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THEORETICAL ASPECTS OF HOMOGENOUS ISOTROPIC TURBULENCE

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Abstract. *We review some recent advances on the problem of fully developed turbulence within the mathematical and physical points of view. From the mathematical perspective, we discuss a framework which has been developed for the rigorous treatment of the conventional statistical theory of turbulence and address some of the rigorous results which have obtained concerning the energy cascade, the energy spectrum, the energy dissipation rate and other physical quantities of turbulent flows. Regarding the physical approach, we focus our attention on the relevance of field theoretical methods in the analysis of dimensionally reduced models (Burgers and two-dimensional incompressible turbulence), the problem of randomly advected scalars, and intermittent fluctuations in homogeneous and isotropic turbulent flows.*

keywords: *Navier-Stokes equations, turbulence, energy cascade, intermittency*

1. Introduction

The conventional theory of turbulence is mainly concerned with relations between mean quantities. The mean quantities can be considered in different ways such as time averages or ensemble averages. In any case, the aim is to relate the physical quantities of a flow taken with respect to the corresponding average. However, most of the relations are usually obtained using heuristic or empirical arguments. It is of fundamental importance to derive such relations in a rigorous way based on first principles, i.e. directly from the Navier-Stokes equations governing the flow. This is the content of the mathematical part of this article, developed in Sec. 2.

The rigorous estimates that we obtain are for both time averages and ensemble averages, and in the context of forced turbulence with homogenous boundary condition, which can be no-slip or periodic with zero space average. The notion of ensemble average is put into a mathematical framework with the concept of statistical solution of the Navier-Stokes equations, which are probability measures in the phase space of the system. On the other hand, time averages are considered for the so-called Leray-Hopf weak solutions of the Navier-Stokes equations. In this note for simplicity we present only the results for ensemble averages, proved in Foias *et al.* (2001). The time-average estimates can be found in Foias *et al.* (to appear).

A fundamental result obtained from heuristic arguments is the Kolmogorov energy dissipation law, which relates the mean energy dissipation rate ϵ to the mean velocity U and a macroscale wavenumber κ_0 through $\epsilon \sim \kappa_0 U^3$. In contrast, we rigorously define these quantities with the help of the statistical solutions of the Navier-Stokes equations and prove that $\epsilon \leq C_1 \kappa_0 U^3$, where κ_0 is taken to be the square-root of the first eigenvalue of the Stokes operator, and C_1 is a parameter which remains bounded as the Reynolds number is increased.

From the energy dissipation law one can derive relations for other quantities such as the Kolmogorov dissipation wavenumber κ_ϵ and the Taylor wavenumber κ_τ, namely $\kappa_\epsilon \sim \kappa_0 \mathrm{Re}^{3/4}$ and $\kappa_\tau \sim \kappa_0 \mathrm{Re}^{1/2}$. In contrast we prove rigorously that $\kappa_\epsilon \leq C_1^{1/4} \kappa_0 \mathrm{Re}^{3/4}$ and $\kappa_\tau \leq C_1^{1/2} \kappa_0 \mathrm{Re}^{1/2}$. And similarly for other relations.

The last estimate concerns the energy cascade process. This is a remarkable feature of turbulence which asserts that within a certain range of scales much lower than the energy injective scales the energy is transferred in average to the smaller scales at a constant rate close to the mean energy dissipation rate. We show that if the Taylor wavenumber is large enough compared with the scales in which the forcing term acts then the energy cascade holds.

In Sec. 3, we introduce the field theory approach to turbulence, which although not based on rigorous mathematical methods, has produced along the last years a number of convincing results related to the intermittency phenomenon in turbulence models. As a concrete starting point, we discuss the pedagogical example of a linear Langevin equation, which has all the basic mathematical features found in several stochastic models designed for the analysis of the turbulence problem. A particular attention is given to the Burgers model of one-dimensional turbulence (Burgers, 1948) , where an extreme degree of intermittency is observed.

