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IMPROVED LUMPED MODELS FOR TRANSIENT HEAT CONDUCTION IN MULTILAYERED COMPOSITE MEDIA

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Abstract. *In this paper we present improved lumped-differential formulations for one-dimensional transient heat conduction in multilayered composite media. Hermite approximations for integrals are used to obtain the average temperatures and heat fluxes in each layer. Average temperatures calculated with improved lumped parameter formulation agree well with reference finite difference solutions. The proposed heat conduction models can be used in fuel dynamics calculation for stability analysis of BWR, simplified model of PWR or real-time simulator of nuclear power plants.*

keywords: *Transient heat conduction, lumped models, composite media, nuclear reactor thermal-hydraulics*

1. Introduction

Transient heat transfer in multilayered composite plates or pipes is of great interest in a number of engineering applications (de Monte, 2000). The use of composite media is necessary when the thermal and mechanical properties of a single layer are not sufficient to fulfil both thermal and mechanical requirements. Various methods are available for the determination of the transient temperature distribution in multilayered composite media, such as Laplace transform method (Carslaw and Jaeger, 1959), the orthogonal expansion technique (Mikhailov et al., 1983), the Green's function approach (Haji-Sheikh and Beck, 1990), and finite integral transform technique (Yener and Özişik, 1974).

From an engineering point of view, most of the analytical or numerical methods are not convenient to be used because of the involved analytical work or numerical computation. In some engineering applications, simplified methods to predict transient history of average temperatures in multilayer composite media are more efficient for the analysis of the system in study. The lumped approach has been widely used in a variety of thermal engineering applications where a simplified formulation of transient heat conduction is sought. The classical lumped parameter approach is in general restricted to problems with low to moderate temperature gradients, typically with Biot number ($Bi = hD/k$) less than 0.1. In most engineering problems, the Biot number is much higher. Cotta and Mikhailov (1997) proposed a systematic formalism to provide improved lumped parameter formulation for steady and transient heat conduction problems based on Hermite approximation for integrals that define averaged temperature and fluxes. Regis et al. (2000) have developed an improved lumped analysis of transient heat conduction in a nuclear fuel rod which is represented by a two-region concentric cylinder. A higher order lumped-differential formulation for the same heat transfer problem has been developed by Su and Cotta (2001) and applied in a simplified model for light water reactor (LWR) thermohydraulic analysis. Su and Cerqueira (2001) carried out numerical simulation of transient thermal events during warm-up and cool-down of subsea oil and gas pipelines, using an improved lumped-differential formulation for the transient heat conduction in multilayered composite pipes.

In this work, we present improved lumped-differential models for the analysis of transient heat conduction in multilayered composite media in cartesian, cylindrical or spherical coordinate systems. The accuracy of the lumped models is shown by comparison with numerical solutions of the distributed models of transient heat conduction.

2. Mathematical Formulation

We consider a composite medium consisting of M parallel layers in cardesian, cylindrical, or spherical coordinates, as shown in Figure 1. The layers are labelled as 1 to M from the left to the right. Let x be the coordinate perpendicular to the layers, $x_i (i = 1, 2, ..., M)$ is its value at the left surface of each layer, and x_{M+1} is the value of x at the right surface of M-th layer. Each layer is considered as homogeneous, isotropic, and with constant thermal properties, but different from the adjacent layers. The thermal contact resistance between adjacent i-th and $(i+1)$-th layers at interface x_{i+1}, $i = 1, ..., M-1$ is modelled by a constant interface heat transfer coefficient h_{i+1}. The volumetric rate of heat generation in i-th layer is $g_i(x,t)$. Initially each layer is at a specified temperature $T_i(x,0) = F_i(x)$, in $x_i < x <_{i+1}$, $i = 1, ..., M$, for $t > 0$. Convective heat transfer occurs at the two outer boundary surfaces $x = x_1$ and $x = x_{M+1}$, with heat transfer coefficient h_1 and h_{M+1}, to environmental fluids with temperatures $T_{\infty 1}$ and $T_{\infty 2}$ respectively.

