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ANALYSIS OF DISPERSION ERRORS IN ACOUSTIC WAVE SIMULATIONS

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Abstract. *The governing equations of the acoustic problem are the compressible Euler equations. The discretization of these equations have to ensure that the acoustic waves are transported with non-dispersive and non-dissipative characteristics. In the present study numerical simulations of a standing acoustic wave were performed. Four different space discretization schemes were tested, namely, a 2^{nd} order finite-differences, a 4^{th} order finite-differences, a 4^{th} order finite-differences compact scheme and a 6^{th} order finite-differences compact scheme. The temporal integration was done with a 4^{th}th order Runge-Kutta scheme. The results obtained were compared with linearized analytical solutions. The influence of the dispersion on the simulation of a standing wave was analyzed. The results confirm that high order accuracy schemes can be more efficient for simulation of acoustic waves, especially the waves with high frequency.*

keywords: *Acoustic wave, dispersion error, compressible flows, standing wave, finite-difference method, high order methods.*

1. Introduction

The engineering research and design requirements of today pose great challenges in computer simulation to engineers and scientists who are called on to analise phenomena in continuum mechanics. The research area of Computational Aeroacoustics (CAA) is an emerging research area. Its major objective is to help the aerospace industry to drastically reduce aircraft noise. There is also an increased interest in other areas such as automobile noise control. As an extension of Computational Fluid Dynamics (CFD), CAA inherits up to date achievements of CFD, nevertheless, it differs from the traditional CFD research in certain ways. Numerical simulations of these phenomena call for computational methods better suited for accurate and efficient calculation of large scale time dependent problems; and most importantly, a better understanding of the influence of the numerical method over the entire physical wavenumber space is indispensable. This involves improvements in design and analysis of both the basic discretization and numerical boundary conditions. One of the challenges in the CAA area is the accurate calculation of the phenomena, especially in the presence of shock waves. The CAA requires numerical schemes of high accuracy, low dispersion and almost non dissipation. The need for high resolution discretization methods is a well known factor in CAA. This is because the numerical study of aeroacoustic requires the correct representation of a large range of spatial and time scales.

In the current work the acoustic waves phenomena was adopted. A very good theory of acoustic waves can be found in Morse and Ingard, 1968.

Tam and Webb, 1992, studied dispersion-relation-preserving finite difference schemes for computational acoustic. Linearized Euler equations were adopted for the numerical model. They emphasize that in uniform mean flow, one has to assure that these partial differential equations support three types of waves: the acoustic, the entropy and the vorticity waves. All three waves are non-dispersive and non-dissipative. The difference between them is that, the acoustic waves propagate with the speed of sound, and the others propagates with the speed of the main flow. Because of this characteristic, one has to assure that the numerical results in acoustic simulation are free of dispersion and dissipation. In order to obtain these results, Tam and Webb, 1992 developed a finite difference approximation in a way that the Fourier transform is preserved. They obtained an optimized fourth-order explicit approximation with a stencil of 7 points. It was shown for these kind of study that the obtained approximation is better that explicit sixth order approximation, for these kind of study. They also optimized the time discretization, and showed outflow boundary conditions that are 'transparent' to the outgoing disturbances. The results obtained by them were very good, showing that non-dispersive methods are important for numerical acoustics.

Vanhille and Pozuelo, 2000, simulate a finite but moderate amplitude standing acoustic wave, using Lagrangian coordinates. In their numerical model a third order partial derivative was obtained. For this derivative a finite-difference scheme of fifth order of truncation error was developed, since the role of this derivative was very important for the formation of the nonlinear standing wave. Their numerical method was validated by comparison with an analytical model. Their results showed the efficiency and the limits of the developed code.

A semi-implicit method for acoustic waves in low Mach number simulations is presented in Wall et al., 2002. The advantage of their proposed method is that the time step is limited only by the convective CFL condition. Their method is 2^{nd} order accurate, both in time and space. An analysis of their results showed that the waves simulated had an average dispersion error of 5%. This was considered by them as not an excessive dispersion error. Their main result is on the gain in computational efficiency, obtained with the semi-implicit method, resulting in a factor of 15 reduction about, as compared with an explicit method.