Operatorial algebras, which became popular in theoretical physics, due to the conformal field theory approach to two-dimensional second order phase transitions (Belavin *et al.*, 1984) could play some role in the statistical theory of turbulence, where an infinite set of coupled Hopf equations are defined. We briefly review Polyakov's attempt to solve two-dimensional turbulence with the help of conformal methods (Polyakov, 1993). It should be clear, however, that our main interest is not to convince the reader about one or other form of the "final theory of two-dimensional turbulence". There are important open problems in the conformal approach, which still wait for a more detailed analysis. The message of the conformal theory is just to point out an interesting way to investigate the Hopf equations, which could yield inspiration for alternative analytical attempts to solve them, based on something else than the usual closure approximations – the keyword, to be discussed in Sec. (3.4), is the "operator product expansion", originally introduced in the context of high energy physics collisions and the statistical mechanics of phase transitions (Zinn-Justin, 1996).

2. Mathematical aspects of turbulence theory

2.1. Mathematical Preliminaries

The starting point for the mathematical theory of turbulence are the Navier-Stokes equations (NSE). As mentioned in the Introduction one attempts to rederive directly from a rigorous mathematical theory of the NSE the results obtained heuristically in the classical statistical theory of turbulence. Different settings can be considered but let us restrict ourselves to forced periodic NSE, with the force mimicking some ficticious mechanism for injecting energy into the large scales of motion. In this case, the incompressible NSE read

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}, \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the kinematic pressure, ν is the kinematic viscosity, and $\mathbf{f}$ is the forcing term. The mass density ρ_0 is constant and does not appear explicitly in the equation. The spatial variable is denoted $\mathbf{x} = (x_1, x_2, x_3)$ and the spatial domain is taken to be $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$, where L_i, $i = 1, 2, 3$, are the periods in each direction x_i. The functions $\mathbf{u}$ and p are assumed to be periodic with period L_i in each direction. It is further assumed that the average of the velocity field and of the forcing term in Ω are zero:

$$\int_\Omega \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0, \qquad \int_\Omega \mathbf{f}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0. \tag{2}$$

In this case, the average of the velocity field is a conserved quantity in time and if the initial velocity field has zero average, it remains zero.

Two appropriate function spaces for the velocity field are those with finite kinetic energy and finite enstrophy (vorticity squared) (besides the boundary and divergence-free conditions). These can be characterized by inner products

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad ((\mathbf{u}, \mathbf{v})) = \int_\Omega \sum_{i=1,2,3} \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{v}}{\partial x_i} \, \mathrm{d}\mathbf{x}, \tag{3}$$

and the associated norms $|\mathbf{u}| = (\mathbf{u}, \mathbf{u})^{1/2}$, $\|\mathbf{u}\| = ((\mathbf{u}, \mathbf{u}))^{1/2}$. Due to the boundary and divergence-free conditions, the $\| \cdot \|$ norm can be directly related to the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ through

$$\|\mathbf{u}\|^2 = |\boldsymbol{\omega}|^2, \tag{4}$$

hence the concept of enstrophy ("energy of rotation"). Then, finite kinetic energy means $(\rho_0/2)|\mathbf{u}|^2 < \infty$ and finite enstrophy means $(\rho_0/2)\|\mathbf{u}\|^2 < \infty$.

We denote by H the space of Ω-periodic, divergence-free velocity fields with finite kinetic energy, and the space V of Ω-periodic, divergence-free velocity fields with finite enstrophy. Then, in an suitable sense the NSE can be written as an evolution equation of the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{F}(\mathbf{u}), \tag{5}$$

for an appropriate nonlinear term $\mathbf{F}$. The pressure disapears and can be regarded as a Lagrange multiplier associated with the divergence-free constraint; it can be fully recovered once the velocity field is known.

2.2. Ensemble Averages and Statistical Solutions of the Navier-Stokes Equations

Turbulence is now regarded as associated with a chaotic evolution of the nonlinear NSE. It is not simply chaos but a particular chaotic motion with some characteristic statistical properties. In a chaotic system, two motions in nearly identical conditions may lead to completely different behaviors in the future. However, some well-defined mean statistical properties appear when averages are taken over a number of experiments. This is the concept of ensemble average. One imagines a number of experiments yielding different velocity fields $\mathbf{u}^{(1)}, \dots \mathbf{u}^{(N)}$ and then an average is taken to find, e.g. the mean flow

$$\mathbf{U}(t, \mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{u}^{(i)}(t, \mathbf{x}). \tag{6}$$

Other characteristic quantities such as energy, enstrophy, rate of energy dissipation, etc., can be represented by some function of the velocity field $\Phi = \Phi(\mathbf{u})$, and its mean quantity is obtained by averaging over the experiments

$$\langle \Phi(\mathbf{u}) \rangle = \frac{1}{N} \sum_{i=1}^{N} \Phi(\mathbf{u}^{(i)}). \tag{7}$$

The symbol $\langle \cdot \rangle$ is the common notation for such ensemble averages.

