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A GENERALIZED FORMULATION FOR THE FALKNER SKAN EQUATION

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Abstract. *The Falkne-Skan similar equation represents one of the greatest successes of the boundary layer theory for the laminar flow case. The domain of validity of this equation, however, is restricted to the region far from the leading edge and to very small pressure gradients.*

In the present work a generalized formulation of the boundary layer theory is used to derive an extended version of the Falkner-Skan equation. This generalized Falkner-Skan equation (GFS) is given by a quasi-similar equation, which incorporates the stream wise coordinate non-similar information, but keeping the ordinary differential equation characteristic. It is shown that the GFS can be used to describe the flow for Reynolds numbers as low as one, and for strong adverse pressure gradients.

Keywords: *Falkne-Skan,Kaplun Limits, Separation*

1. Introduction

The Falkner-Skan equation constitutes one of the classical results of the Prandt's boundary layer theory. The variety of applications and the importance of the Falkner-Skan equation for the understanding of the physical features of the laminar flow, submitted to an strong favorable pressure gradient, have motivate many recent works, most related to the numerical nature of it's solution (Schlichting, 1972).

Being a direct consequence of the classical boundary layer theory, the FSE has a restricted domain of validity. That limitation does not permit the description of the separated flow or the flow near the leading edge ($1<Re_x<1000$) using the similarity FSE approach. The above mentioned restriction can be overcome if a more general boundary layer formulation is used to derive an extended version of the FSE.

In the present work the generalized boundary layer theory is used to obtain a expanded formulation of the FSE. The procedure is similar to one used for the deduction of the classical FSE. After the introduction of a set of similar variables into the generalized boundary layer equation, a quasi-similar equation is obtained. That equation contains the classical FSE as a particular case.

The new generalized Falkner-Skan equation is numerically solved and the results show that the domain of validity of the GFSE can be extended not only to the near leading edge region, but also to the separating flow region. This fact represents a great advantage over the FSE, since the last is not valid for large adverse pressure gradients, witch causes the flow separation. The GFSE is used to correlated the laminar separation point of diffusers, to the imposed pressure gradient (or the diverging angle of the diffuser), indicating that the GFSE can be used as a non expensive simple tool for the project of industrial equipment.

2. The Generalized Boundary Layer Equation

The concept of Kaplun limits (1967) is used to determinate the asymptotic behavior of the Navier-Stokes equation as $Re \to \infty$. The necessary mathematical framework to obtain the high Reynolds number asymptotic behavior of the Navier-Stokes is exhaustively discussed in Cruz (2002) and Silva Freire (1999) here just some of the principal steps are presented. For a laminar incompressible, stationary and two-dimensional flow of a newtonian fluid the continuity and the momentum equations can be written as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{\text{Re}}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{\text{Re}}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \tag{3}$$

In the above equations the variables are made non-dimensional using a characteristic length and a characteristic velocity of the flow. The parameters u and v represent the non-dimensional velocities on the x and y directions respectively and P is the non-dimensional pressure.

The parameter Re represents the Reynolds number which is assumed to be large i.e. (1/Re<<1)

The intermediate variables ere defined as:

$$\hat{y} = \frac{y}{\eta(\varepsilon)} \tag{4}$$

$$\hat{v} = \frac{v}{\eta(\varepsilon)} \tag{5}$$

where $\varepsilon = \frac{1}{\text{Re}}$.

The insertion of Eq. (4) and (5) into Eq. (1), (2) and (3) results in:

$$\frac{\partial u}{\partial x} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{6}$$

$$u\frac{\partial u}{\partial x} + \hat{v}\frac{\partial u}{\partial \hat{y}} = -\frac{\partial P}{\partial x} + \frac{1}{\text{Re}}\left[\frac{\partial^2 u}{\partial x^2} + \frac{1}{\eta(\varepsilon)^2}\frac{\partial^2 u}{\partial \hat{y}^2}\right] \tag{7}$$

$$\eta(\varepsilon)\, u\frac{\partial \hat{v}}{\partial x} + \eta(\varepsilon)\,\hat{v}\frac{\partial \hat{v}}{\partial y} = -\frac{1}{\eta(\varepsilon)}\frac{\partial P}{\partial \hat{y}} + \frac{1}{\text{Re}}\left[\eta(\varepsilon)\frac{\partial^2 \hat{v}}{\partial x^2} + \frac{1}{\eta(\varepsilon)}\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right] \tag{8}$$