Spectral methods can be used to assure that all relevant scales are captured, but high order finite difference are also able to represent short length scales with good accuracy. Lele, 1992, emphasizes the importance of using high order methods schemes for first and second derivatives. Mahesh, 1998, presents high order finite difference schemes, introducing a method that, using the same stencil is more accurate than the standard Padé schemes. The disadvantage of his method is that it requires the solution of first and second derivatives simultaneously. Souza et al., 2002a; Souza et al., 2002b, used high order compact methods for transition phenomen problems. In these studies it was studied the propagation of the Tollmien-Schlichting waves in incompressible flows.

Most sound waves behave as linear waves since they produce pressure fluctuations in air that are very small. A linear waves travels through a medium such as air or water. Fluids such as these can be thought of as consisting of a large number of "particles", each of which consists of a vast number of molecules. Each of these particles moves as the wave travels through and it passes the disturbance on to its neighbors. However, these small parts of the medium do not travel with the wave. Waves transfer energy without transferring matter.

In the current work, the focus is on the evaluation of discretization error. The tests involved the simulation of one dimensional standing wave in a periodic domain. Standing wave may be created from two waves, with equal frequency, amplitude and wavelength, traveling in opposite directions. Using superposition, the resultant wave is the sum of these two waves.

The discretization error was analyzed and tested for different space discretization schemes, namely, a 2^{nd} order finite-differences, a 4^{th} order finite-differences, a 4^{th} order finite-differences compact scheme and a 6^{th} order finite-differences compact scheme. Both centered and non centered schemes were analyzed.

The paper is organized as follows: in section 2 the formulation for the standing wave is shown. The equations adopted are the Euler equation. The numerical method adopted is shown in section 3. In the same section an analysis of the spatial discretization of the finite difference methods used is done. In section 4 numerical results for various test cases are presented. The conclusions about the discretization errors on Computational Aeroacoustics are shown in the last section.

2. Formulation

In the current study, the governing equations are the compressible, isentropic, one-dimensional Euler equations. They consist of the momentum equations for the velocity component (u) in the streamwise direction (x):

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial}{\partial x}\left(\rho u^2 + p\right), \tag{1}$$

and the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x}, \tag{2}$$

where t is the time, ρ is the density and p is the pressure. Finally p and ρ satisfy the idea law gas for a isentropic flow (Sullivan, 1981):

$$p = \left(\frac{\bar{p}_\infty}{\bar{\rho}_\infty \bar{U}_\infty^2} \right) \rho^\gamma, \tag{3}$$

where $\gamma = c_p/c_v$ is the specific heat ratio. The variables used in the above equations are non-dimensional. They are related to the dimensional variables by:

$$x = \frac{\bar{x}}{\bar{L}}, \quad t = \frac{\bar{t}\bar{U}_\infty}{\bar{L}}, \quad u = \frac{\bar{u}}{\bar{U}_\infty}, \quad \rho = \frac{\bar{\rho}}{\bar{\rho}_\infty}, \quad p = \frac{\bar{p}}{\bar{\rho}_\infty \bar{U}_\infty^2},$$

where the terms with an over-bar are dimensional terms, $\bar{L}$ is the reference length, $\bar{U}_\infty$ is the free-stream velocity and $\bar{\rho}_\infty$ is the density of the undisturbed flow.

We can also decomposed the flow in a temporal mean with a small disturbance:

$$u(x,t) = u_0 + u', \tag{4}$$
$$\rho(x,t) = \rho_0 + \rho', \tag{5}$$
$$p(x,t) = p_0 + p', \tag{6}$$

where the index ($_0$) indicates the temporal mean flow and ($'$) indicates of the small disturbance.

