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NUMERICAL INVESTIGATION OF A GALERKIN LEAST-SQUARES MULTI-FIELD FORMULATION FOR VISCOUS FLOWS IN AN AXISYMMETRIC DOMAIN

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Abstract *The investigation of non-Newtonian flows is of paramount relevance in many fields of engineering. Indeed, numerical approximations of non-Newtonian flow models meet some difficulties that may compromise their stability and even prevent physically realistic results. In this article, we exploit the features of a Galerkin least-squares (GLS) finite elements method (FEM) in approximating a multi-field mixed formulation (in extra-stress, velocity and pressure) for isochoric flows of inelastic fluids. We deal about the specific situation when a cylindrical coordinate system is employed to describe non-swirling axisymmetric flows. This treatment adds one degree of freedom to the formulation of 2-D flows, the third normal component of the extra-stress, in the angular direction. We present the mechanical model for such flows, in which the extra-stress tensor is a primal variable, using the equations of mass conservation and momentum balance. We model the stress tensor via a purely viscous constitutive model. We present a GLS approximation which allows the use of equal order finite elements for all variables, overcoming the pressure-velocity and stress-velocity compatibility conditions. The issues of such scheme are expressed by our numerical results. We employ both a Newtonian constant viscosity and a viscosity function given by the Carreau model, which predicts pseudoplastic behavior. We present numerical results for Newtonian and pseudoplastic flows through an axisymmetric contraction. Besides the features of convergence and stability of the GLS approximations, we find good agreement between our results and those from literature. We investigate the role of Carreau's equation parameters and Reynolds number in the flow dynamics near the contraction, interpreting the differences between velocity profiles generated by each fluid with the aid of the extra stress tensor fields, which are directly computed in this formulation.*
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Keywords: *finite elements, Galerkin least-squares, non-Newtonian fluids, stress-velocity-pressure formulation, axisymmetric flow.*

1. Introduction

The investigation of the flow of non-Newtonian fluids is a subject of a great scope of applications in engineering. Examples of non-Newtonian fluids are products involved in the industries of petroleum, food, polymers, cosmetics, etc. Nevertheless, the difficulties in the study of such flows are many, from the modeling of the fluids non-linear material behavior to the mathematical dealing with the constitutive equations coupled to the flow mechanical models.

The analysis of non-Newtonian flows has always been a challenge for fluid mechanists. Specially since the last decades, an extensive research field, the numerical simulation of non-Newtonian fluid flow, has been found of great interest (Crochet *et al.*, 1984; Owens and Phillips, 2002). Prediction of fluid behavior and detailed flow visualization in complex geometry, mostly not accomplishable in experimental analysis, has stimulated the research in this area.

The mathematical modeling for non-Newtonian flows eventually originates non-explicit constitutive equations for the extra-stress tensor. In the numerical standpoint, this feature is a difficulty, since it becomes unfeasible to deal with only two primal variables (velocity and pressure). It is necessary to compute the extra-stress as an additional variable, which increases the number of functional spaces comprised and also the number of degrees of freedom in the numerical model. In the context of finite element methods, the multi-field discrete model consists of the equations of momentum and mass conservation, plus a general constitutive equation for the extra-stress. In such cases, two compatibility conditions arise between the finite element subspaces for the variables: the classical Babuška-Brezzi condition for velocity and pressure subspaces, and the compatibility condition between the extra-stress and velocity subspaces.

As for the mixed formulations in two variables approach, the classical Galerkin method for incompressible fluids suffers from two major difficulties. First, the need to satisfy Babuška-Brezzi condition (Ciarlet, 1978) in order to achieve a compatible combination of velocity and pressure subspaces in mixed formulations. Further, the inherent instability of central difference schemes in approximating advective dominated equations (Brooks and Hughes, 1982).

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In the context of the Stokes equations for Newtonian fluids, the Galerkin least-squares method (GLS) (Hughes *et al.*, 1986) was developed to provide stability to the original Galerkin method by adding mesh-dependent terms to Galerkin formulation, which are functions of the residuals of Euler-Lagrange equations evaluated elementwise. Since the residuals of the Euler-Lagrange equations are satisfied by the exact solutions, consistency is preserved in these methods. This idea was also extended to incompressible Navier-Stokes equations in (Franca and Frey, 1992) employing stability parameters designed to optimize stability and convergence.

Multi-field formulations have been analyzed by several researchers, with the aim of constructing formulations that have the properties of stability and convergence, and also establishing the combinations of finite element subspaces which would be effective for each of those formulations. In the context of the Stokes problem, the three-field formulation is already a challenge, being exploited by authors as Marchal and Crochet (1987), Franca and Stenberg (1991), Baranger and Sandri (1992), Baaijens (1998) and references therein, Carneiro de Araújo and Ruas (1998) and references therein, Bonvin *et al*. (2001), among others. A four field formulation known as *Elastic Viscous Split Stress (*EVSS) is an alternative which employs the rate of strain tensor as an additional variable, in order to appease the compatibility conditions between the tensor and velocity subspaces (Guenette and Fortin, 1995; Sun *et al.*, 1996; Gatica *et al.*, 2004). Stabilized formulations based on the Galerkin lest-squares methodology have been employed to multi-field formulations with two main goals. One is the achieving of the necessary stability and convergence for both Newtonian and non-Newtonian models. The other, which may be considered as the main advantage of such methods, is the circumvention of the compatibility conditions between the functional subspaces. Some authors have been successful in achieving these goals via methodologies that may differ in the possible finite elements to employ, in the design of the stability parameter, in the GLS terms which are present in the formulation, in the sign for these terms, and also in the solution algorithms that effectively converge for such formulations. Among the main works one may cite Franca and Stenberg (1991), who give a first GLS formulation for Stokes flows in three fields; Behr *et al.* (1993), who extend the former formulation to inertial flows; and Bonvin *et al.* (2001), who present two GLS formulations for equal-order triangular elements in two dimensional flows.

In this study, we employ a GLS three-field formulation which is based on the stabilized formulations of Behr *et al.* (1993) and Bonvin *et al*. (2001). We have incorporated features of both works to come up with a GLS three-field formulation for isochoric generalized Newtonian flows. It is important to notice that, for flows of pseudoplastic fluids, even if the global Reynolds number of the flow, calculated usually with the fluid's zero-shear-rate viscosity, is low, the asymmetric inertial operator may be locally important in the flows regions where the shear rates are high. Such thing happens because in these regions the viscosity may decrease much, and the advective transport of momentum becomes dominant comparing to the diffusive one. So, for pseudoplastic flows, the stabilization of the advective operator is necessary even for mild Reynolds flows. In addition, the implementation of such formulation for the approximation of non-swirling axisymmetric flows is an innovation. We present some preliminary numerical tests for this formulation which showed good stability features and comprehensive results for lid-driven cavity flows, validating our finite element code for planar flows. For the axisymmetric case, we employ both a Newtonian constant viscosity and a viscosity function given by the Carreau model, which predicts pseudoplastic behavior to approximate flows through an axisymmetric contraction. Besides the features of convergence and stability of the GLS approximations, we find good agreement between our results and those from literature. We investigate the role of Carreau's equation parameters and Reynolds number in the flow dynamics near the contraction, interpreting the differences between velocity profiles generated by each fluid with the aid of the extra stress tensor fields, which are computed directly in this formulation.