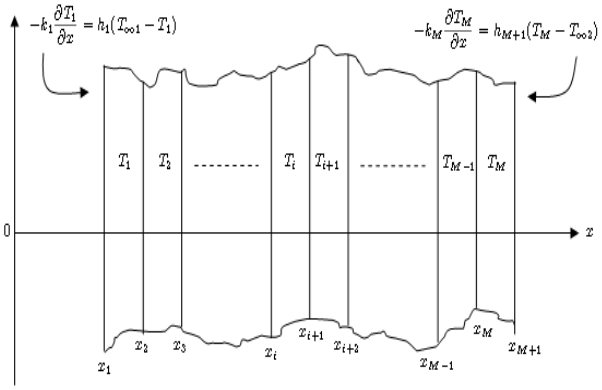

Figure 1: Illustration of M-layers composite medium

The mathematical formulation of the one-dimensional heat conduction problem is given as

$$\frac{\partial T_i(x,t)}{\partial t} = \alpha_i \frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p \frac{\partial T_i(x,t)}{\partial x}\right) + \frac{\alpha_i}{k_i} g_i(x,t) \tag{1}$$

$$\text{in} \quad x_i < x < x_{i+1}, \quad t > 0, \quad i = 1, 2, ..., M$$

where $p = 0, 1, 2$ for planar, cylindrical, and spherical layers respectively. Eq.(1) is to be solved with the following boundary and interface conditions:

$$-k_1 \frac{\partial T_1}{\partial x} = h_1(T_{\infty 1} - T_1), \qquad \text{at} \quad x = x_1 \tag{2}$$

$$-k_M \frac{\partial T_M}{\partial x} = h_{M+1}(T_M - T_{\infty 2}), \qquad \text{at} \quad x = x_{M+1} \tag{3}$$

$$-k_i \frac{\partial T_i}{\partial x} = h_{i+1}(T_i - T_{i+1}), \qquad \text{at} \quad x = x_{i+1} \quad i = 1, ..., M-1 \tag{4}$$

$$k_i \frac{\partial T_i}{\partial x} = k_{i+1} \frac{\partial T_{i+1}}{\partial x}, \qquad \text{at} \quad x = x_{i+1} \quad i = 1, ..., M-1 \tag{5}$$

and the initial conditions for temperatures in each layer

$$T_i(x,0) = F_i(x), \qquad x_i < x < x_{i+1}. \quad i = 1, ..., M \tag{6}$$

where $T_i(x,t)$ is the temperature in the i-th layer, $\alpha_i = k_i/\rho_i c_{pi}$ its thermal diffusivity, k_i the thermal conductivity, ρ_i the density, and c_{pi} the specific heat.

Introducing the following dimensionless variables and parameters,

$$X = \frac{x}{L} \quad X_i = \frac{x_i}{L} \quad \kappa_i = \frac{k_i}{k_{ref}}$$

$$\delta_i = \frac{\alpha_i}{\alpha_{ref}} \quad \tau = \frac{\alpha_{ref}\, t}{L^2} \quad \Phi_i = \frac{L^2\, g_i(x,t)}{k_{ref}\, \Delta T_{ref}}$$

$$Bi_i = \frac{h_i L}{k_{ref}} \quad \theta_i = \frac{T_i(x,t) - T_{ref}}{\Delta T_{ref}}$$