Two major problems are of concern in the statistical theory of turbulence. One is evolving turbulence, such as in decaying processes of flows past objects. The other is turbulence in statistical equilibrium in time. There is also a mathematical framework for time-dependent turbulence (see Foias, 1972, Foias *et al.*, 2001) but we will restrict ourselves to the case of statistical equilibrium in time, also called stationary turbulence.

In the mathematical formulation of the long-time statistics of chaotic systems one considers averages with respect to the so-called invariant measures. They are supposed to contain all the statistical information in the case of equilibrium in time. One mathematical problem with the three-dimensional NSE is the lack of any proof of well-posedness of the equations. This jeopardizes the use of classical dynamical systems tools, such as invariant measures. This difficulty has been bypassed with the notion of statistical solution of the Navier-Stokes equations, introduced by Foias (1972, 1973). In the case of statistical equilibrium, meaningful averages are taken with respect to probability measures which are the stationary statistical solutions of the NSE, defined essentially as measures μ in the space H satisfying the Liouville-type equation

$$\int_H (\mathbf{F}(\mathbf{u}), \Phi'(\mathbf{u}))\, \mathrm{d}\mu(\mathbf{u}) = 0, \tag{8}$$

for appropriate test functions Φ, and where Φ' is the functional derivative of Φ in a suitable sense. The reason for this equation can be thought in the following way: The evolution of the statistical information $\langle \Phi(\mathbf{u}) \rangle$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \Phi(\mathbf{u}) \rangle = \frac{1}{N} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} \Phi(\mathbf{u}^{(i)}) = \frac{1}{N} \sum_{i=1}^{N} (\frac{\mathrm{d}\mathbf{u}^{(i)}}{\mathrm{d}t}, \Phi'(\mathbf{u}^{(i)}))$$

$$= \frac{1}{N} \sum_{i=1}^{N} (\mathbf{F}(\mathbf{u}^{(i)}), \Phi'(\mathbf{u}^{(i)})) = \langle ((\mathbf{F}(\mathbf{u}), \Phi'(\mathbf{u})) \rangle. \tag{9}$$

Then, the flow is in statistical equilibrium if

$$\langle (\mathbf{F}(\mathbf{u}), \Phi'(\mathbf{u})) \rangle = 0. \tag{10}$$

In the case the averages are taken with respect to the probability measure μ, this relation can be expressed more explicitly by (8). From here on, the mathematical expression of ensemble average is

$$\langle \Phi(\mathbf{u}) \rangle = \int_H \Phi(\mathbf{u})\, \mathrm{d}\mu(\mathbf{u}) \tag{11}$$

for a given quantity $\Phi = \Phi(\mathbf{u})$ of the flow, with respect to a given stationary statistical solution μ. The equation (8) assures that such probability measures are meaningful objects to express physical statistical quantities of real flows governed by the Navier-Stokes equations.

2.3. Estimates for Some Physical Characteristic Quantities

Some important characteristic mean quantities can be obtained with the help of the ensemble average. For instance, the mean rate of energy dissipation is

$$\epsilon = \nu\kappa_0^3\langle\|\mathbf{u}\|^2\rangle. \tag{12}$$

The quantity κ_0 is a characteristic wavenumber for the macroscales. For instance, κ_0 can be taken to be the inverse of the largest period $\kappa_0 = 1/L_0 = 1/\max\{L_1, L_2, L_3\}$. In this way, ϵ is in fact the mean rate of energy dissipation per unit mass and unit time; the term κ_0^3 compensates for the integration in space implicit in the definition of $\|\cdot\|^2$. Similarly, the mean kinetic energy per unit mass is

$$e = \kappa_0^3\langle|\mathbf{u}|^2\rangle. \tag{13}$$

The root-mean-square velocity is

$$U = \sqrt{2e}. \tag{14}$$

The Kolmogorov and Taylor wavenumbers and the Reynolds number are respectively

$$\kappa_\epsilon = \left(\frac{\epsilon}{\nu^3}\right)^{1/4}, \qquad \kappa_\mathcal{T} = \left(\frac{\epsilon}{2\nu e}\right)^{1/2}, \qquad \mathrm{Re} = \frac{U}{\kappa_0\nu}. \tag{15}$$