2. Mechanical model

The mechanical modeling presented herein concerns a material body $\mathcal{B}$ for which flow is defined by the triple velocity, mass density and stress tensor fields, ($\mathbf{v}$, ρ, $\mathbf{T}$), and the associated system of contact and body forces, ($\mathbf{t}(\mathbf{n})$, $\mathbf{f}$).

Principle of Mass Conservation: The mass of a mechanical body $\mathcal{B}$ does not change with time: Mathematically, this primer principle may be stated as

$$\frac{d}{dt}\int_{\mathcal{P}} \rho dV = 0 \tag{1}$$

where ρ is the mass density, $\mathcal{P}$ is a part of a configuration $\mathcal{B}_t$ of the body $\mathcal{B}$ at the time t. Applying Reynolds transport theorem (Gurtin, 1981) to Eq.(1), and assuming an incompressible fluid model, i.e., constant ρ, a variational principle for isochoric motion may be derived:

$$\int_{\mathcal{P}} q \operatorname{div} \mathbf{v} dV = 0 \qquad \forall q \in L^2(\mathcal{B}_t) \tag{2}$$

where $\mathbf{v}$ denotes a virtual velocity field of the flow, and $L^2(\mathcal{B}_t)$, accounts for the functional space of the pressure field.

Principle of Power Expended (Gurtin, 1981): This major dynamic principle is equivalent to the laws of conservation of momentum, formulated in a variational sense. It asserts that, for any part $\mathcal{P}$, with $H^1(\mathcal{B}_t)^{nsd}$ denoting the space of virtual velocities associated to $\mathcal{B}_t$, the power expended on $\mathcal{P}$ by external body forces $\mathbf{f}$ and surface forces $\mathbf{t}(\mathbf{n})$ is equal to the stress power plus the rate of change of kinetic energy:

$$\int_{\mathcal{P}} \rho \mathbf{f} \cdot \mathbf{v}\, dV + \int_{\partial \mathcal{P}} \mathbf{t}(\mathbf{n}) \cdot \mathbf{v}\, dA = \int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D}\, dV + \int_{\mathcal{P}} \rho \dot{\mathbf{v}} \cdot \mathbf{v}\, dV \qquad \forall \mathbf{v} \in H^1(\mathcal{B}_t)^{nsd} \tag{3}$$

where $\mathbf{D}$ is the strain rate tensor, and $\mathbf{T}$ stands for a second-order symmetric tensor, the stress tensor. The first term on the right side of Eq. (3) accounts for the stress power, i.e., the power expended due to the work of internal contact forces, and the second is the rate of change of kinetic energy.

3. Material Behavior

Although Cauchy theorem (Gurtin, 1981) describes the form of contact forces for any continuous body, the way in which materials deform or flow when submitted to any dynamic condition is not stated by this theorem. Besides, the behavior of continuous bodies submitted to arbitrary conditions differs drastically, due to the material dependent relation between contact forces within the body upon its motion and deformation. This relation is described by the rheological constitutive equations, which are mathematical models for the stress tensor, $\mathbf{T}$. These equations are constructed in order to obey certain axiomatic principles (Astarita and Marrucci, 1974): determinism, local action and frame indifference. A functional dependence of $\mathbf{T}$ with the strain rate tensor, $\mathbf{D}$, is acceptable in view that this could represent a frame indifferent model, as $\mathbf{D}$ is frame indifferent (Gurtin, 1981). The most general linear relation between $\mathbf{T}$ and $\mathbf{D}$ tensors may be given as the following isotropic function:

$$\mathbf{T} = (-P + \varpi \operatorname{div} \mathbf{v})\mathbf{I} + 2\mu \mathbf{D} \tag{4}$$

where μ is the fluid viscosity and the parameter ϖ is related to the scalar function κ, called bulk coefficient of viscosity. For an incompressible fluid, the divergence of the velocity field is null, and Eq. (4) may be written as function of a mean pressure, p, the mean of the normal components of $\mathbf{T}$, as:

$$\mathbf{T} = -p\mathbf{I} + 2\mu \mathbf{D} = -p\mathbf{I} + \boldsymbol{\tau} \tag{5}$$

where $\boldsymbol{\tau}$ is the extra-stress tensor.

The practically observed phenomena of shear-thinning, viscoplasticity and shear-thickening in pure shear flows give rise to the construction of the generalized Newtonian models (Bird *et al.*, 1987). These models apply the empirical viscosity functions which fit the behavior stress versus strain rate in viscometric flows for modeling the stress tensor. They maintain the mathematical structure of a Newtonian fluid, in the form that $\mu=\eta(\dot{\gamma})$, being the non-Newtonian viscosity of the fluid, which is a function of the strain rate, $\dot{\gamma}$. For flows in general, $\dot{\gamma}$ is defined as a Frobenius norm of $\mathbf{D}$:

$$\dot{\gamma} = (2 II_{\mathbf{D}})^{1/2} = (2 \operatorname{tr} \mathbf{D}^2)^{1/2} \tag{6}$$

An example of an empirical model for the viscosity is the Carreau model (Bird *et al.*, 1987), employed to model the behavior of pseudoplastic fluids. This model is given by the following constitutive equation:

$$\frac{\eta(\dot{\gamma}) - \eta_\infty}{\eta_0 - \eta_\infty} = \left[1 + (\lambda \dot{\gamma})^2\right]^{\frac{n-1}{2}} \tag{7}$$

The Reynolds number for a generalized Newtonian fluid may be defined, for a general characteristic viscosity, η_c, which depends on the model employed, as follows,

$$\mathrm{Re} = \frac{\rho L u_0}{\eta_c} \tag{8}$$

in which L and u_0 are the characteristic length and velocity of the flow. Note that for the Newtonian model $\eta_c = \mu$.

For viscoelastic fluids, i.e., fluids with memory, the most common models are the differential constitutive equations (Astarita and Marrucci, 1974; Bird *et al.*, 1987). In general, they are given as functions of the objective derivatives (Astarita and Marrucci) of the extra stress tensor, in order to maintain the necessary feature of frame indifference. Among these viscoelastic models, two are of great importance in numerical simulation of non-Newtonian flows, due to their widespread employment: the Maxwell-B and the Oldroyd-B models, given as:

$$\boldsymbol{\tau} + \lambda \overset{\nabla}{\boldsymbol{\tau}} = 2(\mu_1 \mathbf{D} + \mu_2 \overset{\nabla}{\mathbf{D}}) \tag{9}$$

where λ and μ_i are the material functions for these models, with μ_2=0 for the Maxwell-B model. The symbol ∇ represents the upper convected derivative of the respective variable. In numerical approximations, the Oldroyd-B constitutive equation (Eq. (9)) is sometimes decomposed in the following form (Crochet and Keunings, 1982):

$$\begin{aligned} &\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 \\ &\boldsymbol{\tau}_1 + \lambda_1 \overset{\nabla}{\boldsymbol{\tau}_1} = 2\mu_1 \mathbf{D} \\ &\boldsymbol{\tau}_2 = 2\mu_2 \mathbf{D} \end{aligned} \tag{10}$$

Thus, the model is viewed as the sum of an elastic ($\boldsymbol{\tau}_1$) portion of $\boldsymbol{\tau}$ and a viscous and one ($\boldsymbol{\tau}_2$).