The mathematical formulation is cast in dimensionless form

$$\frac{\partial \theta_i(X,\tau)}{\partial \tau} = \frac{\delta_i}{X^p}\frac{\partial}{\partial X}\left(X^p \ \frac{\partial \theta_i(X,\tau)}{\partial X} \right) + \frac{\delta_i}{\kappa_i}\Phi_i(X,\tau) \tag{7}$$

with the respective dimensionless boundary and interface conditions,

$$-\kappa_1 \frac{\partial \theta_1(X,\tau)}{\partial X} = Bi_1 \left[\theta_{\infty 1} - \theta_1(X,\tau)\right] \tag{8}$$

$$\text{at} \quad X = \gamma_1 \quad \text{and} \quad \tau > 0$$

$$-\kappa_i \frac{\partial \theta_i(X,\tau)}{\partial X} = Bi_{i+1}[\theta_i(X,\tau) - \theta_{i+1}(X,\tau)] \tag{9}$$

$$\text{at} \quad X = X_{i+1} \text{ and } \tau > 0 \quad i = 1, ..., M-1$$

$$\kappa_i \frac{\partial \theta_i(X,\tau)}{\partial X} = \kappa_{i+1} \frac{\partial \theta_{i+1}(X,\tau)}{\partial X} \qquad \text{at} \tag{10}$$

$$X = \gamma_{i+1} \quad \text{and} \quad \tau > 0 \quad i = 1, ..., M-1$$

$$-\kappa_M \frac{\partial \theta_M(X,\tau)}{\partial X} = Bi_{M+1}[(\theta_{\infty 2} - \theta_M(X,\tau)] \tag{11}$$

$$\text{at} \quad X = X_{M+1} \quad \text{and} \quad \tau > 0$$

and the dimensionless intial conditions,

$$\theta_i(X,\tau) = F_i^*(X) \qquad \text{in} \quad X_i < X < X_{i+1} \quad \text{at} \quad \tau = 0 \tag{12}$$

The reference length L might be chosen as the thickness of one of the layers, say the first layer, $L = x_2 - x_1$, or as the inner radius in the cases of concentric cylindrical or spherical layers, $L = x_1$, or as the outer radius in the cases of solid concentric cylinder or sphere, $L = x_{M+1}$. The reference thermal conductivity and diffusivity are taken as that of one of the layers, say the first layer, $k_{ref} = k_1$ and $\alpha_{ref} = \alpha_1$.

3. Improved Lumped-Differential Formulation

We introduce the definition of spatially averaged dimensionless temperature for each layer as

$$\theta_{av_i}(\tau) = \frac{p+1}{X_{i+1}^{p+1} - X_i^{p+1}} \int_{X_i}^{X_{i+1}} X^p \ \theta_i(X,\tau) \ dX \quad i = 1, ..., M \tag{13}$$

and operate Eq.(7) with the the operador $(p+1)/(X_{i+1}^{p+1} - X_i)^{p+1} \int_{X_i}^{X_{i+1}} X^p \, dX$, to yield

$$\frac{d\theta_{av_i}(\tau)}{d\tau} = \frac{(p+1)\delta_i}{X_{i+1}^{p+1} - X_i^{p+1}} \left[X_{i+1}^p \left. \frac{\partial \theta_i(X,\tau)}{\partial X} \right|_{X=X_{i+1}} \right. \tag{14}$$

$$\left. - X_i^p \left. \frac{\partial \theta_i(X,\tau)}{\partial X} \right|_{X=X_i} \right] + \frac{\delta_i}{\kappa_i} G_i(X,\tau)$$

where the heat source term is defined as,

$$G_i(\tau) = \frac{p+1}{X_{i+1}^{p+1} - X_i^{p+1}} \int_{X_i}^{X_{i+1}} X^p \Phi_i(X,\tau) \ dX$$

Eq.(14) is an equivalent integro-differential formulation of the original mathematical formulation without any introduced approximation.

In classical lumped-differential analysis, the boundary temperatures are assumed to be the same as the average temperatures. The approach is limited to low Biot numbers.

