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SCALING LAWS FOR TURBULENT BOUNDARY LAYERS SUBJECT TO ADVERSE PRESSURE GRADIENT AND SEPARATION

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Abstract. *The behaviour of a turbulent flow near a separation point is a very complex phenomenon. Despite the numerous works aimed at understanding this task, it remains still as an unsolved problem. Indeed, the reduction to zero of the main scaling parameters of the turbulent boundary layer flow, e. g. the skin friction velocity, completely breaks down the classical two-deck structure.The purpose of this work is to perform an analyses of the scaling parameters used in different law of the wall formulations which account for the adverse pressure gradient field. The predictions of seven different scaling laws will be compared with three sets of independent experimental results. The characteristic scales of the turbulent boundary layer will be evaluated for the upstream, recirculation and downstream regions. The outcome of this analysis may shed some light on the development of more accurate scaling laws for turbulent boundary layers subjected to large adverse pressure gradients and separation.*

keywords: *Turbulence, Separation, Adverse pressure gradient, Law of the wall, Experiments.*

1. Introduction

Turbulent flow near a separation point is a very complex problem. Despite the many works that have been published particularly in the last forty years, flow separation still remains a great challenge to researchers. The reason for this is very simple, near to a separation point the reduction to zero of the main velocity scaling parameter for the turbulent boundary layer, the skin friction velocity, completely breaks down the classical two-layered asymptotic structure. As a result, analyses where alternative characteristic scaling parameters are advanced must be proposed. These analyses must then explain how the classical former asymptotic structure for zero pressure gradient boundary layers can be modified such that an equation analogous to Stratford's law can replace the log-law at a separation point.

Despite the large number of references that can be found in literature, the present work will focus on a small selected group of works. These are considered representative enough of the state of the art by the present authors. In fact, seven different law of the wall formulations will be analysed here. Five laws that were developed mainly through asymptotic reasoning, namely, the log-laws of Mellor (1966), of Afzal (1983), of Nakayama and Koyama (1984), of Durbin and Belcher (1992) and of Cruz and Silva Freire (1998). Additionally, two laws that were derived mainly on empirical bases, the laws of Simpson (1983) and of Nickels (2004) will be considered.

Specifically, the purpose of this work is to perform a detailed analysis on the applicability of the scaling parameters used in different law of the wall formulations for flow description under an adverse pressure gradient field. Predictions given by the seven above mentioned different scaling laws will be compared with three sets of independent experiments, the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006). The characteristic scales of the turbulent boundary layer will be tested for both the regions of attached and of detached flows. Special attention will be given to an evaluation and comparison of the velocity characteristic scales. The outcome of this analysis might shed some light onto the development of more accurate scaling laws for turbulent boundary layers subjected to large adverse pressure gradients and resulting flow separation.

This manuscript is organized as follows: the next section will present the main derivations required to introduce each scaling law. The section of results will present the selected four sets of experimental data compared with the analytical predictions. Results for the characteristic scales will also be shown.

2. Scaling Laws

This section introduces the different law of the wall formulations that will be evaluated in the present work. Since our main concern is to carry out an analysis of the ability of the chosen laws in predicting flow separation, just the main parts of the original derivations will be presented here. We must warn the reader that some of the derivations are quite complex, a recurring feature that has really prevented us from going into too much detail. For a complete account of the formulations, the reader is referred to the original references.

2.1. Asymptotic correlations

2.1.1. The law of the wall formulation of Mellor (1966)

The effect of pressure gradients on the behaviour of turbulent boundary layers without restriction to equilibrium was investigated by Mellor (1966) through dimensional arguments. When a large external pressure gradient is applied to a boundary layer, no portion of the defect profile overlaps the logarithmic law. In fact, as previously suggested by Coles (1956) and by Stratford (1959), very near a separation point the logarithmic part of velocity profile ceases to exist.

However, if Millikan's (1939) arguments are recast and a new pressure gradient parameter is included in the analysis, an equation can be derived which satisfies the required limiting form as a separation point is approached. Assuming that in the viscous sublayer the stress terms should be balanced only by the pressure term in the motion equations, and by considering some overlap arguments, Mellor (1966) found that for outer layer the velocity profile can be written as

$$u^+ = \xi_{p^+} + \frac{2}{\varkappa}\Big(\sqrt{1+p^+z^+}-1\Big) + \frac{1}{\varkappa}\ln\Big(\frac{4z^+}{2+p^+z^+ + 2\sqrt{1+p^+z^+}}\Big), \tag{1}$$

where $z^+ = zu_{p\nu}/\nu$, $u^+ = u/u_{p\nu}$, $u_{p\nu} = [(\nu/\rho)(dp/dx)]^{1/3}$ and $p^+ = [(\nu/\rho)(dp/dx)]/u_*^3$. Eq. (1) follows different asymptotic behaviours in the limiting cases $p^+ \to 0$ or ∞, tending respectively to the classical logarithmic law or to Stratford's equation. Function ξ_{p^+} is a known parameter having been determined numerically for a range of p^+'s (see a specific table in Mellor (1966)).

2.1.2. The law of the wall formulation of Afzal (1983)

Afzal (1983) studied the asymptotic behaviour of a boundary layer near a separation point supposing a two-deck asymptotic structure to hold. As main non-dimensional parameters, Afzal took $R_p = \delta U_p/\nu$; $\Lambda = \tau_w/\delta p_x$, $p_x = \partial_x p =$ - UU_x; where, $U_p = (\nu p_x/\rho)^{1/3}$, $\delta =$ boundary layer thickness, and $\tau_w =$ wall shear stress.