4. Multi-field finite element formulations

The formulation that we are about to present assumes problems defined on a bounded domain $\Omega \subset \Re^{nsd=2,3}$, with a polygonal or polyhedral boundary Γ, formed by the union of Γ_g, where Dirichlet conditions are imposed, and Γ_h, subjected to Neumann boundary conditions. As usual, $L^2(\Omega)$, $L_0^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$ stand for Hilbert and Sobolev functional spaces (Ciarlet, 1978). Finally, $\|\,,\|_0$ denotes the $L^2(\Omega)$ norm and $\|\,,\|_{0,K}$ the $L^2(\Omega_K)$ one.

Multi-field formulations are those that employ as primal variables, besides the usual velocity ($\mathbf{u}$) and pressure (p), any of the fields of extra stress ($\boldsymbol{\tau}$), strain rate ($\mathbf{D}$) or velocity gradient (grad $\mathbf{u}$). In this section we make some basic comments on mixed multi-field formulations for isochoric flows and also present the stabilized formulation employed in this study. We use the following notation: C_h is a partition of the closed domain $\overline{\Omega}$ into elements, R_k denote the polynomial spaces of degree k , and $(\cdot\,,\cdot)$ represents the L^2 inner product (Ciarlet, 1978).

The Galerkin formulation for the isochoric flow of a Newtonian fluid may be stated as: given $\mathbf{f}$, find the triple ($\boldsymbol{\tau}^h$, p^h, $\mathbf{u}^h$)$\in \mathbf{W}^h \times P^h \times \mathbf{V}^h$ such as:

$$B_1(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h; \mathbf{S}, q, \mathbf{v}) = F_1(\mathbf{S}, q, \mathbf{v}), \quad (\mathbf{S}, q, \mathbf{v}) \in \mathbf{W}^h \times \mathbf{V}^h \times P^h \tag{11}$$

where

$$\begin{aligned} &B_1(\boldsymbol{\tau}, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) = \frac{1}{2\mu}(\boldsymbol{\tau}, \mathbf{S}) - (\mathbf{D}(\mathbf{u}), \mathbf{S}) + \rho([\operatorname{grad}\mathbf{u}]\mathbf{u}, \mathbf{v}) + (p, \operatorname{div}\mathbf{v}) - (\boldsymbol{\tau}, \mathbf{D}(\mathbf{v})) + (\operatorname{div}\mathbf{u}, q) \\ &F_1(\mathbf{S}, q, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \end{aligned} \tag{12}$$

introducing the finite element subspaces:

$$\begin{aligned} &P^h = \{q \in C^0(\Omega) \cap L_0^2(\Omega) \,\big|\, q_{|K} \in R_l(K), K \in C_h\} \\ &\mathbf{V}^h = \{\mathbf{v} \in H_0^1(\Omega)^{nsd} \,\big|\, \mathbf{v}_{|K} \in R_k(K)^{nsd}, K \in C_h\} \\ &\mathbf{W}^h = \{\mathbf{S} \in C^0(\Omega)^{nsd\times nsd} \cap L^2(\Omega)^{nsd\times nsd} \,\big|\, \mathbf{S}_{ij} = \mathbf{S}_{ji}, i = 1,\ldots,nsd \,\big|\, \mathbf{S}_{|K} \in R_j(K)^{nsd\times nsd}, K \in C_h\} \end{aligned} \tag{13}$$

In the context of the Stokes problem, i.e., when the inertial term is negligible, Franca and Stenberg (1991) proposed a GLS formulation which is stable for any combinations of finite elements. They established the convergence and stability lemmas for such formulation. Behr *et al.* (1993) improved these results presenting a stabilized formulation very similar to this former, but also incorporating the inertia terms which had been neglected by Franca and Stenberg (1991). Behr *et al.* (1993) used a design of the stability parameter which incorporates the local Reynolds number and the mesh size parameter h_K, as in Franca and Frey (1992).

A mixed formulation which is largely employed (Baaijens, 1998) is the one based on a linear version of the Oldroyd-B model, using the split of Eq. (10). The model is linear in view that the parameter λ_1 in Eq. (10) is null. The Galerkin finite element formulation for such model is given as: find the triple ($\boldsymbol{\tau}_1^h$, p^h, $\mathbf{u}^h$)$\in \mathbf{W}^h \times P^h \times \mathbf{V}^h$ such as

$$B_2(\boldsymbol{\tau}_1^h, p^h, \mathbf{u}^h; \mathbf{S}, q, \mathbf{v}) = F_2(\mathbf{S}, q, \mathbf{v}), \quad (\mathbf{S}, q, \mathbf{v}) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h \tag{14}$$

with

$$B_2(\boldsymbol{\tau}_1, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) = \frac{1}{2\eta_p}(\boldsymbol{\tau}_1, \mathbf{S}) - (\mathbf{D}(\mathbf{u}), \mathbf{S}) + \rho([\mathrm{grad}\,\mathbf{u}]\mathbf{u}, \mathbf{v}) + 2\eta_s(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v})) + (\boldsymbol{\tau}_1, \mathbf{D}(\mathbf{v})) - (p, \mathrm{div}\,\mathbf{v}) + (\mathrm{div}\,\mathbf{u}, q)$$
$$F_2(\mathbf{S}, q, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \tag{15}$$

Bonvin *et al.* (2001) prove uniqueness and existence of the problem of Eq. (14), neglecting the inertial term ($\rho = 0$). They also establish the stability lemma for the finite element formulation, which depends on two compatibility conditions, one between the finite element subspaces of stress and velocity, and other between the subspaces of velocity and pressure. The former is the classical Babuška-Brezzi condition for the Stokes problem. Bonvin *et al.* (2001) derive Galerkin least-squares methods that circumvent those compatibility conditions, adding terms to the forms B_2 and F_2, which correspond to the least squares forms of the residual equations. These formulation is valid for equal-order triangular elements.