The basic idea of the improved lumped-differential approach is to provide reasonably accurate relations between the boundary fluxes and the averaged temperatures, which are to be developed from Hermite approximations for the integrals that define the average temperatures and heat fluxes. The general Hermite

approximation for an integral, based on the values of the integrand and its derivatives at the integration limits, is written in the following form

$$\int_a^b y(x)dx = \sum_{\nu=0}^{\alpha} C_\nu y^{(\nu)}(a) + \sum_{\nu=0}^{\beta} D_\nu y^{(\nu)}(b) \tag{15}$$

where $y(x)$ and its derivatives $y^{(\nu)}(x)$ are defined for all $x \in (a,b)$. It is assumed that the numerical values of $y^{(\nu)}(a)$ for $\nu = 0,1,...,\alpha$, and $y^{(\nu)}(b)$ $\nu = 0,1,...,\beta$ are available. The general expression for the $H_{\alpha,\beta}$ approximation is given by

$$\int_a^b y(x)dx = \sum_{\nu=0}^{\alpha} C_\nu(\alpha,\beta)h^{\alpha+1}y^{(\nu)}(a) \tag{16}$$

$$+ \sum_{nu=0}^{\beta} C_\nu(\beta,\alpha)h^{\alpha+1}y^{(\nu)}(b) + O(h^{\alpha+\beta+3})$$

where $h = b - a$, and

$$C_\nu(\alpha,\beta) = \frac{(\alpha+1)!(\alpha+\beta+1-\nu)!}{(\nu+1)!(\alpha-\nu)!(\alpha+\beta+2)!} \tag{17}$$

For example, the plain trapezoidal rules ($H_{0,0}$ approximation) for averaged temperature and heat flux integrals in each layer are given in the following form:

$$\theta_{av_i} = \frac{(p+1)}{(X_{i+1}^{p+1} - X_i^{p+1})}\int_{X_i}^{X_{i+1}} \theta_i X^p dX = \frac{p+1(X_{i+1}-X_i)}{2(X_{i+1}^{p+1} - X_i^{p+1})}(X_i^p \theta_i \mid_{X_i} + X_{i+1}^p \theta_i \mid_{X=X_{i+1}}) \quad i = 1,...,M \tag{18}$$

$$\int_{X_i}^{X_{i+1}} \frac{\partial \theta_i}{\partial X} dX = \theta_i \mid_{X=X_{i+1}} - \theta_i \mid_{X_i} = \frac{X_{i+1}-X_i}{2}(\frac{\partial \theta_i}{\partial X} \mid_{X=X_{i+1}} + \frac{\partial \theta_i}{\partial X} \mid X = X_i) \quad i = 1,...,M \tag{19}$$

For each layer, we have two unknown boundary temperatures, $\theta_i \mid_{X_i}$ and $\theta_i \mid_{X=X_{i+1}}$, and two unknown heat fluxes, $\partial\theta_i/\partial X \mid_{X=X_i}$ and $\partial\theta_i/\partial X \mid_{X=X_{i+1}}$. Now, we have $2M$ equations provided by Eqs. (18, 19), $2(M-1)$ equations by Eqs. (10, 11), and two equations by Eqs. (8, 9). That is, we have exactly $4M$ equations for $4M$ unknowns which are solved to give the sought relations between the boundary temperatures and heat fluxes and the averaged temperatures in the multilayered composite pipe. These relations are then used in the Equations (4) to close the M ordinary differential equations for the averaged temperatures, to be solved with the initial conditions for the averaged temperatures

$$\theta_{av_i}(X) = \theta_{av0_i}, \qquad i = 1,...,M. \tag{20}$$

4. Results and Discussions

The proposed higher order lumped parameter model for one-dimensional transient heat conduction is solved numerically with a fourth order Runge-Kutta method, implemented in Compact Visual Fortran 6.6. The solution is then compared with a finite difference solution of the original partial differential equations and with the classical lumped parameter model as described in Levy (1999).

Numerical results were obtained for typical parameter values encountered in nuclear reactor enginnering applications (Levy, 1999). The parameters used in the test cases are given in Tables 1 and 2.