For a boundary layer subject to an adverse pressure gradient, the adequate limiting forms can be written as:

$$\Lambda \to 0; \qquad R_p \to \infty; \qquad \Lambda R_p \to O(1). \tag{2}$$

Appropriate scales for the outer region of the flow in the streamwise direction, x, were taken to be $L =$ -U/U_x, and U. For the transversal direction, y, Afzal proposed to consider δ and $U_\delta = (\delta p_x/\rho)^{1/2}$. For the inner layer, Afzal proposed to consider $l = \nu/U_p$ and U_p as the important scales (x-direction). In y-direction, ν/U_p and U_p^2 were considered. Then, through matching arguments, limiting forms for the inner and outer solutions were proposed.

For the inner layer, the resulting solution was

$$\frac{u}{u_p} = A(\Lambda)\sqrt{\zeta} + C(\Lambda R_p), \qquad \zeta = \frac{zU_p}{\nu}, \;\; \zeta \to \infty. \tag{3}$$

For the outer layer, Afzal found

$$\frac{u}{U} = F_0'(x,0) + (U_\delta/U)[A(\Lambda)\sqrt{Z} + D], \qquad Z = \frac{z}{\delta}, \;\; Z \to 0. \tag{4}$$

Finally, after comparing his results with the experiments of nine different authors, Afzal proposed:

$$A = 3.5 + 19\,\Lambda, \qquad 0 \leq \lambda \leq 0.2$$
$$C = 2.5\,\Lambda R_p, \qquad 0 \leq \lambda R_p \leq 2.5 \qquad (5)$$
$$C = 2.5\,\Lambda R_p - 0.012(\Lambda R_p)^2, \qquad 0 \leq \lambda R_p \leq 100$$

2.1.3. The law of the wall formulation of Nakayama and Koyama (1984)

Nakayama and Koyama (1984) obtained a law of the wall for boundary layers subjected to adverse pressure gradients by conducting an one-dimensional analysis on the turbulent kinetic energy equation with assumptions of local similarity. Considering the two possible limiting cases of a constant stress layer and of a zero wall stress layer, the authors propose a turbulent kinetic energy equation that upon integration yields,

$$u^+ = \frac{1}{\varkappa^+}\Big[3(\zeta - \zeta_s) + \ln\big(\frac{\zeta_s + 1}{\zeta_s - 1}\frac{\zeta - 1}{\zeta + 1}\big)\Big], \qquad (6)$$

where

$$\zeta = \big(\frac{1 + 2\tau^+}{3}\big)^{1/2}. \qquad (7)$$

The above formulation introduces a von Kármán modified constant, $\varkappa^+$, and the slip value, ζ_s. For a boundary layer subjected to an adverse pressure gradient,

$$\tau^+ = 1 + p^+ z^+, p^+ = \nu\rho^{1/2}(d\tau/dz)_w/\tau_w^{3/2}, z^+ = (\tau_w/\rho)^{1/2} z/\nu \qquad (8)$$

The von Kármán modified constant was estimated to be

$$\varkappa^+(p^+) = \frac{0.419 + 0.539p^+}{1 + p^+}. \qquad (9)$$

The slip value ζ_s was determined from the condition that in the limiting case $p^+ \to 0$ the above formulation reduces to the classical law of the wall. It follows that

$$\zeta_s(p^+) = (1 + (2/3)e^{-\varkappa A}p^+)^{1/2} \approx (1 + 0.074p^+)^{1/2}. \qquad (10)$$

Nakayama and Koyama (1984) considered their analysis general in the sense that velocity was related to the local shear stress instead of to the distance from the wall. Additionally, the analysis does not have to be restricted to a linear velocity-stress relation but can be applied for any monotonically increasing shear stress layer.

2.1.4. The formulation of Durbin and Belcher (1992)

Durbin and Belcher (1992) performed a scale analysis for turbulent boundary layers in strong adverse pressure gradients. The authors considered a three-layered asymptotic structured for the boundary layer so that an extra blending layer was inserted between the wall viscous region and the outer region. In the blending layer the mean mean velocity profile was found to follow a $z^{1/2}$ behaviour.

The relevant scales to the problem, were considered by Durbin and Belcher to depend on, α = -$U_\infty U_\infty$; L = U_∞^2/α; $u_p = (\nu\alpha)^{1/3}$.

Based on these definitions, a small parameter can be defined,

$$\epsilon = \frac{u_p}{U_\infty} = (\nu\alpha/U_\infty^3)^{1/3} = R_L^{-1/3}. \qquad (11)$$

Please note that α denotes the external pressure gradient.

For the inner and blending layers, Durbin and Belcher found the adequate scaling to be

$$\hat{z} = \frac{z u_p}{\nu}, \qquad \hat{u} = \frac{u}{u_p}, \qquad \hat{\nu}_t = \frac{\nu_t}{\nu}, \qquad \hat{\kappa} = \frac{\kappa}{u_p^2}, \qquad \hat{\varepsilon} = \frac{\varepsilon \nu}{u_p^4}. \tag{12}$$

Then, using the κ-ϵ model to close the approximate equations of motion, a solution was proposed for the blending region:

$$\hat{U}_{\hat{z}} = \frac{\hat{z}}{\hat{\nu}_t}, \qquad \hat{U} = A_u \, \hat{z}^n, \tag{13}$$

where $A_u = 7.65$ and $n = 1/2$. The value of A_u is obtained directly from the standard constants of the κ-ϵ model.

The authors thus concluded that a graph of U versus $y^{1/2}$ in inner or in transition coordinates should present a linear region, with all curves collapsing onto each other.