In this study, we employed a GLS formulation based on the Galerkin scheme of Eq. (14), based on the stabilized formulations of Behr *et al.* (1993) and Bonvin *et al.* (2001): Find the triple $(\boldsymbol{\tau}_1^h, p^h, \mathbf{u}^h) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h$ such as:

$$B_{GLS}(\boldsymbol{\tau}_1^h, p^h, \mathbf{u}^h; \mathbf{S}, q, \mathbf{v}) = F_{GLS}(\mathbf{S}, q, \mathbf{v}) \quad \forall\ (\mathbf{S}, q, \mathbf{v}) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h \tag{16}$$

where

$$\begin{aligned}
B_{GLS}(\boldsymbol{\tau}_1, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) &= B_2(\boldsymbol{\tau}_1, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) + \sum_{K \in C_h} \tau(\mathrm{Re}_K)([\mathrm{grad}\,\mathbf{u}]\mathbf{u} - 2\eta_s\,\mathrm{div}\,\mathbf{D}(\mathbf{u}) + \mathrm{grad}\,p - \mathrm{div}\,\boldsymbol{\tau}_1, \\
&[\mathrm{grad}\,\mathbf{v}]\mathbf{u} - 2\eta_s\,\mathrm{div}\,\mathbf{D}(\mathbf{v}) + \mathrm{grad}\,q - \mathrm{div}\,\mathbf{S})_K + 2\eta_p\beta\left(\frac{1}{2\eta_p}\boldsymbol{\tau}_1 - \mathbf{D}(\mathbf{u}), \frac{1}{2\eta_p}\mathbf{S} - \mathbf{D}(\mathbf{v})\right) \\
F(\mathbf{S}, q, \mathbf{v})_{GLS} &= F_2(\mathbf{S}, q, \mathbf{v}) + \sum_{K \in C_h} \tau(\mathrm{Re}_K)(\mathbf{f}, [\mathrm{grad}\,\mathbf{v}]\mathbf{u} - 2\eta_s\,\mathrm{div}\,\mathbf{D}(\mathbf{v}) + \mathrm{grad}\,q - \mathrm{div}\,\mathbf{S})_K
\end{aligned} \tag{17}$$

where $\tau(\mathrm{Re}_K)$ is the stability parameter, given as suggested by Franca and Frey (1992) and Behr *et al.* (1993):

$$\begin{aligned}
&\tau(\mathrm{Re}_K) = \frac{h_K}{2|\mathbf{u}|_\infty}\xi(\mathrm{Re}_K) \\
&\xi(\mathrm{Re}_K) = \begin{cases} \mathrm{Re}_K, 0 \le \mathrm{Re}_K < 1 \\ 1\ , \mathrm{Re}_K \ge 1 \end{cases} \\
&\mathrm{Re}_K = \frac{m_k |\mathbf{u}|_\infty h_K}{4\eta_p(\dot{\gamma})/\rho} \\
&m_k = \min\{1/3, 2C_k\} \\
&\sum_K h_K^2 \|\mathrm{div}\,\mathbf{T}\|_{0,K} \le C_k \|\mathbf{T}\|_0^2
\end{aligned} \tag{18}$$

REMARK 1: The differences between our formulation (Eq. (16)) and the formulation of Behr *et al.* (1993) are that ours represents a truly Galerkin least-squares formulation, in the sense that the stabilizing terms are added as the least squares forms of the residual equations; and that the term containing the solvent viscosity η_s is also present. The differences between the formulation of Bonvin et. al. (2001) and ours are the design of the stability parameter and that our formulation also accounts for inertia effects. In addition, this formulation is designed to circumvent the compatibility conditions between all functional subspaces even for quadrilateral elements, allowing the use of structured meshes.

REMARK 2: In this paper we give emphasis to the case of non-swirling axisymmetric flows. The geometric models are made in a two-dimensional manner. The Cartesian components of the frame of reference are x_1, the axial component, and x_2, the radial component. Symmetry in the angular direction of x_3 is assumed, so that all velocity and body force components in this direction are null, and so are all the derivatives of velocity, pressure and forces. Nevertheless, the third normal component of the stress and strain rate tensors may not be assumed as nulls. Being so, in two-dimensional planar problems there are six degrees of freedom per node, namely u_1, u_2, p, τ_{12}, τ_{11} and τ_{22}, and in axisymmetric problems they are seven: u_1, u_2, p, τ_{12}, τ_{11}, τ_{22} and τ_{33}.

5. Numerical results

We implemented the formulation of Eq. (16) in the finite element code named FEM, developed by our laboratory group. We present some results for 2-D isochoric flows which were obtained using meshes of quadrilateral bilinear elements for all variables ($Q_1/Q_1/Q_1$ - τ_1-p-$\mathbf{u}$). We have also obtained similar results for the combinations of elements $Q_2/Q_1/Q_2$ and $Q_1/Q_1/Q_2$, which are not shown here. To solve the resulting algebraic system of equation, we have implemented a Newton-based method (Dalquist and Bjorck, 1969).

5.1. Newtonian flow in a square lid-driven cavity

In this section, we present the results obtained with formulation of Eq. (16) for Newtonian flow in a lid-driven cavity flow (see Fig. 1 for the problem statement), using a 40x40 mesh, which the problem statement is given as in Fig. 1(a).

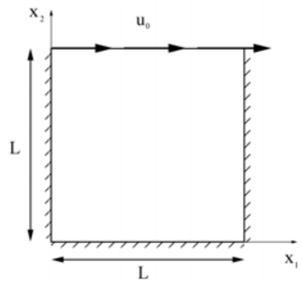
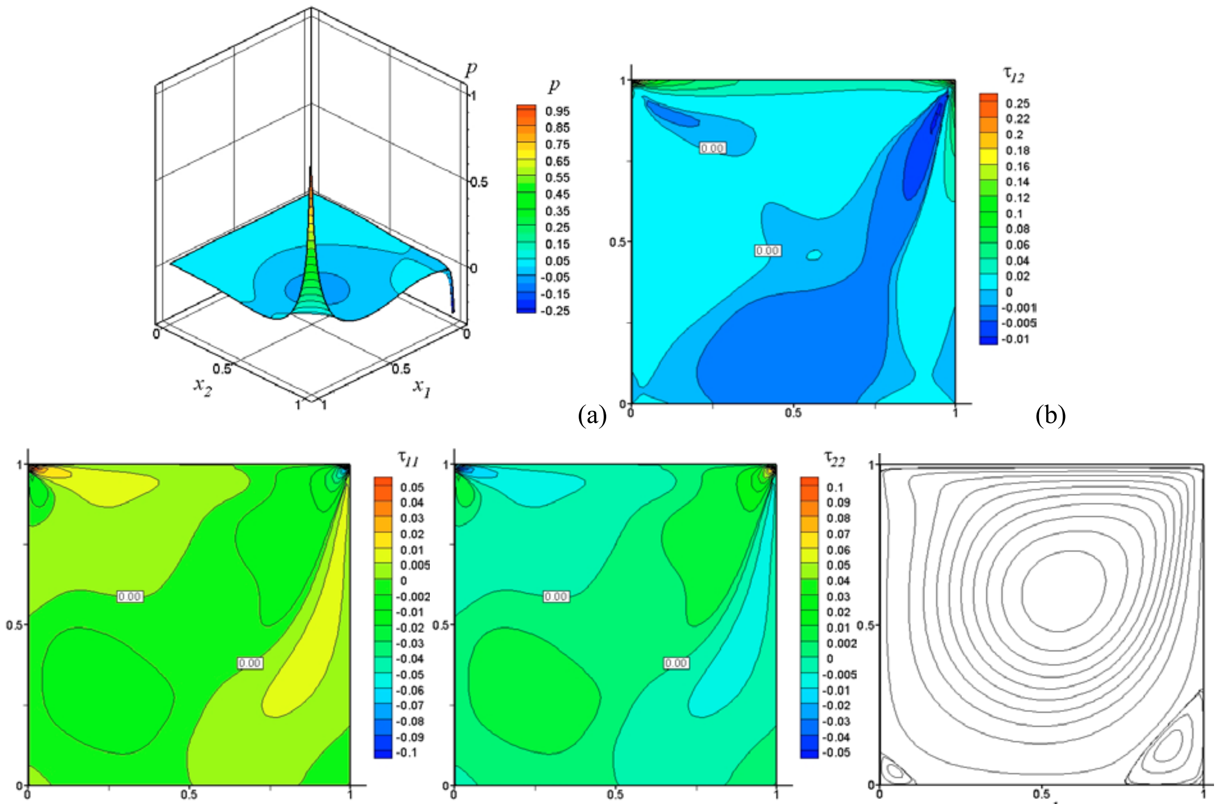