Applying the η-limit onto Eq. (7) and (8) respectively, it is obtained:

For the momentum equation on x-direction:

$$ord(\eta) = ord(1) : \quad u\frac{\partial u}{\partial x} + \hat{v}\frac{\partial u}{\partial \hat{y}} = -\frac{\partial P}{\partial x} \tag{9}$$

$$ord(1) > ord(\eta) > ord\left(\sqrt{\varepsilon}\right) : \quad u\frac{\partial u}{\partial x} + \hat{v}\frac{\partial u}{\partial \hat{y}} = -\frac{\partial P}{\partial x} \tag{10}$$

$$ord(\eta) = ord\left(\sqrt{\varepsilon}\right) : \quad u\frac{\partial u}{\partial x} + \hat{v}\frac{\partial u}{\partial \hat{y}} = -\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial \hat{y}^2} \tag{11}$$

$$ord(\eta) < ord\left(\sqrt{\varepsilon}\right) : \quad \frac{\partial^2 u}{\partial \hat{y}^2} = 0 \tag{12}$$

For the momentum equation in the y-direction:

$$ord(\eta) = ord(1) \; : \; u \frac{\partial \hat{v}}{\partial x} + \hat{v} \frac{\partial \hat{v}}{\partial y} = - \frac{\partial P}{\partial \hat{y}} \tag{13}$$

$$ord(\eta) < ord(1) \; : \; \frac{\partial P}{\partial \hat{y}} = 0 \tag{14}$$

In each of the above two sets of differential Eq. (9) to (12) and (13) to (14) there is only one main equation according to Kaplun's definition. Equation (11) represents the main equation for the *x* momentum equation and Eq. (13) is the main equation for the *y* momentum equation. It should be noted that the terminology "main", is related to the fact that the above mentioned equations exhibit some specific characteristics. In both cases, the Eq. (11) and (13) contains the other equations and are not contained by any other of the remaining expressions (Cruz, 2002). This fact indicates that in the limit as Re→ ∞ the behavior Navier-Stokes equations is adequately described by the following set of equations

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial P}{\partial x} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u}{\partial y^2} \tag{15}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial P}{\partial y} \tag{16}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{17}$$

The Eq. (15) to (16) represent a generalized form of the boundary layer theory which combines the Euler inviscid flow equations and the Prandtl classical boundary layer formulation into a single and more widespread formulation.

3. Flow past a wedge and the quasi-similar Falkner-Skan equation

The classical approach to describe the flow past a wedge problem consists of the use of the Falkner-Skan equation, and its solutions were investigated in detail by Hartree (1937). An important characteristic of that equation, corresponds to the fact that the potential flow is proportional to a power of the length coordinate measured from the stagnation point, i. e. for

$$U(x) = u_1 x^m \tag{18}$$

The transformation of the independents variables *y* and *x*, which leads to an ordinary equations, is:

$$\tilde{\eta} = y \sqrt{\frac{(m+1)}{2} \frac{U}{\nu x}} = y \sqrt{\frac{(m+1)}{2} \frac{u_1}{\nu}} x^{\frac{m-1}{2}} \tag{19}$$

The stream function and the velocity components are:

$$\Psi(x,y) = \sqrt{\frac{2}{m+1}} \sqrt{\nu u_1} \, x^{\frac{m+1}{2}} f(\eta) \tag{20}$$

$$u = u_1 x^m f'(\eta) = U f'(\eta) \tag{21}$$

$$v = -\sqrt{\frac{m+1}{2} \nu u_1 x^{m-1}} \left\{ f + \frac{m-1}{m+1} \eta f' \right\} \tag{22}$$

Introduction these values into the equation of motion (16) and dividing by $m u_1 x^{2m-1}$, it is obtained:

$$m = \frac{\beta}{2-\beta} \tag{23}$$

$$\beta = \frac{2m}{m+1} \tag{24}$$

Using the stream function and of the Eq. (15) to (17) can be rewritten and the resulting transformed equation is obtained:

$$\begin{aligned}
&\left[-F - \frac{(m-1)\eta F'}{m+1}\right]\left[\frac{1}{2}(m+1)F''' + \frac{(m-1)mF' - \left(m-\frac{3}{4}\right)\eta F'' + \frac{1}{4}\eta^2 F'''}{\xi_0}\right] + \\
&F'\left\{\frac{1}{2}(m+1)\left[\left(1+\frac{2(m-1)}{m+1}\right)F'' + \frac{(m-1)\eta F'''}{m+1}\right] - \frac{1}{\xi_0}\left\{\frac{1}{4}F(m-3)(m-1) + \right.\right. \\
&\left[\frac{(m-3)(m-1)^2\eta}{4(m+1)} - \frac{1}{2}(m-1)\left(1+\frac{m-1}{m+1}\right)\eta + \frac{1}{4}\left(1+\frac{m-1}{m+1}\right)\eta^2\right]F'' + \\
&\left.\left.\left[\frac{1}{2}(m-1)\eta^2 + \frac{1}{4}\left(1+\frac{2(m-1)}{m+1}\right)\eta^2 + \frac{\eta^3}{4}\right]F'' + \frac{(m-1)\eta^3 F'''}{4(m+1)}\right\}\right\} = \frac{1}{2}(m+1)F^{IV}
\end{aligned} \tag{25}$$

Where,

$$F^{(j)} = \frac{\partial^{(j)} F}{\partial\, \tilde{\eta}^{(j)}} \tag{26}$$

$$\xi_0 = \mathrm{Re}_x \tag{27}$$

Where $\tilde{\eta}_\infty$ represents a value of η assumed to be far enough of the solid wall and Re_x is the local Reynolds number. The local Reynolds number represents the non dimensional distance from the leading edge

Equation (25) must be solved according to the following boundary conditions:

$$F'(0,\xi_0) = 0 \tag{28}$$

$$F(0,\xi_0) = 0 \tag{29}$$

$$F'(\tilde{\eta}_\infty,\xi_0) = 1 \tag{30}$$

$$F''(\tilde{\eta}_\infty,\xi_0) = 0 \tag{31}$$

4. Results

Figures 1 to 5 show the velocity profiles obtained from the solution of Eq. (25), for various values of the local Reynolds number. Near the leading edge the results clearly show the overshot of the velocity profiles near the wall, this phenomenon is caused by the abrupt change of the velocity form the undisturbed free stream flow to the flow submitted to the no slip condition. It is important to note that the classical boundary layer theory cannot predict the flow near the leading edge since, the momentum equation perpendicular to the solid surface is not considered in the theory. It is also important to mention that far from the leading edge, the velocity profile assumes the classical FSE shape. It can be show that Eq (25) contains the FSE as a particular case in the limit as $\mathrm{Re}_x \to \infty$ as show below:

$$\lim_{\xi_0\to\infty}\left\{\begin{array}{l}\left[-F-\dfrac{(m-1)\eta F'}{m+1}\right]\left[\dfrac{1}{2}(m+1)F'''+\dfrac{(m-1)mF'-\left(m-\dfrac{3}{4}\right)\eta F''+\dfrac{1}{4}\eta^2 F'''}{\xi_0}\right]+\\ F'\left\{\dfrac{1}{2}(m+1)\left[\left(1+\dfrac{2(m-1)}{m+1}\right)F''+\dfrac{(m-1)\eta F'''}{m+1}\right]-\dfrac{1}{\xi_0}\left\{\dfrac{1}{4}F(m-3)(m-1)+\right.\right.\\ \left[\dfrac{(m-3)(m-1)^2\eta}{4(m+1)}-\dfrac{1}{2}(m-1)\left(1+\dfrac{m-1}{m+1}\right)\eta+\dfrac{1}{4}\left(1+\dfrac{m-1}{m+1}\right)\eta^2\right]F''+\\ \left.\left.\left[\dfrac{1}{2}(m-1)\eta^2+\dfrac{1}{4}\left(1+\dfrac{2(m-1)}{m+1}\right)\eta^2+\dfrac{\eta^3}{4}\right]F''+\dfrac{(m-1)\eta^3 F'''}{4(m+1)}\right\}\right\}\end{array}\right\}=\lim_{\xi_0\to\infty}\left[\frac{1}{2}(m+1)F^{IV}\right]$$