3. Numerical Method

The Eqs.(1) and (2) are solved numerically using a uniform grid. The number of points used in each simulation depended on the discretization method adopted. Therefore, a efficient selection of the grid points numbers and the accuracy of a code can be evaluated by comparison with analytical solution. Finding general analytical solutions of the Euler equations is not easy due to its non linearity. Because of this, the waves considered had small amplitudes. Taking this into account the equations were be linearized to obtain an analytical solution. This assumption is realistic if the fluctuations are small. The temporary mean flow adopted was null ($u_0 = 0$). With these two assumptions one can obtain the following solutions of the Euler equations (1 and 2):

$$u'(x,t) = A\,\sin(\alpha x - \omega t) + A\,\sin(\alpha x + \omega t), \tag{7}$$
$$\rho'(x,t) = \frac{\rho_0}{c}\left[u' - 2\,A\,\sin(\alpha x + \omega t)\right], \tag{8}$$
$$p'(x,t) = \left(\frac{p_0}{\rho_0^\gamma} \right) \rho^\gamma, \tag{9}$$

where α is the disturbance wavenumber in the x-direction, ω is the disturbance frequency and A is the disturbance amplitude.

The acoustics wave propagation develops from specified initial conditions. In the present computations the analytical solutions at time equal zero were used as initial conditions. These conditions were:

$$u'(x,t) = 2Asin(\alpha x) \tag{10}$$
$$\rho'(x,t) = 0.0 \tag{11}$$
$$p'(x,t) = 0.0 \tag{12}$$

The time-advance of the computational variables (ρ and u) was obtained by a 4^{th} order Runge-Kutta method. The right-hand side of the equations (1) and (2) require the evaluation of spatial derivatives. A periodic boundary was used at the inlet and the outlet of the domain. For the spatial derivatives, different schemes were used. Bellow the discretization used for each method is present. The letter i represents the grid position in x-direction, which varies from 0 to N.

2^{nd} order explicit derivatives for $0 < \mathrm{i} < \mathrm{N}$:

$$f_i' = \frac{1}{2dx}(-f_{i-1} + f_{i+1}) \tag{13}$$

4^{th} order explicit derivatives for 0 < i < N:

$$f'_i = \frac{1}{12dx^2}(f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}) \quad (14)$$

4^{th} order implicit (compact) derivatives for 0 < i < N. To find the values of implicit derivatives a matrix must be solved, where all derivatives in a grid line are solved simultaneously. The system can be obtained using the following equation:

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{dx}(-f_{i-1} + f_{i+1}) \quad (15)$$

6^{th} order implicit (compact) derivatives for 0 < i < N. Here the derivatives are calculated as in the last scheme, solving a system composed by:

$$f'_{i-1} + 3f'_i + f'_{i+1} = \frac{1}{12dx}(f_{i+2} + 28f_{i+1} - 28f_{i-1} - f_{i-2}) \quad (16)$$

Table 3 shows the round-off error for the first derivatives schemes and the stencil size of 2^{nd}, 4^{th} order explicit, 4^{th} order compact and 6^{th} order compact methods:

Schemes	Max L.H.S. Stencil Size	Max R.H.S. Stencil Size	Truncation Error
2^{nd} order	1	3	$\frac{1}{3}h^2 f^3$
4^{th} order	1	5	$\frac{1}{120}h^4 f^5$
4^{th} order compact	3	3	$\frac{4}{5!}(3\alpha - 1)h^4 f^5$
6^{th} order compact	3	5	$\frac{4}{7!}h^6 f^7$

A Fourier analysis of the finite-difference methods adopted in the current study was performed. This analysis and the notion of the modified wavenumber provides a convenient means of quantifying the error associated with the differencing schemes. Mahesh, 1998, gives a good explanation of the modified wavenumber considering the test function $f = e^{ikx}$ on a periodic domain. Discretize this function on a domain of length 2π, using a uniform mesh of N points. The mesh spacing is therefore giving by $h = 2\pi/N$. The exact value of the first derivative of f is ike^{ikx}. However, the numerically computed derivatives will be of the form $ik'e^{ikx}$. The k' is the modified wavenumber.