Figure 1. Problem statement for the lid-driven cavity.

We tested three different cases: Stokes flow where the Newtonian viscosity was split in two parts, $\eta_p=\eta_s=0.5$; flows with inertia $\eta_p>\eta_s$; flows with inertia with $\eta_s=0$. For the first two cases, the results were compared to those of Behr *et al.* (1993) and they were found to be in agreement (Zinani and Frey, 2005). The third case was deeply investigate in order to validate our FEM code. The Reynolds number was calculated with the constant viscosity $\eta=\mu$.

Reynolds numbers from 1 to 1000 were approximated. Here we show some results for Re=1, Re=100 and Re=400. These results are compared with some authors: Ghia *et al.* (1982), Screiber and Keller (1983), Ku and Hatziavramidis (1985), Sivaloganathan and Shaw (1988), Jurjevic (1999), for the horizontal velocity and pressure profiles in the line $x_1=0.5L$ and for the eye of vortex position. For all Reynolds number the fields of extra-stress, pressure and velocity were stable and physically comprehensive, as depicted in Figure 2.

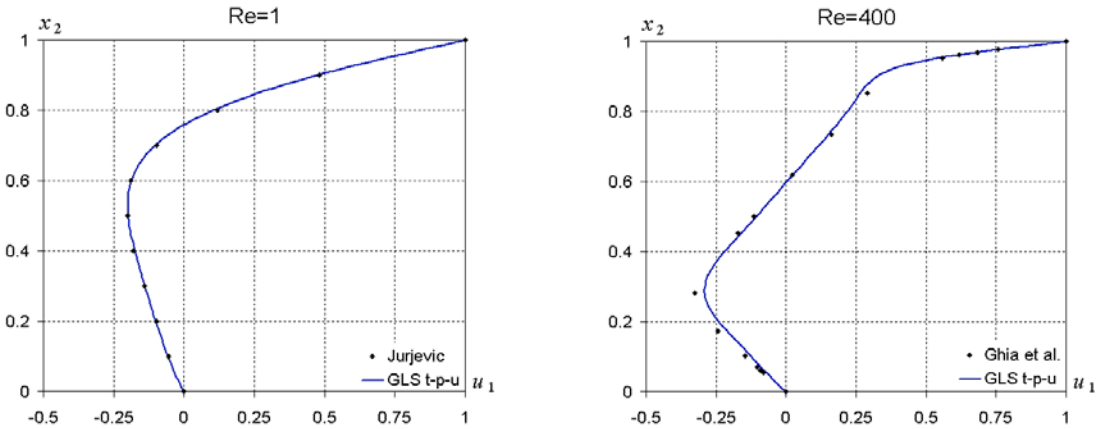


(a) (b)

(c) (d) (e)

Figure 2. (a) pressure elevation, (b) contours of τ_{12}, (c) contours of τ_{11}, (d) contours of τ_{22}, (e) streamlines

Figure 3 shows our results of the horizontal velocity u_1 versus x_2 in x_1=0.5L, for Re=1 and Re=400, comparing these results with the literature.



(a) (b)

Figure 3. Horizontal velocity versus position in x_2 (a) Re=1, (b) Re=400.

In Tab. 1 we compare the eye of the vortex position with results found in literature, for Re=100 and Re=400.

Table 1. Eye of the vortex position

Re	Reference	x_1	x_2
100	Screiber and Keller, 1983	0.61667	0.74167
	Ku and Hatziavramidis, 1985	0.62109	0.73752
	Sivaloganathan and Shaw, 1988	0.61	0.73
	FELAB τ-p-**u** GLS	**0.618**	**0.736**
400	Ku and Hatziavramidis, 1985	0.55463	0.60415
	Sivaloganathan and Shaw, 1988	0.56	0.61
	Jurjevic, 1999	0.564	0.6055
	FELAB τ-p-**u** GLS	**0.561**	**0.605**

All the numerical results for the lid-driven cavity Newtonian flow agreed with the literature not differing plus than 1% in all cases investigated.

5.2. Pseudoplastic flow through a sudden axisymmetric contraction

In this section we present results for Newtonian and non-Newtonian flows trough an axisymmetric 4:1 sudden contraction. We employed the same Galerkin Least-Squares formulation from Eq. (16), developed in a cylindrical coordinate system with null angular velocity and null gradients in the angular direction. The characteristic length of the problem is the outflow's diameter, $L=D_0$, the characteristic velocity is the outflow mean velocity, u_0, and the characteristic viscosity is η_0 for non-Newtonian flows and μ for Newtonian flows. The solvent viscosity η_s is set as zero for all cases. The polymeric viscosity function used to predict fluid's shear thinning was the Carreau model (Eq. (7)), for which we varied the parameters λ, n and η_0, setting η_∞=0. The problem statement is given as in Fig. 4, we use only half of problem's actual domain to build the geometric model.