Table 1: Table 1. Fixed Parameters for Selected Test Cases

k_f	(W/m K)	4.1	k_c	(W/m K)	12.8
ρ_f	(kg/m^3)	10980.0	ρ_c	(kg/m^3)	6570.0
c_f	(J/kg^{o}C)	299.0	c_c	(J/kg^{o}C)	330.0
r_{fo}	(m)	0.00445	r_{ci}	(m)	0.00455
r_{co}	(m)	0.00503	h_g	(W/m^2 ^{o}C)	5670.0
g	(W/cm^3)	482.256			

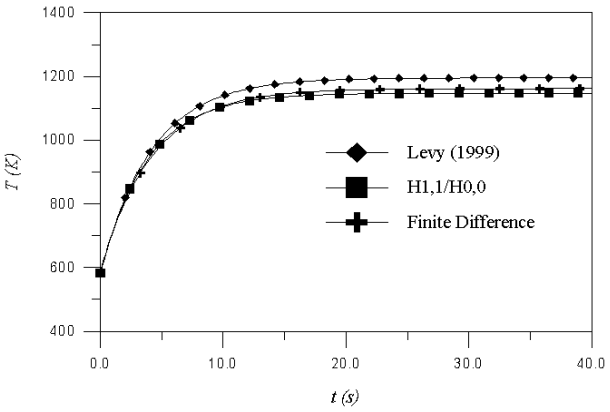

Figure 2: Average fuel temperatures for $Bi = 4.18838$.

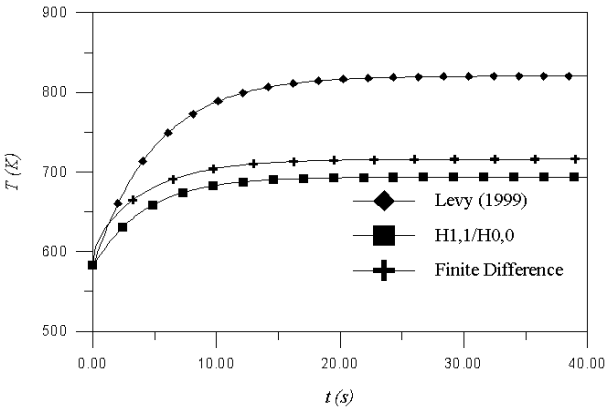

Figure 3: Average clad temperatures for $Bi = 4.18838$.

Table 2: Experimental results of bending properties of materials MAT1 and MAT2. Average values obtained in 20 experiments

Case	h (W/m^2 ^{o}C)	Bi
1	10658.3	4.18838
2	21316.6	8.37676
3	31974.9	12.5651

In Figures 2 and 3 it is shown the averaged fuel and cladding temperatures for case 1 ($Bi = 4.18838$) obtained by the improved and classical lumped formulations, compared with a finite difference solution with fully implicit scheme and central difference in spatial coordinate. The steady state solution obtained by the finite difference method agrees perfectly with the exact analytical solution. We can observe that the averaged fuel temperature obtained by the improved lumped parameter formulation is in excellent agreement with the finite difference solution, while the agreement for averaged cladding temperature is still quite reasonable. The cladding temperatures are shown in the lowermost set of curves in each graph. There is a significant improvement over the classical lumped parameter formulation also for the averaged cladding temperature. In Table 1, we give the parameters used in all the three test cases. In Table 2, the cladding-coolant heat transfer coefficients are given for the three test cases.

With increased Biot numbers, there is an improvement in the agreement between the improved lumped parameter formulation and the finite difference solution, as can be observed in Figures 4 and 5 for $Bi = 8.37676$ and in Figures 6 and 7 for $Bi = 12.5651$.