2.1.5. The law of the wall formulation of Cruz and Silva Freire (1998, 2002)

Cruz and Silva Freire (2002) proposed the law of the wall for a separating flow to be written as

$$u = \frac{\tau_w}{|\tau_w|} \frac{2}{\varkappa} \sqrt{\frac{\tau_w}{\rho} + \frac{1}{\rho}\frac{dP_w}{dx} z} + \frac{\tau_w}{|\tau_w|} \frac{u_*}{\varkappa} \ln\left(\frac{z}{L_c}\right), \tag{14}$$

where

$$L_c = \frac{\sqrt{\left(\frac{\tau_w}{\rho}\right)^2 + 2\frac{\nu}{\rho}\frac{dP_w}{dx} u_R} - \frac{\tau_w}{\rho}}{\frac{1}{\rho}\frac{dP_w}{dx}}, \tag{15}$$

$\varkappa = 0.4$, u_* is the friction velocity, and u_R ($= \sqrt{\tau_p/\rho}$, τ_p = total shear stress) is a reference velocity.

Equation (14) is a generalization of the classical law of the wall and replaces the three expressions advanced in Cruz and Silva Freire (1998), Eqs. (25, 26, 27). Eq. (15) is an expression for the near wall region characteristic length, which is assumed to be valid in the attached and in the reverse flow regions.

The generalization provided by Eq.(14) implies that the friction velocity, u_*, used in the definition of L_c had to be replaced by the reference velocity u_R. Please, note that the characteristic length in the reverse flow region is different from the classical characteristic length given by the classical law of the wall. This in agreement with Simpson et al. (1981), who suggested that a characteristic length for the backflow region should be directly proportional to the absolute value of the wall shear stress.

2.2. Empirical correlations

2.2.1. The contribution of Simpson (1983)

From an analysis of available experimental data Simpson proposed velocity expressions for a description of reverse flow. As basic scaling parameters, he proposed to take the maximum negative velocity, U_N, and its distance from the wall, δ_N.

The resulting expressions were:

1) Viscous sublayer.

$$u^+ = z^+ + 0.5(u_p/u_\tau)^3 (z^+)^2, \qquad 0 \le z < z_1. \tag{16}$$

2) Fully turbulent region.

$$\frac{\overline{U}}{U_N} = A\left[\left(\frac{z}{\delta_N}\right) - \log\left(\frac{z}{\delta_N}\right) - 1\right] - 1, \qquad z \ge z_1. \tag{17}$$

The following definitions apply to the above equations:

$$u_p = \left(\frac{\nu}{\rho}\frac{\partial p}{\partial x}\right)^3, \quad A = 0.3, \quad z_1 = 0.02\delta_N. \tag{18}$$

2.2.2. The contribution of Nickels (2004)

Nickels (2004) developed an appropriate functional form for the mean velocity profile to parameterize the DNS data of Spalart (1988) e Osterlund (1999) on flows subject to adverse pressure gradients. The model was based on the concept of a universal critical Reynolds number for the viscous sublayer. In theory, this procedure can explain the shift in the apparent log-law due to pressure-gradient effects and gives an appropriate scaling for the Reynolds stresses. The results are supposed to hold for values of p_x^+ in the range -0.02 $< p_x^+ <$ 0.06 and are presented considering the flow to be divided into three distinct regions: a viscous layer, a blending layer and an external layer.

The model of Nickles for the inner flow in adverse-pressure-gradient flow uses the idea advanced by Clauser (1956) that the viscous sublayer grows until it reaches a critical Reynolds number. Thus, at a certain distance from the wall the sublayer should reach a state in which it becomes unstable and perturbations grow. This critical height is denoted in non-dimensional form by z_c^+ $(=zu_\tau/\nu)$ and is normally quoted in literature to be equal to 12 for a zero-pressure-gradient flow.

For flows under an adverse-pressure-gradient u_τ approaches zero so that an alternative scaling velocity has to be chosen. Nickels proposes to consider this scale to be U_T, with $U_T = \sqrt{\tau_T/\rho}$, $\tau_T =$ maximum shear stress in the viscous layer.

We may then define a critical Reynolds number,

$$R_c = \frac{U_T z_c}{\nu} \equiv 12. \tag{19}$$

As the wall shear stress becomes zero, the velocity scales reduce to

$$U_T = R_c^{1/3} U_p, \qquad \text{with} \qquad U_p = \left(\frac{\nu}{\rho}\frac{\partial p}{\partial x}\right)^{1/3}. \tag{20}$$

The critical height is then given by

$$z_c = \frac{\nu R_c^{2/3}}{U_p}, \qquad R_c = 12. \tag{21}$$

Nickels defines $z_p = zU_p/\nu$, where ν/U_p represents the thickness of the viscous sublayer.

The resulting law of the wall for adverse-pressure-gradient driven flows is given by ($\varkappa_0 = 0.4$)

$$\frac{u}{u_p} = \frac{R_c^{1/3}}{\varkappa_0} \ln\left(\frac{0.6 z_p}{R_c^{2/3}}\right). \tag{22}$$

3. Results

The above theories will be tested against the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006).

The data of Simpson et al. (1981) were concerned with a two-dimensional, separating turbulent boundary layer for an air-foil type flow. In a wind tunnel with an adjustable wall the flow was accelerated and then decelerated until separation. Measurements were made with the help of hot-wire anemometry, surface hot-wire skin-friction gauges and laser-Doppler anemometry. Profiles of U, W, $\overline{u^2}$, $\overline{w^2}$ and of -$\overline{uw}$ were given at six different stations.

Dengel and Fernholz (1990) studied the effects of small changes in the static pressure distribution on the development of an axisymmetric, incompressible, turbulent boundary layer in the vicinity of separation. The boundary layer was made to develop in a wind tunnel over an inner circular cylinder with an elliptical nose cone. Three types of flows were studied to produce finite regions of approximately zero, slightly positive and slightly negative skin-friction. Measuring techniques consisted of hot-wire anemometry, pulsed-wire anemometry and laser-Doppler anemometry. Mean and fluctuating profiles were presented for five stations.

The experiments of Loureiro et al. (2006) were performed in a water channel. The flow over a smooth, two-dimensional steep hill was characterized at thirteen stations. The large region of reverse flow on the lee of the hill was particularly scrutinized through five stations. Measurements were conducted by laser-Doppler anemometry. Profiles of U, W, $\overline{u^2}$, $\overline{w^2}$ and of -$\overline{uw}$ were presented for all stations.

