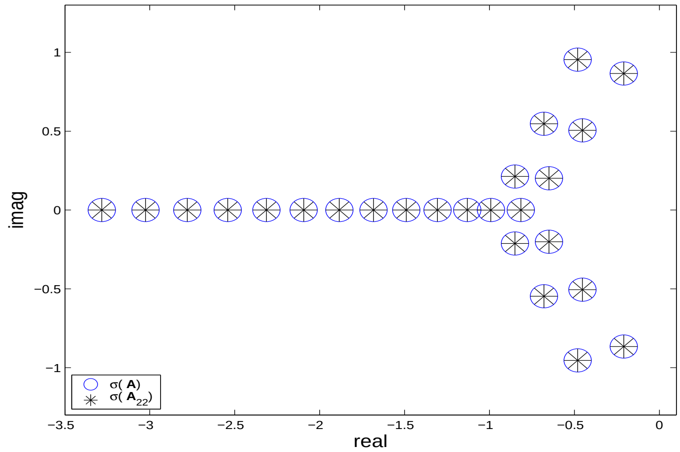

Figure 6: Comparison of the spectrum computed by solving the GEVP using the QZ method and by solving the reduced EVP.

The method presented in this work reduces significantly the time of computation. Table 1 presents the CPU time, in seconds, required to solve the original GEVP by the QZ method t_{GEVP} and to solve the reduced EVP by the LAPACK ZGEEV routine t_{EVP} for different meshes. The later includes the time to compute all the operations necessary to obtain the reduced EVP, which, as mentioned before, consists of inverting two $m \times m$ matrices, one $(2n-m-b) \times (2n-m-b)$ matrix, and some matrix-matrix products. In the particular case of the one-dimensional problem used as an example here, a simple row and column permutations transforms the sub-blocks into diagonal matrices, making the method even more efficient. The CPU time using the permutations, t^P_{EVP},is also presented in Table 1. All calculations were performed on a machine with 1.00 GB of RAM and 1.83 GHz Intel T-2400 processor using MatLab, version 6.5.

Table 1: CPU time, in seconds, required to compute the eigenvalues by the different methods: (a) solving the original GEVP by QZ method; (b) solving the reduced EVP using LAPACK routine; (c) solving the reduced EVP taking advantage of the matrix structure to invert the sub-blocks.

# ele	Size of original matrix	Size of transformed matrix	GEVP time	EVP time	EVP-Perm time	$\frac{t_{GEVP}}{t_{EVP}}$	$\frac{t_{GEVP}}{t^P_{EVP}}$
100	602	198	16.250	1.031	0.625	15.761	26.000
150	902	298	55.547	3.281	1.921	16.934	28.915
200	1202	398	131.438	7.031	4.219	18.694	31.1538
300	1802	598	474.750	27.422	14.547	17.312	32.635
350	2102	698	791.657	46.906	21.578	16.883	36.688

The proposed method is faster by a factor of approximately 17 for $N > 600$. The speed up is even greater (factor of approximately 35) if the matrix structure of the one-dimensional problem is taken into account to optimize the inversion of the two sub-matrices during the process of obtaining the reduced EVP.

5. Final Remarks

A new method to eliminate the "infinite eigenvalues" of the generalized eigenvalue problem that arises from linear stability analysis of incompressible flows is presented. The algorithm transforms the original generalized eigenproblem (GEVP) into an equivalent simple eigenvalue problem (EVP), whose dimension is approximately 1/3 of the original problem. The eigenvalues of the transformed EVP correspond exactly to the finite eigenvalues of the original GEVP.

The main advantages of the proposed methods are:

- Eliminates the "infinite eigenvalues" without the need of mapping or preconditioning techniques, which are computationally expensive;
- Reduces the size of the eingenproblem without loss of accuracy. Previous methods that reduced the size of the eigenproblem were restricted to creeping flow analysis (zero Reynolds number) or penalty methods;
- The transformed and smaller mass matrix is non-singular and consequently, the original GEVP can be easily re-written as a simple EVP.

All the above features bring significant reduction on the computational cost required to evaluate the eigenspectrum of an incompressible flow. In the example presented here, the proposed method was faster by an order of magnitude (factor of approximately 17) when compared to the solution of the original GEVP.

The analysis also shows that the number of infinite eigenvalues of a incompressible viscous flow is actually larger than that proposed by Christodoulou and Scriven, 1988; it is equal to twice the number of residual equations associated with the mass conservation equations (twice the number of degrees of freedom associated with the pressure field) plus the number of residuals associated with essential boundary conditions on velocity.

Although the formulation and the example used in this work was of a linear stability analysis of incompressible flow, this procedure may be also used to any generalized eigenproblem that comes from linear stability analysis with algebraic restriction (like incompressibility, for example).

6. Acknowledgements

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