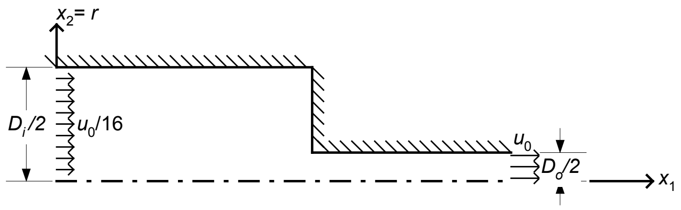


Figure 4. Problem statement: flow through an axisymmetric contraction

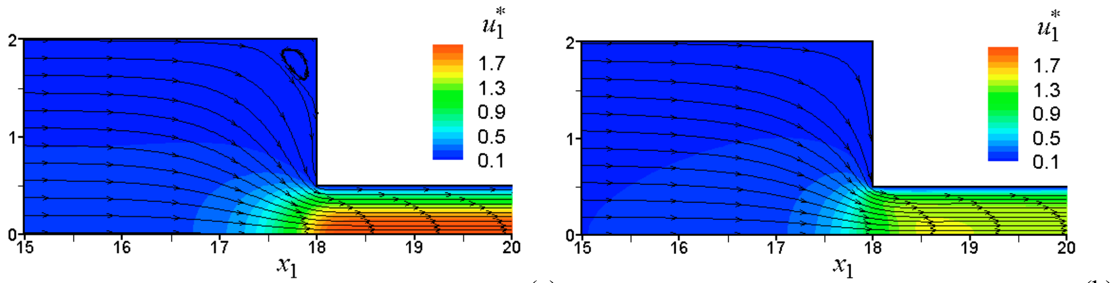
We based our investigation in the paper of Kim *et al.* (1983). In this paper, a dimensionless Carreau number, Cu, is defined in order to accomplish for the effects of varying the parameter λ:

$$\mathrm{Cu} = \frac{\lambda u_0}{D_0/2} \tag{19}$$

Kim *et al.* (1983) investigate the flows of Carreau fluid for very low Reynolds numbers, Re=0 to Re=2. In this paper we extend this investigation to a grater range of Re, varying it from Re=0 to Re=10. The Carreau numbers tested were Cu=0.01, Cu=1 and Cu=100, the former predicting the most pseudoplastic behavior because the viscosity shear-thins at lower levels of shear rates. The parameter n tested were n=0 (Newtonian), n=0.4 and n=0.2, where the former is the most pseudoplastic one. A mesh dependency analysis was performed over four meshes, comprised of 1934 to 6266 elements. The results were found to be mesh independent for any mesh from 4368 elements, and the results presented herein correspond to that 4368 finite element mesh, with 4653. In following figures, the components of the frame of reference are dimensionless as: $r^*=x_2/D_0$ and $x_1^*=x_1/D_0$.

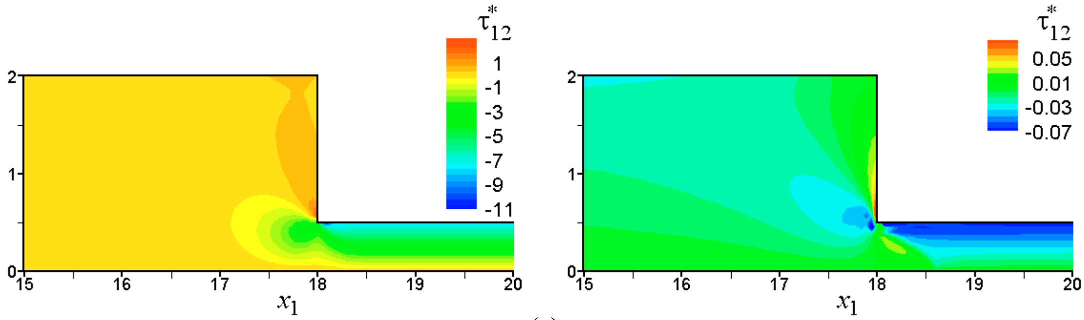
Figure 5 deals with the Carreau fluids with 5.(a) Cu=10, n=0.2 and 5.(b) Cu=100, n=0.2, both with Re=2. We show the contours of u_1^*, where $u_1^*=u_1/u_0$, superposed by the flow streamlines. The differences between the axial velocity fields are easily perceived: for lower Cu, this field is similar to that of a Newtonian fluid, in which the axial velocity increases quadratically with the distance from the wall, and experiments a smooth acceleration in the contraction. For the high Cu flow, axial velocity gradients occur close to the walls, and are almost null in the symmetry line, characterizing the flat velocity profile typically pseudoplastic. In the contraction, advective acceleration happens more harshly, because high deformation rates reduce the viscosity in this region and allow the quick acceleration of the fluid. The streamlines show that, for the low Cu, a mild recirculation appears in the contraction corner, as in Newtonian low Reynolds number flows. For the high Cu the recirculation does not occur, the behavior is similar to high Reynolds number Newtonian flows.

Figure 6 and shows the contours for the component τ_{12} (dimensionless as $\tau_{12}^*=(\tau_{12}L)/u_0)$ of the extra-stress tensor obtained as a variable of the numerical problem. Figure 7 shows the field of u_2^*, where $u_2^*=u_2/u_0$. One may notice the lower values of τ_{12} and higher values of u_2 in the contraction are obtained for the most pseudoplastic flows, due to the low viscosity that appear in regions of high deformation rates, reducing considerably the resistance to flow. High deformation rates in the contraction promote a great viscosity thinning near to the its corner, and high viscosity gradients in the same place, which may even compromise numerical stability locally, but not compromising the global results. In Fig. 8 we depict the dimensionless viscosity fields, $\eta^*=\eta/\eta_0$, obtained for those cases. These fields showed the expected features of low viscosity in the high strain rate regions, close to the walls of the most narrow duct, with such features being more pronounced in the high Cu flow.



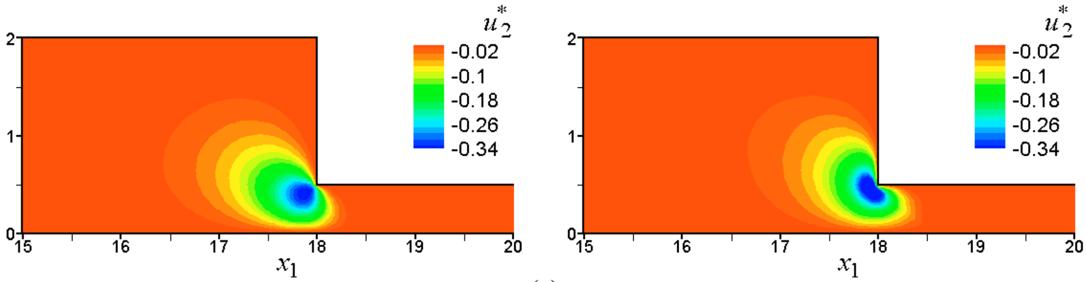
(a) (b)

Figure 5: Axial velocity contours and streamlines, Re=2, (a) Cu=10, n=0.2 and (b) Cu=100, n=0.2.



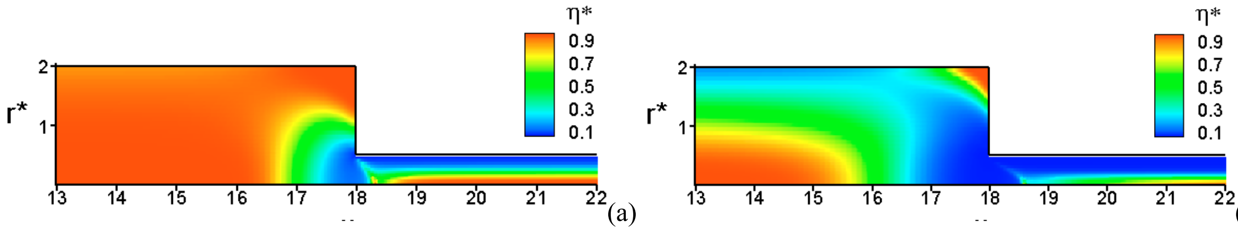
(a) (b)

Figure 6: Shear component of the extra stress tensor, Re=2, (a) Cu=10, *n*=0.2 and (b) Cu=100, *n*=0.2.