An assessment of the theories presented in Section 2 will be made by comparing selected profiles of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006) with the several proposed expressions.

Graphs will be presented so as evaluate the performance of the formulations in four distinct regions: (a) upstream of separation, (b) near to the separation point, (c) in the reverse flow region and (d) downstream of the separation bubble.

The formulation of Mellor, given by Eq. 1, is shown in Fig. 1 in coordinates

$$U_M = u^+ \qquad \text{and} \qquad Z_M = \frac{4z^+}{2 + p^+z^+ + 2\sqrt{1 + p^+z^+}}. \tag{23}$$

Agreement with experimental data was found to be fair just in the vicinity of a separation point (Fig. 1b). In particular, both the near wall data of Simpson et al. (1981) and of Loureiro et al. (2006) agree well with predictions.

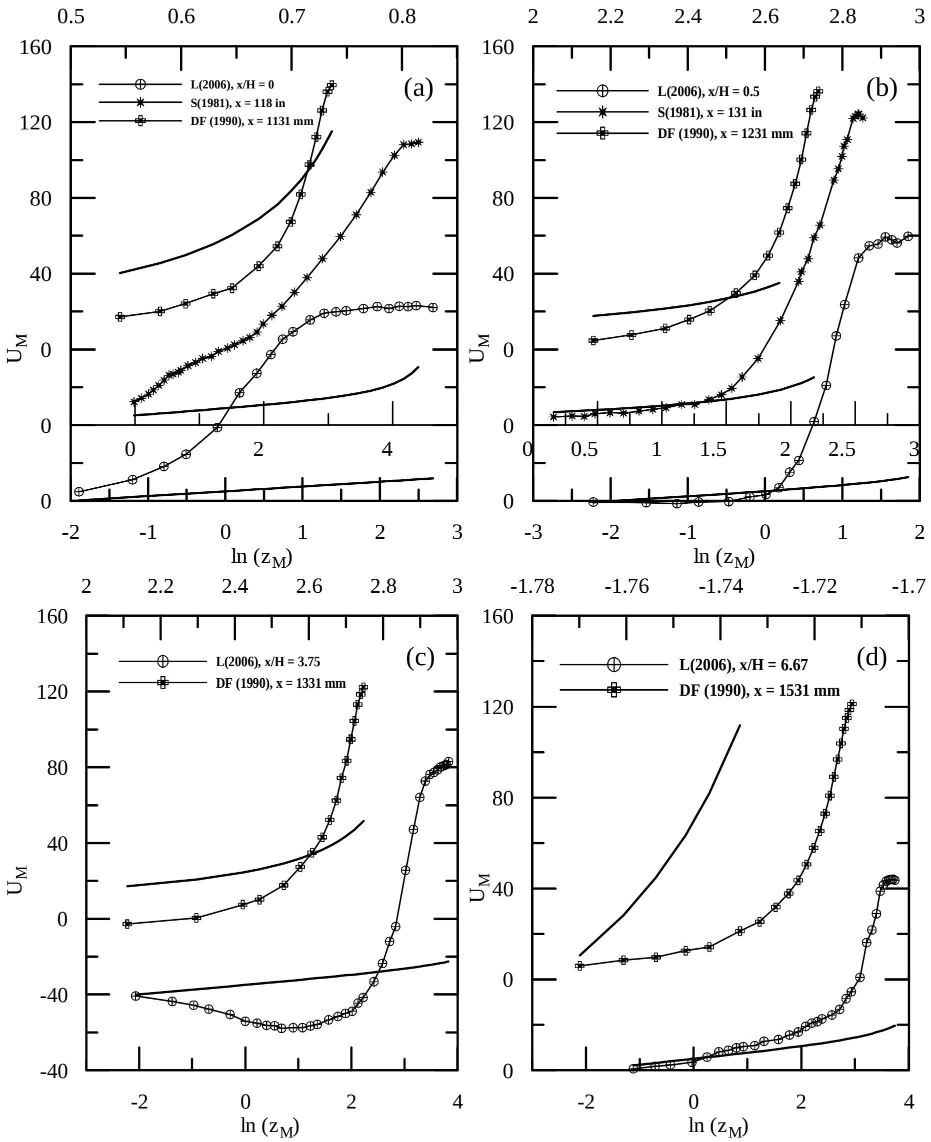

Figure 1: Comparison of Mellor's (1966) theory with the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006).

To test Afzal's (1982) equation, Eq. 3, the following co-ordinate system was used

$$U_{Afzal} = \frac{u}{u_p} \qquad Z_{Afzal} = \zeta = \frac{zU_p}{\nu}. \tag{24}$$

Because Afzal's formulation is supposed to hold for mild adverse pressure gradients, just the data of Loureiro et al. (2006) could be used here to test its applicability. The results shown in Fig. 2. The overall agreement of the theory is very good but for the description of the reverse flow region.