Resulting

$$-FF''+\left(\frac{3m-1}{m+1}\right)F'F''=F^{IV} \tag{32}$$

Integrating the Eq. (32),

$$F'''+FF''-\beta\left(F'\right)^2+C=0 \tag{33}$$

Where C is the integration constant. The boundary conditions (28) to (29), furnishes $C=\beta$. The FSE can now be obtained as follows:

$$F'''+FF''+\beta\left[1-\left(F'\right)^2\right]=0 \tag{34}$$

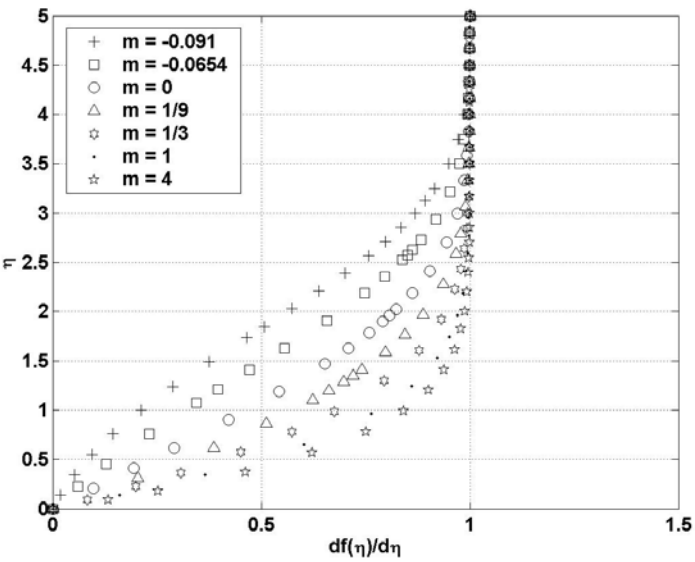

Figure 1 – The nondimensional velocity profiles assumes the classical FSE shape in the limit as $\mathrm{Re}_x \to \infty$. In this case, result for $\mathrm{Re}_x = 10^9$.

The boundary layer flow subjected an adverse pressure gradient are presented in Fig. (2) and (3). Figure (3) show the FSE limiting case (*m* = -0.091) it is clear that the present formulation can predict the whole flow region. An interesting feature of the present formulation can be observed in Fig. (2), it is show that the Eq. (25) can adequately predict the flow submitted to an adverse pressure gradient that can cause flow separation. This fact is possible due to the quasi-similar character of the proposed Eq. (25) that permits a changeable velocity profile witch can take into account not only the effects of the leading edge but also the influence of the strong pressure gradients and/or separation. Equation (25) can easily be used to determinate the separation point of a diffuser and to calculate its shape for engineering ends.

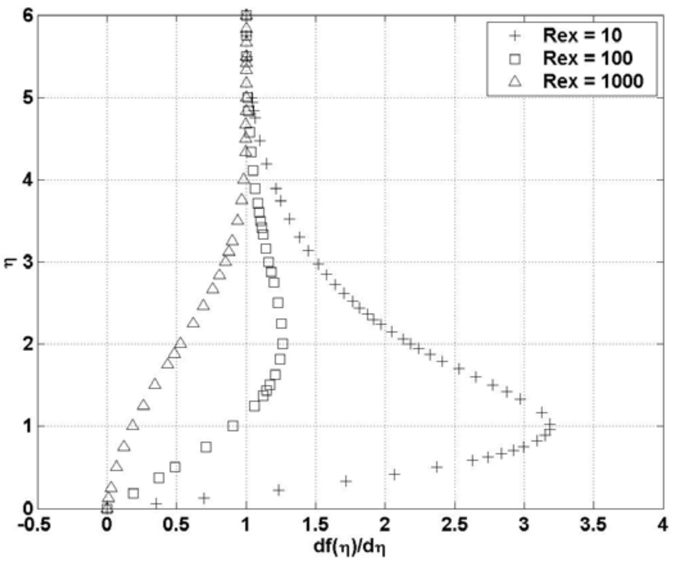

Figure 2 – Nondimensional velocity profiles, result for m = -0.1.

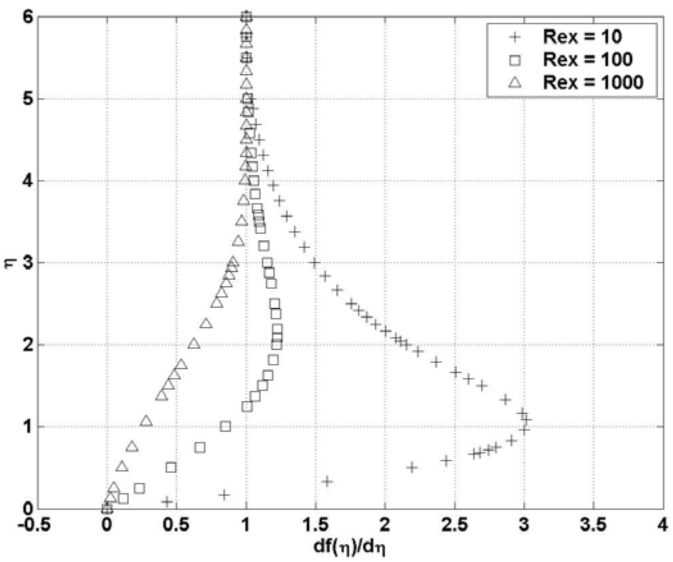

Figure 3 – Nondimensional velocity profiles, Result for m = -0.091.