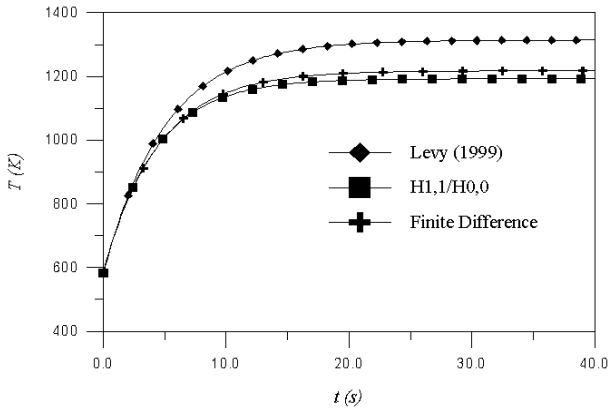

Figure 4: Average fuel temperatures for $Bi = 8.37676$.

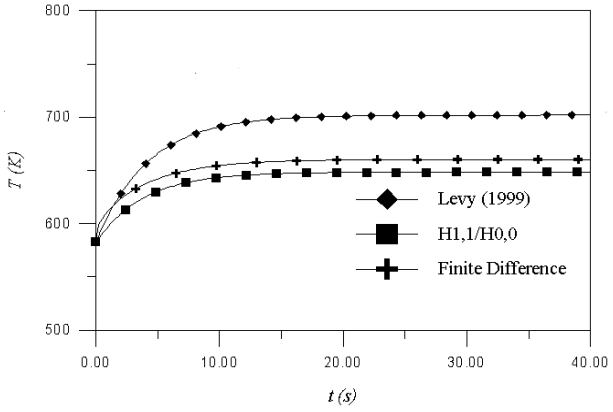

Figure 5: Average cladding temperatures for $Bi = 8.37676$.

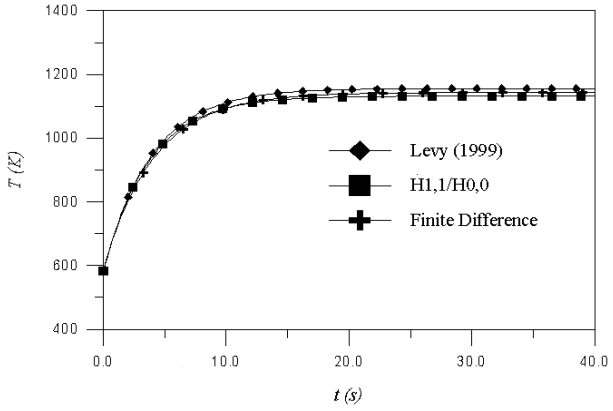

Figure 6: Average fuel temperatures for $Bi = 12.5651$.

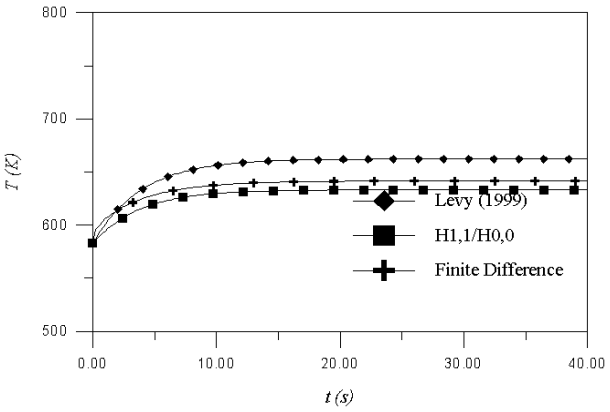

Figure 7: Average cladding temperatures for $Bi = 12.5651$.

5. Conclusions

In this paper we presented improved lumped-differential models for one-dimensional transient heat conduction in multilayered composite media. Hermite approximations for integrals are used to obtain the average temperatures and heat fluxes in each layer. The partial differential equations are reduced to a system of ordinary differential equations that can be readily solved numerically. The proposed models are applied to a PWR fuel rod analysis and the average temperatures calculated with improved lumped parameter formulation agree well with finite difference solutions. The proposed heat conduction models can be used in fuel dynamics calculation for stability analysis of BWR, simplified model or real-time simulator of nuclear power plants.

6. Acknowledgements

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