Plots of the modified wavenumber k' against wavenumber k are presented in Fig. 1. In this figure the precision of the different schemes can be compared. The wavenumber was normalized by $k_{max} = \pi/\Delta x$. According to Lele, 1992, the difference of the modified wavenumber to the exact value is associated with the dispersion error. The wavenumber of Fig. 1 and 2 is related to the number of points (N) per wavelength by:

$$k = \frac{1}{\frac{N-1}{2}}. \quad (17)$$

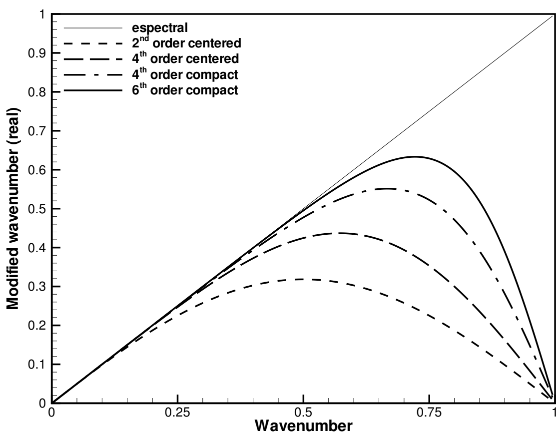

Figure 1: Modified wavenumber vs. wavenumber for first derivative approximations.

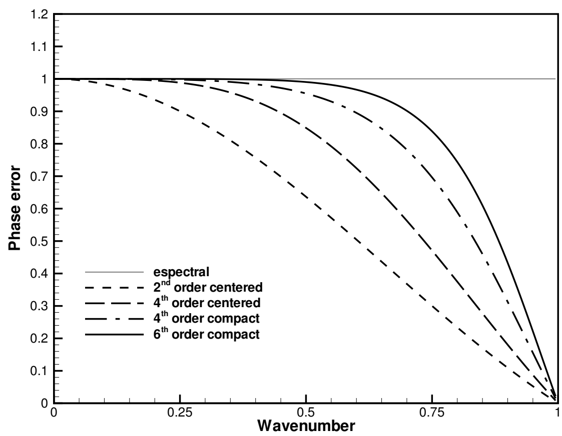

Figure 2: Phase speed vs. wavenumber for first derivative approximations.

In Fig. 2 the plots of phase speed against wavenumber of the analyzed difference schemes are shown. These plots were obtained by considering exact time advancement of the advection equation:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{18}$$

In this figure the error in phase speed can be more clearly observed. The next section presents results of a standing wave simulation of infinitesimal amplitude and the results were related with the modified wavelength and phase speed plots.

4. Numerical Results

The propagation of standing acoustic waves with a small amplitude was used to analyze the numerical method. The number of points used for discretization of the mesh interferes directly in the precision of the amplitude and phase speed waves, generating a numerical error. This error is called of dispersion error. The study of this error is very important for problems with two or three dimensions, where the computational cost is very high. With this analysis, one can choose the number of points for which the dispersion error is negligible. In this work, the 2^{nd} and 4^{th} order explicit finite-difference methods and the 4^{th} and 6^{th} order compact finite-difference methods were analyzed.

The parameters for the temporary mean flow adopted were those at sea level according to the International Standard Atmosphere. The initial amplitude A of the perturbation was approximately 10^{-4} m and the phase speed c was approximately 340.21 m/s. The wavenumber α of the disturbance selected for the simulation was 2π. The numerical results were simulated for over 100 time periods. The spatial domain extended from $x = 0$ to $x = 1$, where $x = 1$.

In the dispersion error analysis a very small time step was adopted. This choice assured that the temporal discretization error was much smaller than the space discretization error. The order of the time step adopted was 10^{-6}. In the simulations meshes with 5, 9 and 13 points per wavelength were used in the x discretization. The mesh with 5 points corresponds to the wavenumber value of $k = 0.5$ in the Figs. 1 and 2. The 9 points mesh corresponds to the wavenumber value of $k = 0.25$ in the same figures, and the mesh with 13 points corresponds to the wavenumber value of $k = 0.167$.