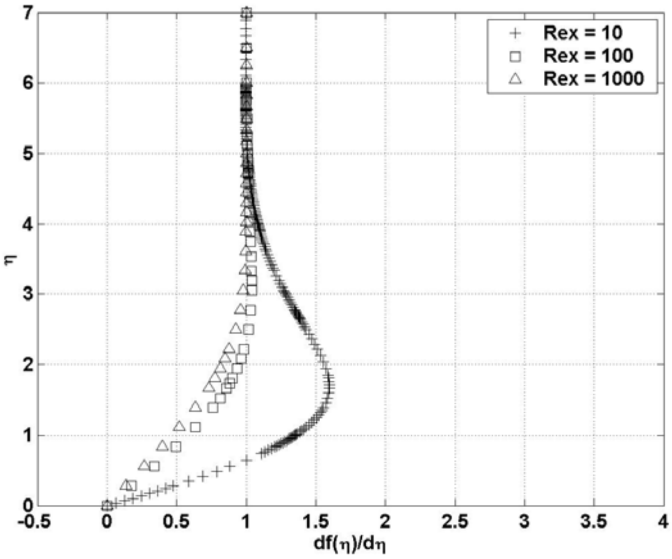

Figure 4 – Nondimensional velocity profiles, Result for m = 0.0.

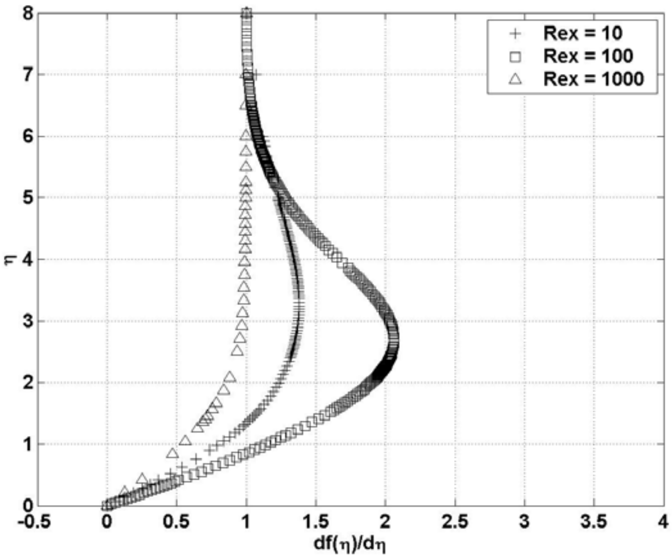

Figure 5 – Nondimensional velocity profiles, Result for m = 0.1.

5. Conclusion

In this work the asymptotic behavior of the Navier-Stokes equation was analyzed using the Kaplun limits-intermediate variable technique. A set of partial differential equations was obtained, which represents a generalization of the classical Boundary Layer Theory. The equations developed here represent a self contained theory, making unnecessary any type of viscid-inviscid interactive process. A quasi-similar equation for the flow over a flat plate was developed witch contains the Falner-Skan formulation as a particular case. A numerical solution of the quasi-similar equation was presented, showing some characteristics behaviors of the numerical solutions of the Navier-Stokes equation for the flow over a flat plat and other geometries.

The deduction of the equations set (15) to (17) is the central result of the present work. The Kaplun limits approach was used to determine the asymptotic behavior of a set of partial differential equations, resulting in a generalized Boundary Layer formulation.

The main difference between the classical Boundary Layer formulation and the approach used here is that in the later, the central focus of the analysis is to describe the near wall asymptotic behavior of the flow and not exactly, the asymptotic behavior of Navier Stokes equation, at the fluid region as a whole. The use of Kaplun limits permitted to obtain the principal asymptotic equation for each component of the velocity and to compose a "principal set" set of equations witch are in fact, the combination of two well known formulations (the nonlinear inviscid flow and the Boundary Layer) into a single more general theory, witch describes the asymptotic solution of the Navier-Stoke equations for the entire flow region.

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