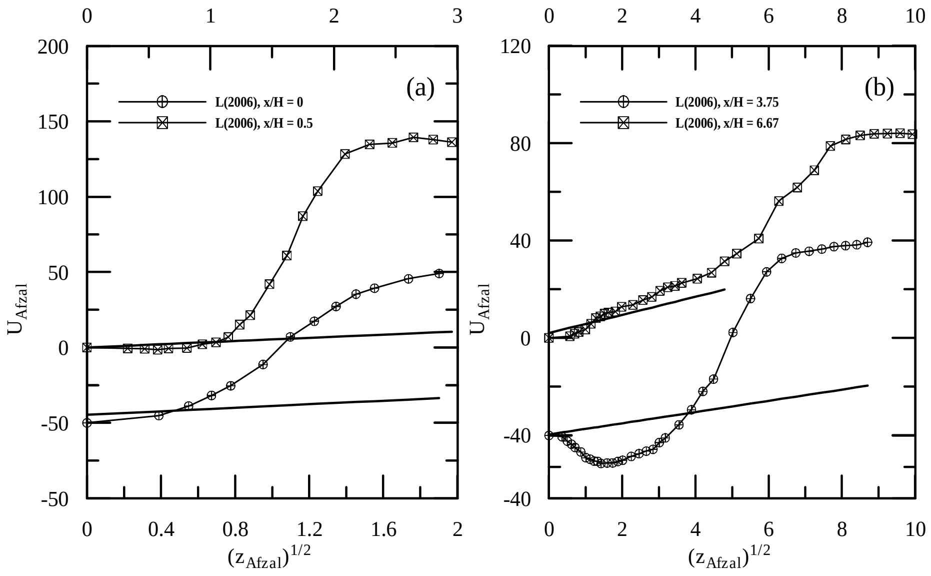

Figure 2: Comparison of Afzal's (1983) theory with the data of Loureiro et al. (2006).

To compare the theory of Nakayama and Koyama (1984) we use the co-ordinate system:

$$U_{NK} = u^{+} \qquad Z_{NK} = \frac{\zeta_s + 1}{\zeta_s - 1}\frac{\zeta - 1}{\zeta + 1} \tag{25}$$

Predictions for the data of Simpson et al. (1981) both upstream and in the vicinity of the separation point are reasonable. In the region of reverse flow, the data of Dengel and Fernholz (1990) is fairly adjusted. Comparison to other data is not good.

The arguments of Durbin and Belcher (1992) imply for the existence of a power-law in a region near to separation. This layer would exist to provide matching between the inner and outer solutions. Graphs in 1/2-coordinates are shown in Fig. 4, where the following definitions were used $U_{DB} = u/u_p$, $Z_{DB} = zu_p/\nu$. In the region near to a separation point, Fig. 4b, the data of Simpson suggest to the existence of a possible power-law region very close to the law. The data of Loureiro et al. also provide a good fit in the region of reverse flow. However, that was not the intention of the authors. Their formulation is supposed to hold in regions of attached flow. Overall the agreement with the other data is very poor.

The theory of Cruz and Silva Freire is considered next. This theory was specially advanced to describe reverse flow. Consider $U_{CSF} = u/u_\tau$ and $Z_{CSF} = z/L_c$ so that Eq. 14 can be represented in Fig. 5. The agreement between the theory of Cruz and Silva Freire (2002) and the data of Simpson et al. (1981) and of Loureiro et al. (2006) is remarkable. Note specifically the good agreement near to the separation point and in the reverse flow region. The general trend of the reverse flow is indeed well captured by the theory.

The reverse flow data of Loureiro et al. (2006) is compared in Fig. 6 with the theory of Simpson (1983) for the outer flow description. The agreement is good.

The empirical theory of Nickels (2004) is compared with the data of other authors in Fig. 7. The co-ordinates are defined according to

$$U_{Ni} = u/u_p, \qquad Z_{Ni} = \frac{0.6 z_p}{R_c^{2/3}}. \tag{26}$$

The comparison is not favorable to Nickel's formulation. But for a good agreement regarding the data of Dengel and Fernholz (1990), the other comparisons are very disappointing.

4. Final Remarks

The present work has conducted a detailed analysis on the description of turbulent flow near a separation point, including its region of reverse flow. Predictions given by the seven different scaling laws were compared