(a) (b)

Figure 7: Radial velocity contours and streamlines, Re=2, (a) Cu=10, *n*=0.2 and (b) Cu=100, *n*=0.2.

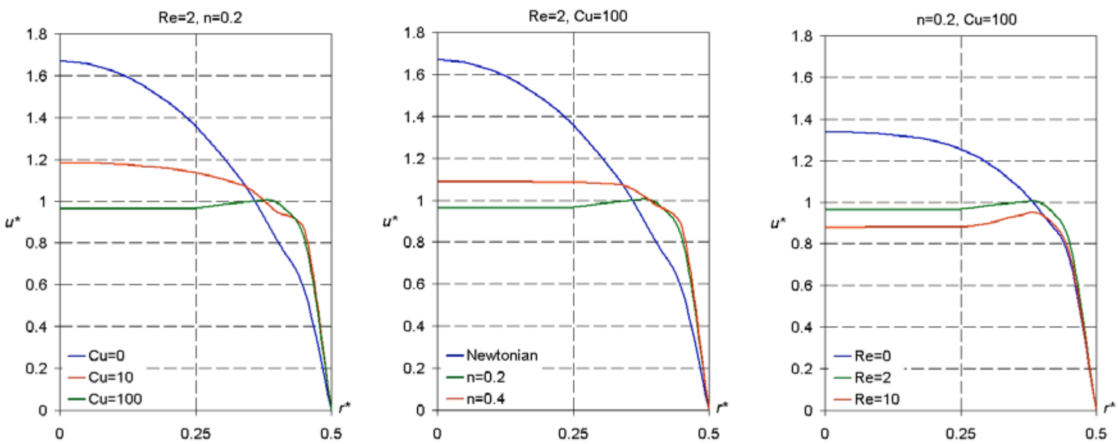


(a) (b)

Figure 8: Viscosity contours, Re=2, (a) Cu=10, *n*=0.2 and (b) Cu=100, *n*=0.2.

In the following graphics we investigate the effects of inertia, through the Reynolds number, and the fluid's material behavior, through the Carreau number, in the flow dynamics.

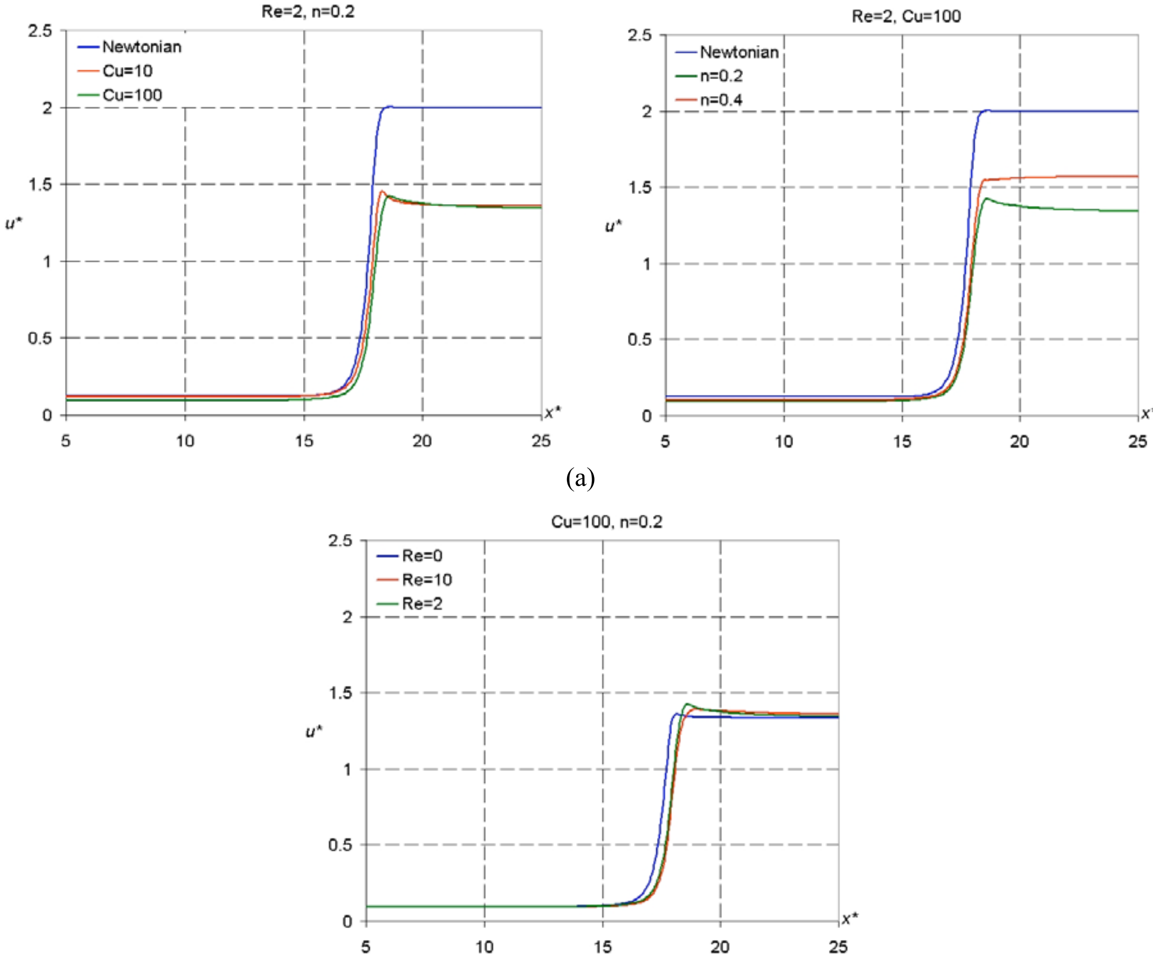
In Fig. 9 we investigate the axial velocity profile in the contraction plane. In Fig. 9.(a) we investigate different Carreau numbers for the same *n*=0.2 and Re=2, plotting this profile for Cu=0 (Newtonian), Cu=10 and Cu=100. In Fig. 9.(b) we investigate different *n* for the same Cu=100 and Re=2, *n*=0 (Newtonian), *n*=0.2 and *n*=0.4. In both cases we observe that the most shear-thinning fluid tend to form a more flattened profile. In Fig. 9.(c) we investigate different Reynolds number for Cu=100 and *n*=0.2, Re=0, Re=2, and Re=10. We observe that, as for Newtonian flows, the increase of the Reynolds number tends to flatten the velocity profile. The tendency of a non-centerline maximum velocity may be seen in the high Reynolds flow (Re=10).