For those quantities, one can obtain the following estimates for large Reynolds number:

$$\epsilon \leq c\kappa_0 U^3, \quad \kappa_\epsilon \leq c\kappa_0\mathrm{Re}^{3/4}, \quad \kappa_\mathcal{T} \leq c\kappa_0^{1/3}\kappa_\epsilon^{2/3}, \quad \kappa_\mathcal{T} \leq c\kappa_0\mathrm{Re}^{1/2}. \tag{16}$$

This are rigorous estimates to be compared with the heuristic estimates from Kolmogorov theory: $\epsilon \sim \kappa_0 U^3$ $\kappa_\epsilon \sim \kappa_0\mathrm{Re}^{3/4}$, $\kappa_\mathcal{T} \sim \kappa_0^{1/3}\kappa_\epsilon^{2/3}$, and $\kappa_\mathcal{T} \sim \kappa_0\mathrm{Re}^{1/2}$.

2.4. Energy Cascade

One of the main mechanisms in the Kolmogorov theory is the cascade of energy from the large, energy-containing scales to the small, energy-dissipative scales. This is usually treated formally by expanding the velocity field in a Fourier series

$$\mathbf{u} = \sum_{\kappa=\kappa_0}^{\infty} \mathbf{u}_\kappa, \tag{17}$$

where κ are the wavenumbers which are discrete in the periodic case, and $\mathbf{u}_\kappa$ is the Fourier component of the flow with wavenumber κ. For two wavenumbers $\kappa_0 \leq \kappa' < \kappa'' \leq \infty$, we denote by $\mathbf{u}_{\kappa',\kappa''}$ the components with wavenumber in the interval $[\kappa', \kappa'')$, i.e.

$$\mathbf{u}_{\kappa',\kappa''} = \sum_{\kappa' \leq \kappa < \kappa'} \mathbf{u}_\kappa. \tag{18}$$

By multiplying the Navier-Stokes equations with $\mathbf{u}_{\kappa',\kappa''}$, for $\kappa'' < \infty$, and integrating over Ω one obtains the energy-budget equation for those modes:

$$\frac{\kappa_0^3}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_{\kappa',\kappa''}|^2 + \nu\kappa_0^3\|\mathbf{u}_{\kappa',\kappa''}\|^2 = \kappa_0^3(\mathbf{f}_{\kappa',\kappa''}, \mathbf{u}_{\kappa',\kappa''}) + \kappa_0^3\mathfrak{e}_{\kappa'}(\mathbf{u}) - \kappa_0^3\mathfrak{e}_{\kappa''}(\mathbf{u}), \tag{19}$$

where

$$\mathfrak{e}_\kappa(\mathbf{u}) = -((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \mathbf{u}_{\kappa,\infty}). \tag{20}$$

The first term in the equation is the rate of change of kinetic energy per unit mass per unit time; the second term is the rate of energy dissipation per unit mass per unit time; the third term is the rate of energy injection per unit mass per unit time; the fourth term is the energy flux per unit mass and unit time from wavenumbers below κ' to higher wavenumbers; and the last term is like the fourth but for κ''.

In statistical equilibrium, the time-derivative term drops out and, in average with respect to a stationary statistical solution μ we find

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa',\kappa''}\|^2\rangle = \kappa_0^3\langle(\mathbf{f}_{\kappa',\kappa''}, \mathbf{u}_{\kappa',\kappa''})\rangle + \kappa_0^3\langle\mathfrak{e}_{\kappa'}(\mathbf{u})\rangle - \kappa_0^3\langle\mathfrak{e}_{\kappa''}(\mathbf{u})\rangle. \tag{21}$$

For the energy cascade, an important quantity is the mean energy flux $\kappa_0^3\langle\mathfrak{e}_\kappa(\mathbf{u})\rangle$, and it is expected on heuristic grounds that within the inertial range the mean energy flux is exactly the mean rate of energy dissipation, i.e $\kappa_0^3\langle\mathfrak{e}_\kappa(\mathbf{u})\rangle \approx \epsilon$.