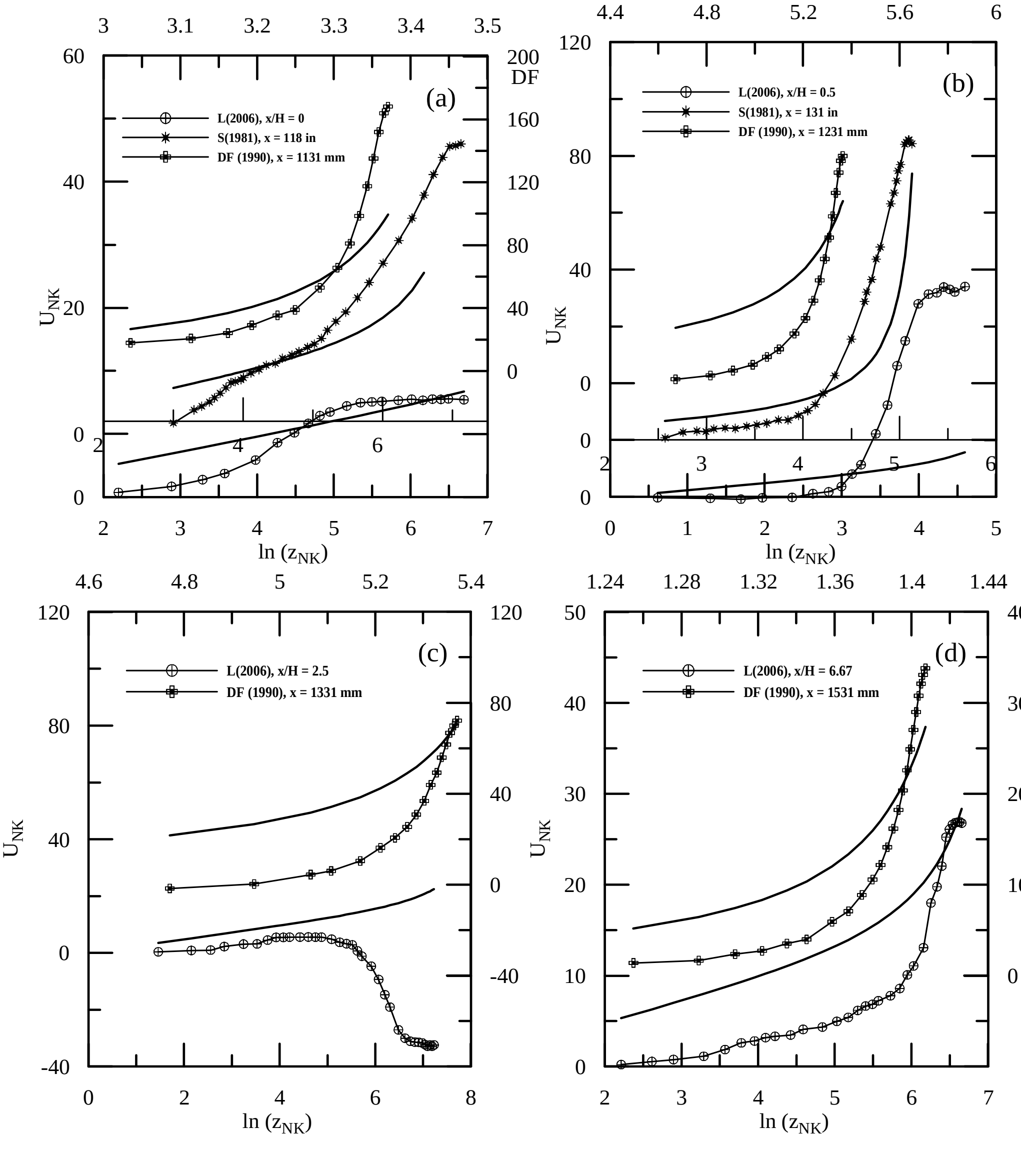

Figure 3: Comparison of Kakayama and Koyama's (1984) theory with the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006).

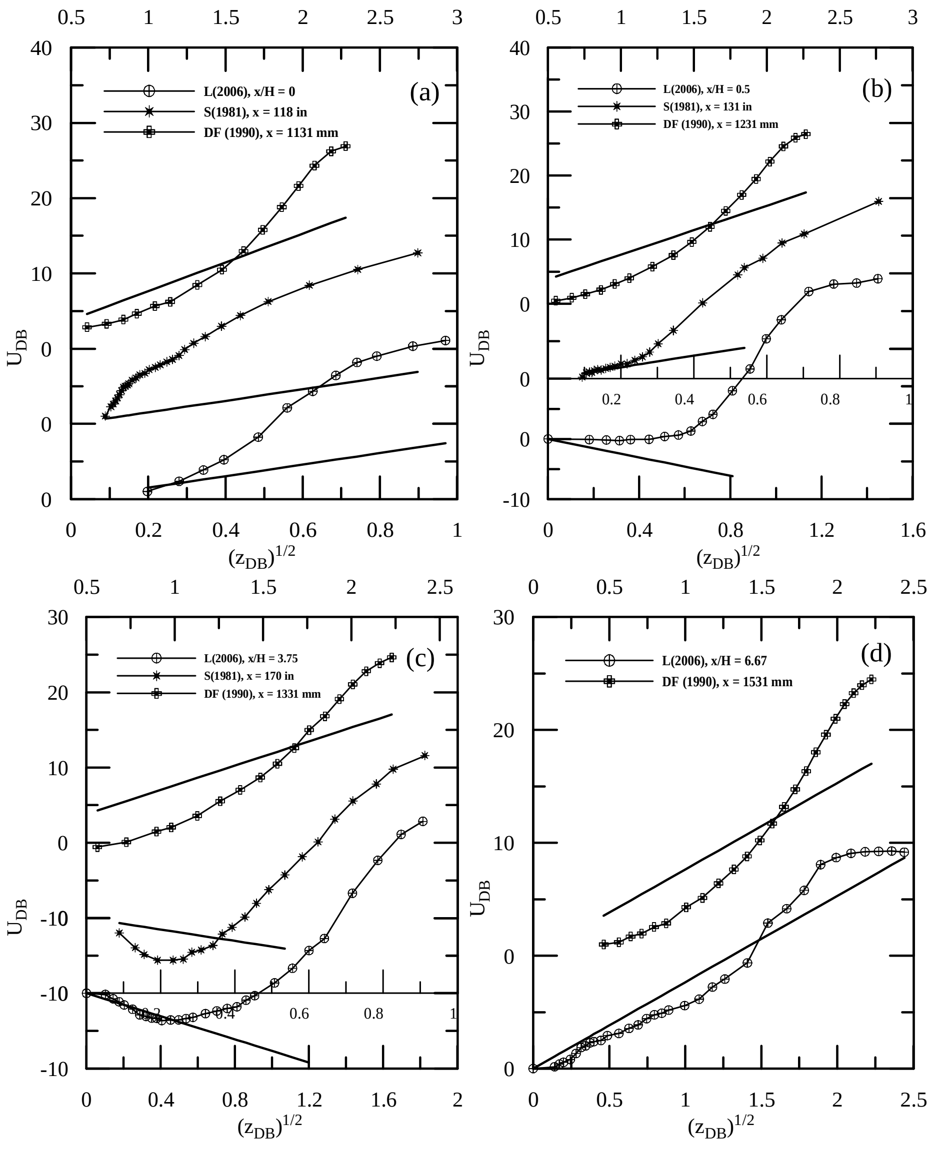

Figure 4: Comparison of Durbin and Belcher's (1992) theory with the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006).