(a) (b) (c)

Figure 9. Axial velocity profile in the contraction.

In Fig. 10 we investigate the axial velocity profile in the symmetry line along the domain, for the same cases as above: Fig. 10.(a) n=0.2 and Re=2, Fig. 10.(b) Cu=100 and Re=2, Fig. 10.(c) Cu=100 and n=0.2. We observe that pseudoplasticity decreases the value of the centerline axial velocity. This is understood when we remember the flattened velocity profile that occurs due to shear-thinning, different from the parabolic profile of Newtonian flows. Although, when comparing pseudoplastic fluids with the same Cu and n, in part (c), we notice that the Reynolds number does not have much influence on the value of the centerline velocity, which was expected due to the velocity profile depends only on the fluids parameters.



(a) (b) (c)

Figure 10. Axial velocity profile in the symmetry line.

In Fig. 11 we investigate the pressure drop in the symmetry line along the domain, for similar cases as above: Fig. 11.(a) n=0.2 and Re=2, Fig. 11.(b) Cu=100 and Re=2, Fig. 11.(c) Cu=100 and n=0.2. In part (a), we may observe the pressure levels of different orders of magnitude when comparing pseudoplastic to Newtonian flows. Pseudoplasticity reduces the total pressure drop. Another interesting feature is that the slope of the pseudoplastic curves differ much from the Newtonian in the region downstream the contraction, i.e., where deformation by shear is more severe and shear-thinning is more pronounced. The pressure drop in the region upstream the contraction is very similar for Newtonian and pseudoplastic fluids, because in this region the Carreau number is locally small (remember that the Carreau number was defined using the velocity and lengths from the most narrow duct). In part (b) the pressure drops for the high Cu fluids are shown, and we observe that the levels of pressure fall much lower than the Newtonian fluid flowing with Re=2, and that they are even lower for the most shear-thinning fluid, with n=0.2. In part (c) the total pressure drop is compared for two equally pseudoplastic fluids flowing with different Reynolds numbers, showing that the increase of Re reduces the pressure drop as expected and similarly to Newtonian flows.

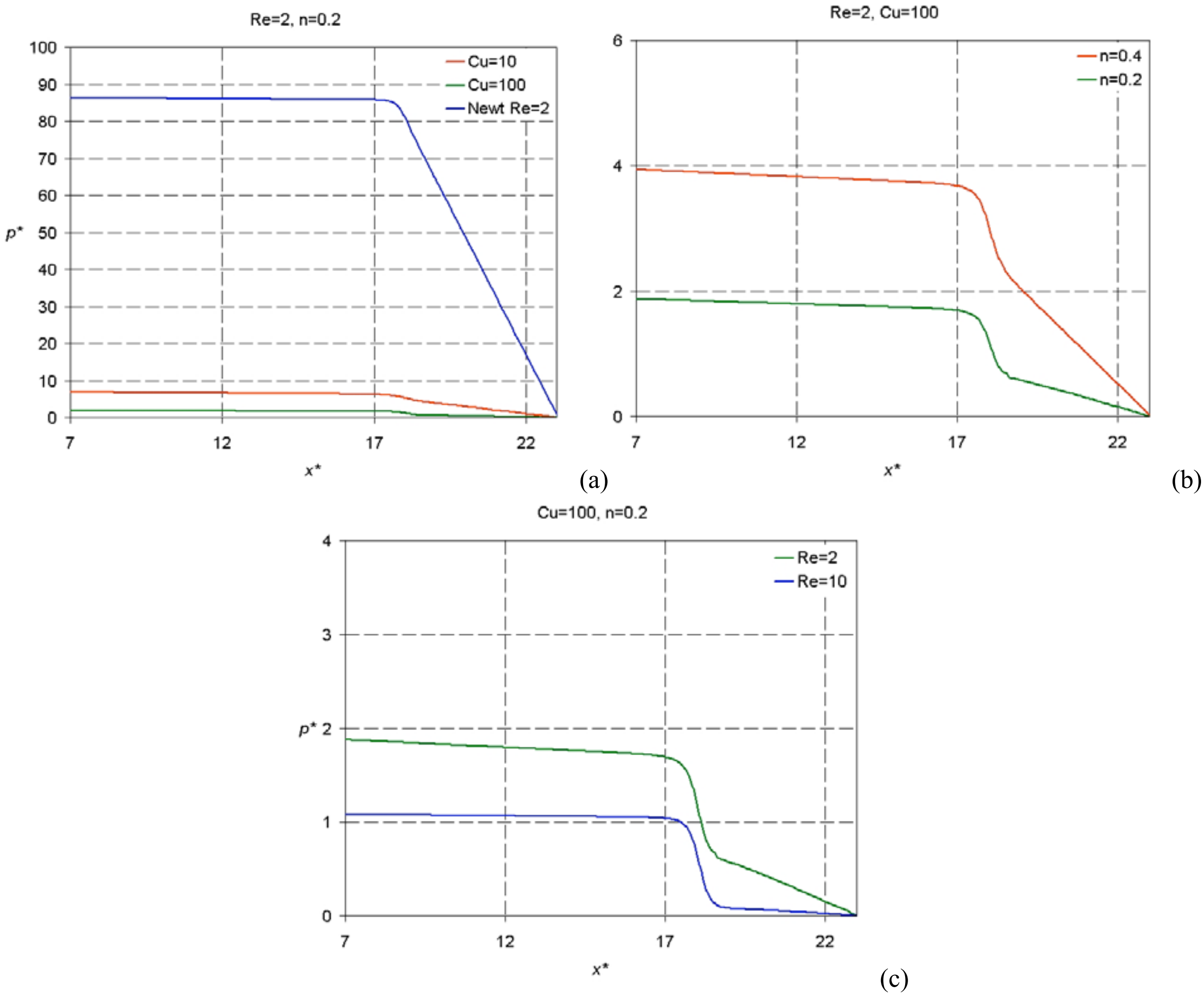


Figure 11. Pressure profile in the symmetry line.

6. Final remarks

We have presented some comprehensive results for Newtonian and generalized Newtonian flows using a three-field finite element formulation stabilized by a GLS scheme. We employed that formulation to approximate non-swirling flows in an axisymmetric domain. We used equal-order finite elements, which were able to circumvent the Babuška-Brezzi conditions required in the classical Galerkin method. We studied the features of pseudoplastic fluids flowing through an axisymmetric contraction, investigating the effects of shear-thinning in the flow dynamics. We found that the main effects that occur due to viscosity reduction are the flattening of the axial velocity profile in the contraction plane and the friction decrease, features that may be accomplished by increasing the Reynolds or Carreau numbers, or by decreasing the n parameter in the fluid model. This study is still in progress and we intend to proceed on investigations in the mathematical analysis of the formulation presented, and in the extension of this GLS formulation to non-linear viscoelastic models as Maxwell-B and Oldroyd-B.

7. Acknowledgements

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