However, a mathematical difficulty appears when we take $\kappa'' = \infty$. Formally, one would have an equality, but rigorously it is not known whether the so-called global weak solutions of the NSE are regular enough and, instead, an inequality appears in the time-dependent energy-budget equations as well as in the averaged equations:

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa',\infty}\|^2\rangle \le \kappa_0^3\langle(\mathbf{f}_{\kappa',\infty}, \mathbf{u}_{\kappa',\infty})\rangle + \kappa_0^3\langle\mathfrak{e}_{\kappa'}(\mathbf{u})\rangle. \tag{22}$$

This inequality is not sufficient for a precise estimate of the mean energy flux $\kappa_0^3\langle\mathfrak{e}_\kappa(\mathbf{u})\rangle$. A remedy is to recover an equality by considering a possible loss of energy "to infinity". The limit

$$\langle\mathfrak{e}(\mathbf{u})\rangle_\infty = \lim_{\kappa\to\infty}\langle\mathfrak{e}_\kappa(\mathbf{u})\rangle \tag{23}$$

is rigorously proved to exist. Then, we consider the restricted energy flux

$$\mathfrak{e}_\kappa^*(\mathbf{u}) = \mathfrak{e}_\kappa(\mathbf{u}) - \langle\mathfrak{e}(\mathbf{u})\rangle_\infty. \tag{24}$$

For this object, we have

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa',\infty}\|^2\rangle = \kappa_0^3\langle(\mathbf{f}_{\kappa',\infty}, \mathbf{u}_{\kappa',\infty})\rangle + \kappa_0^3\langle\mathfrak{e}_{\kappa'}^*(\mathbf{u})\rangle. \tag{25}$$

Now, the assumption that the injection of energy is concentrated on large-scale wavenumber is usually accomplished by assuming that the forcing term $\mathbf{f}$ only contains large-scale wavenumber components. This means that $\mathbf{f}_{\overline{\kappa},\infty} = 0$ for some large-scale wavenumber $\overline{\kappa}$. Thus, for $\kappa' \ge \overline{\kappa}$, the forcing terms drops out of the energy-budget equation and we are left with

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle = \kappa_0^3\langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle, \tag{26}$$

for $\kappa \ge \overline{\kappa}$. From this equation we find on the one hand

$$\kappa_0^3\langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle = \nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle \le \nu\kappa_0^3\langle\|\mathbf{u}_{\kappa_0,\infty}\|^2\rangle = \epsilon, \tag{27}$$

and on the other hand

$$\begin{aligned}\kappa_0^3\langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle &= \nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle = \nu\kappa_0^3\langle\|\mathbf{u}_{\kappa_0,\infty}\|^2\rangle - \nu\kappa_0^3\langle\|\mathbf{u}_{\kappa_0,\kappa}\|^2\rangle \ge \epsilon - \nu\kappa_0^3\kappa^2\langle|\mathbf{u}_{\kappa_0,\kappa}|^2\rangle \\ &\ge \epsilon - \nu\kappa_0^3\kappa^2\langle|\mathbf{u}_{\kappa_0,\infty}|^2\rangle = \epsilon - 2\nu\kappa^2 e = \epsilon - \kappa^2\frac{2\nu e}{\epsilon}\epsilon = \left(1 - \left(\frac{\kappa}{\kappa_\tau}\right)^2\right)\epsilon. \end{aligned} \tag{28}$$

Thus, we find

$$1 - \left(\frac{\kappa}{\kappa_\tau}\right)^2 \le \frac{\kappa_0^3\langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle}{\epsilon} \le 1 \tag{29}$$

This means that provided the square of the Taylor wavenumber is much larger than the square of energy-injective scales, i.e.

$$\kappa_\tau^2 \gg \overline{\kappa}^2, \tag{30}$$

then the energy cascade

$$\kappa_0^3\langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle \approx \epsilon \tag{31}$$

occurs for $\overline{\kappa} \le \kappa \ll \kappa_\tau$. For more details, see Foias *et al.*(2001)and Rosa (2002).

2.5. Estimates for Finite-Time Averages

Physicists and engineers are often taking finite-time averages instead of ensemble averages in pratice in laboratories. For this reason, mathematical estimates for finite-time average characteristic quantities are desirable. These have been obtained in Foias *et al.* (to appear). The estimates mimick those for ensemble averages, except that lower bounds for the averaging time are needed. In some cases, these lower bounds yield short times, depending only in the macroscale wavenumber and the kinematic viscosity. In other cases, such as the energy cascade, the averaging time sufficient in theory is very large, depending on the forcing term.