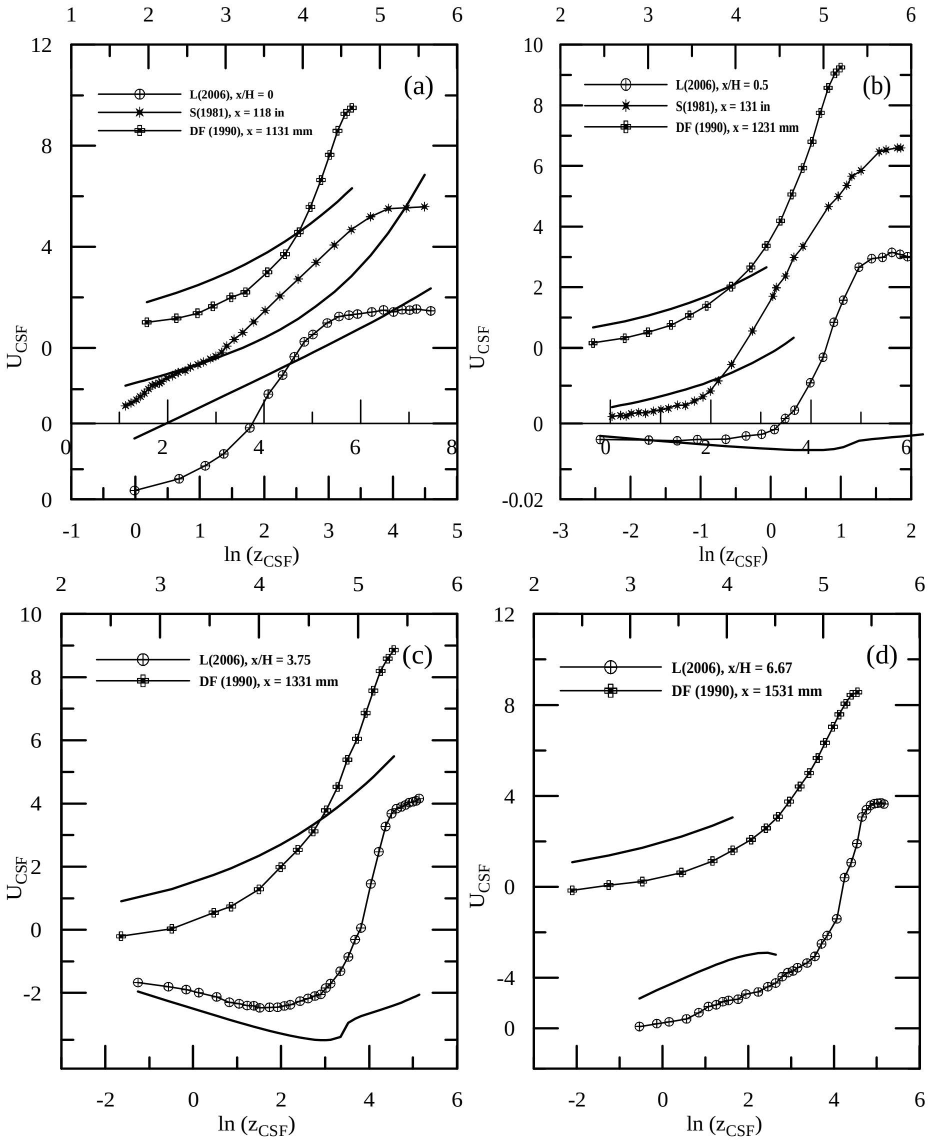

Figure 5: Comparison of Cruz and Silva Freire's (2002) theory with the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006).

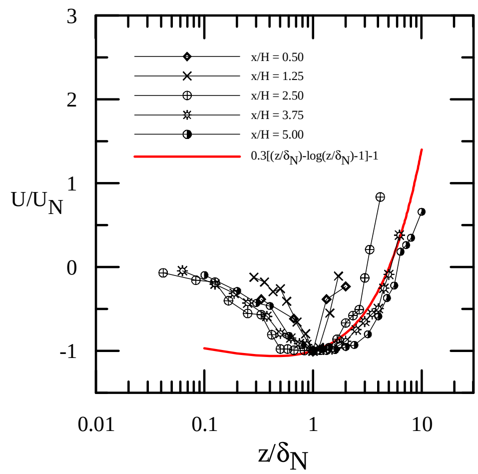

Figure 6: Comparison of Simpson's (1983) theory with the data of Loureiro et al. (2006).