In the figures below, the solid line corresponds the amplitude of the analytical solution and the dotted line corresponds the amplitude obtained with the numerical simulation at time equals 100 periods. The square, diamond and circle symbols correspond to numerical results obtained with mesh discretization of 5, 9 and 13 points for wavenumber.

Figure 3 shows the numerical results with a 2^{nd} order centered explicit method. The result obtained with the mesh of 5 points shows a large error as compared with the analytical solution. This result can be explained with the help of Fig. 1. There is a difference between the spectral (exact) solution and the 2^{nd} order centered approximation for the wavenumber of $k = 0.5$. This difference is directly connected with the dispersion error. This error is reduced increasing the number per wavenumber to 9 and 13 of points. It can be observed that even with wavenumber of $k = 0.167$, which that corresponds to 13 points per wavelength one can see a dispersion error. This effect can be understood with the help of Fig. 2, which shows that even with these number of points per wavelength there exists a phase error.

The numerical results obtained with the 4^{th} order centered explicit method is presented in Fig. 4. It can be observed that in the result obtained with 5 points per wavelength, the dispersion error is much smaller the one obtained with a 2^{nd} order centered explicit method. The results obtained with 9 and 13 points per wavelength are better and the dispersion error can not be quantified in this figure. This results are consistent with the graphics shown in Figs. 1 and 2.

Figure 5 shows the numerical results obtained with the 4^{th} order compact method. By comparing Figs. 4 and 5 it can be observed that the result obtained with the 4^{th} order compact method with 5 points per wavelength was more accurate than the result obtained with the 4^{th} order explicit method. For the results obtained with 9 and 13 points per wavelength, the numerical error cannot be quantified by comparing these two figures.

The last simulation results, using a 6^{th} order compact method is shown in Fig. 6. This method presented the best result for the simulations when compared with the other methods. The amplitude error obtained for the simulation with 5 points per wavelength was very small and cannot be quantified in this figure. The numerical result obtained with 9 and 13 points per wavelength was very good and the dispersion error was almost null. It can be seen in Fig. 2 that the phase error with wavelength of $k = 0.5$ is very small for the 6^{th} order compact method. Fig. 6 is consistent with the result.

Figure 7 quantifies the dispersion error obtained in each numerical simulation with different number of points per wavelength. It can be observed that even with a small number of points per wavelength, the 6^{th} order compact is accurate and has low dispersion error. The results obtained with the 4^{th} order compact method

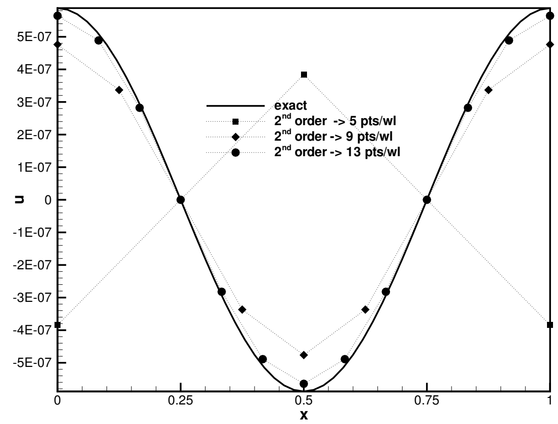

Figure 3: Results of standing wave with 2^{nd} order approximation.

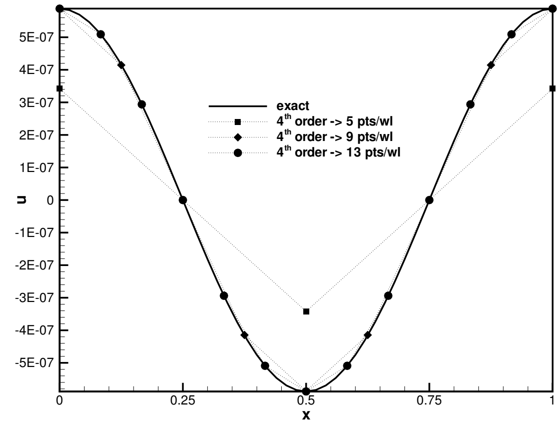

Figure 4: Results of standing wave with 4^{th} order explicit approximation.

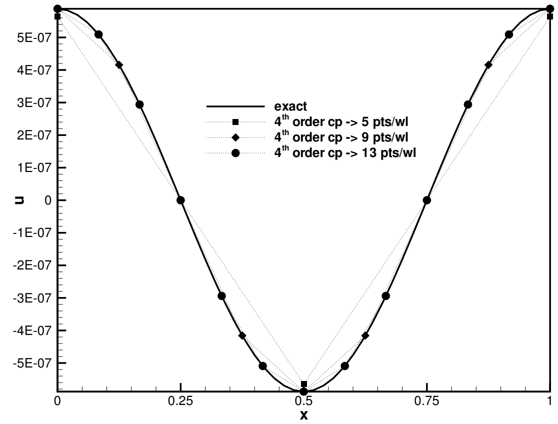

Figure 5: Results of standing wave with 4^{nd} order compact approximation.

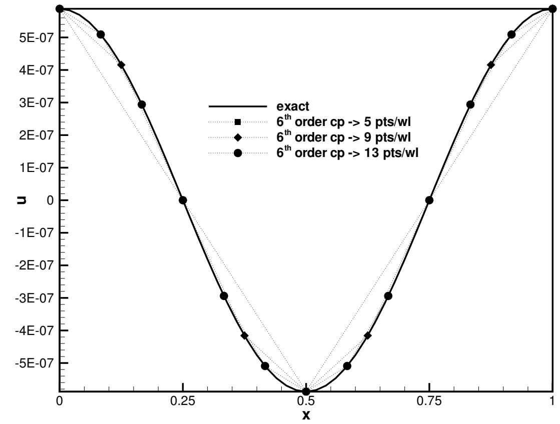

Figure 6: Results of standing wave with 6^{th} order compact approximation.

shows the advantages in using compact methods. This can be seen by comparing with the 4^{th} order explicit method. This last method shows a large dispersion error if compared with results of the 4^{th} order compact method. The 2^{nd} order explicit method gave the worst results in these simulations. The result obtained with 31 points per wavelength was worse than the result obtained with 6^{th} order compact using 9 points per wavelength.

5. Conclusions

In this work numerical simulation of a standing acoustic wave was performed. The adopted equation was the Euler equation for a isentropic one-dimensional flow. The method adopted was accurate in time, using a 4^{th} order Runge-Kutta scheme, and adopting a very small time step. This assured a very small error in time discretization, allowing a comparison of the dispersion errors of the spatial discretization. The amplitude of the wave was very small in order to present nonlinear behavior. This allows the comparison of the numerical results with the linear analytical solution.

The results showed that the 2^{nd} order explicit method analyzed was dispersive even using many points per wavelength. The results also showed that the 4^{th} order compact method was better than 4^{th} order explicit method analyzed. The results confirmed that high order methods are better in wave transport phenomena and can be interesting in acoustical numerical studies.

6. Acknowledgments

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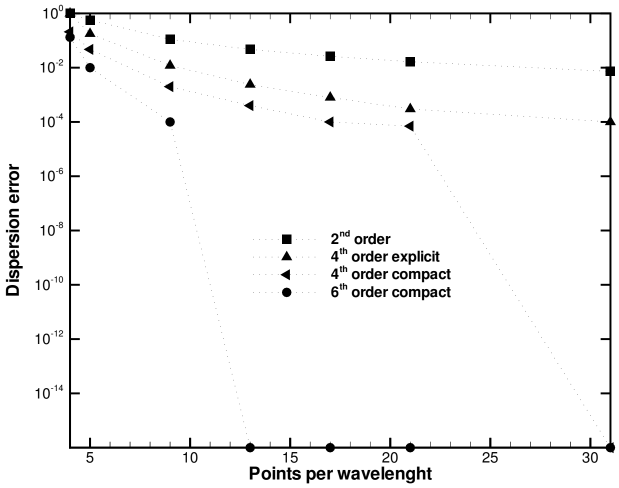

Figure 7: Dispersion of the approximations with different number of points per wavelength.

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