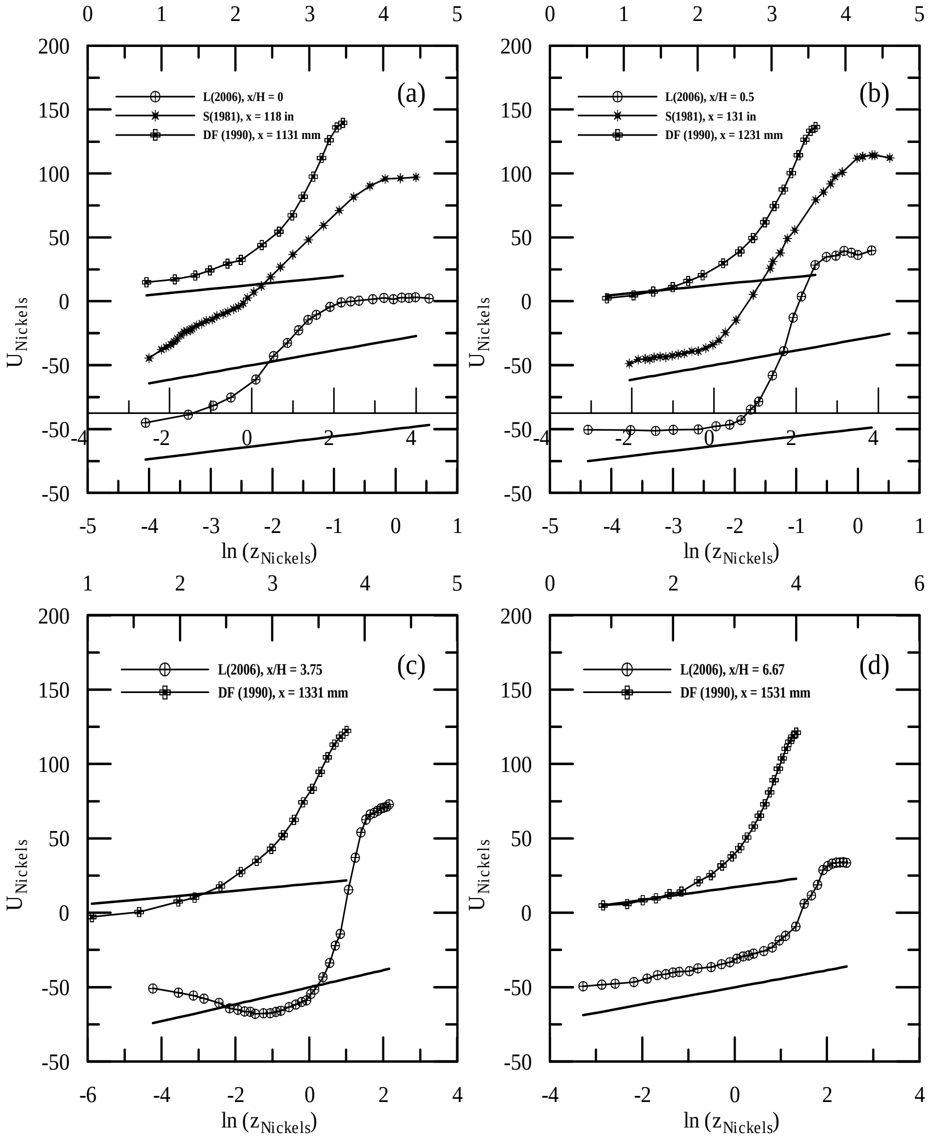

Figure 7: Comparison of Nickels´s (2004) theory with the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006).

with three sets of independent experiments, the data of Simpson et al. (1981), of Dengel and Fernholz (1990) and of Loureiro et al. (2006). The performance of the formulations were evaluated in four distinct regions: (i) upstream of separation, (ii) near to the separation point, (iv) in the reverse flow region and (v) downstream of the separation bubble. Consequently, this study provides a broad and independent analysis of the problem, and investigates the ability of the scaling laws to predict the flow behaviour from upstream of the separation region, along to the recirculation bubble and downstream of the reattachment point.

The results have clearly demonstrated that the more sophisticated laws, where adjustable scaling parameters are used, do perform much better than the others. In this sense, the formulation of Cruz and Silva Freire (2002) appears to be much superior than the others. In fact, this formulation has shown to be adequate for use in regions of attached and of detached flows. This a very unique feature, that makes it a very useful tool.

The formulations which are based solely on the pressure gradient and on the velocity scale derived from this parameter show poor results. These laws are observed to be incapable of providing good results for all the four distinct regions studied. The different performances of the formulations are naturally due to the different characteristic scales used and on the arguments which the derivations are based on.

A further refinement of the present analysis is currently under construction. The seven proposed near wall formulations will be tested against a larger set of experimental data.

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5. References

Afzal, N.: 1983, Analysis of a turbulent boundary-layer subjjected to a strong adverse pressure-gradient, "Int. J. Engng. Sci.", 21, 6, 563–576.

Coles, D.: 1956, The law of the wake in the turbulent boundary layer, "J. Fluid Mech.", 1, 191–226.

Clauser, F. H.: 1956, The turbulent boundary layer, "Adv. Appl. Mech.", 4, 1-51.

Cruz D. O. A. and Silva Freire A. P.: 1998, On single limits and the asymptotic behaviour of separating turbulent boundary layers. "Int. J. Heat Mass Transfer", 41, 2097–2111.

Cruz D. O. A. and Silva Freire A. P.: 2002, Note on a thermal law of the wall for separating and recirculating flows. "Int. J. Heat Mass Transfer", 45, 1459–1465.

Dengel, P. and Fernholz, H. H.: 1990, An experimental investigation of an incompressible turbulent boundary layer in the vicinity of separation. "J. Fluid Mech.", 212, 615–636.

Durbin, P. A. and Belcher, S. E.: 1992, Scaling of Adverse-Pressure-Gradient Turbulent Boundary Layers. "J. Fluid Mech.", 238, 699–722.

Loureiro, J. B. R., Soares, D. V., Fontoura Rodrigues, J. L. A., Pinho, F. T. and Silva Freire, A. P.: 2006, Water tank and numerical model studies of flow around steep smooth two-dimensional hills. "Boundary-layer Meteorol.", (in press).

Mellor, G. L.: 1966, The effects of pressure gradients on turbulent flow near a smooth wall. "J. Fluid Mech.", 24, 255–274.

Millikan, C. B.: 1939, A critical discussion of turbulent flow in channels and tubes. "Proc. 5th Int. Cong. App. Mech.", J. Wiley, N. Y., 386–392.

Nakayama, A. and Koyama, H.: 1984, A wall law for turbulent boundary layers in adverse pressure gradients. "AIAA J.", 22, 1386–1389.

Nickels, T. B.: 2004, Inner scaling law for wall-bounded flows subject to large pressure gradients. "J. Fluid Mech.", 521, 217–239.

Österlund, J. M.: 1999, Experimental studies of zero pressure-gradient turbulent boundary layer flow. "PhD thesis", Royal Institute of Stockolm.

Simpson, R. L., Chew, Y. T. and Schivaprasad, B. G.: 1981, The structure of a separating boundary layer. Part 1: Mean flow and Reynolds stresses. "J. Fluid Mech.", 113, 23–51.

Simpson, R. L.: 1983, "AIAA J.", 41, 142–144.

Spalart, P. R.: 1988, Direct numerical study of a turbulent boundary layer up to $Re_\theta = 1410$. "J. Fluid Mech.", 249, 337–371.

Stratford, B. S.: 1959, The prediction of separation of the turbulent boundary layer. "J. Fluid Mech.", 5, 1–16.