2.6. Further Results

We treated above only the forced periodic cased in three-dimensions. The two-dimensional periodic case has also been treated in detail (see Foias *et al.*, 2002, Foias *et al.*, to appear). Estimates for the energy dissipation rate have been proved in three-dimensional shear flows in Constantin and Doering (1994) and in three-dimensional pressure gradient flows in Constantin and Doering (1995). Further results can be found in Foias *et al.*(2001), Rosa (2002), Bercovici *et al.* (1995), Doering and Foias (2002), Foias (1997), Foias *et al.* (2001) .

3. An outline of the field theory approach to turbulence

3.1. Stochastic Hydrodynamics and Turbulence

It is a challenging problem to find the probability measures describying the decay/stationary states of turbulent flows, directly from Eq. (8). The idea would be to follow in turbulence the same line of reasoning that has been successfully carried out in equilibrium statistical mechanics, where the Liouville equation is in fact solved, leading to the usual Boltzmann-Gibbs distribution (Huang, 1987). However, even though a closed analytical solution of turbulent Liouville equations is presently not available, we have witnessed in the last ten years or so a considerable progress towards a theoretical understanding of intermittency, the hallmark of the small scale flow fluctuations – a central issue in turbulence research. The stage where these recent studies have been performed is set by stochastic and dimensionally reduced models (i) the Burgers model (Burgers, 1948, Polyakov, 1995, Gurarie and Midgal, 1996, Cheklov and Yakhod, 1996, Khanin *et al.*, 1997, Eijdnen and Eijdnen, 2000, Moriconi and Dias, 2001), (ii) Kraichnan model of passive random advection (Kraichnan, 1994, Gawedski and Kupianen, 1995, Shraiman and Siggia, 1996, Falko *et al.*, 1996 and Balkovski and Lebedev, 1998) , iii) stochastic Navier-Stokes equations (Wyld, 1961, Orszag *et al.*, 1996, Moriconi and Nobre, 2002, Moriconi, 2004, Moriconi and Nobre, 2004), and (iv) the conformal theory of two-dimensional turbulence (Polyakov, 1993, Lowe, 1993 and Moriconi, 1996). As the theory of conformal turbulence is related to a somewhat diverse approach, we discuss it in a separate section. Let us start by defining models (i), (ii) and (iii).

3.1.1. Burgers Model

This is a model of "one-dimensional turbulence", where, of course, there is no incompressibility constraint and no pressure term in the evolution equations. In the stochastic version, the velocity field $u = u(x,t)$ evolves according to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}u = f, \tag{32}$$

where $f = f(x,t)$ is a gaussian stochastic force, with zero mean, and two-point correlation function

$$\langle f(x,t)f(x',t')\rangle = D(x-x')\delta(t-t'). \tag{33}$$

The function $D(x-x')$ is assumed to be correlated at large length scales. Usually one defines

$$D(x-x') = D_0\exp[-\frac{(x-x')^2}{L^2}] \ . \tag{34}$$

A theorem by E. Novikov (1964) tells us that the energy is injected into the system at a rate $D_0/2$ (this results holds also for the incompressible case in any spatial dimension). The flow is highly intermittent due to the presence of shock waves, as formerly noticed by Burgers himself, and corroborated through direct numerical simulations (Chekhlov and Yakhod, 1996 and Gotoh and Kraichnan, 1998) .

3.1.2. Kraichnan's model of Passive Advection

We have, here, the problem of a passive scalar $\theta = \theta(\mathbf{x}, t)$ (temperature, dye density, etc.), defined in d-dimensional space, which, besides diffusing, is advected by a random velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, and is intensified/depleted along lagrangian trajectories by an external stochastic agent $f = f(\mathbf{x}, t)$:

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\theta - \kappa \nabla^2 \theta = f. \tag{35}$$

Observe that there is no back reaction of the scalar field on the velocity field. The $\mathbf{u}$ and f functions are regarded to be independent gaussian stochastic fields. The velocity structure function $S_2 \equiv \langle |\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}', t)|^2 \rangle$ has the general scaling form, $S_2(r) \sim r^{1-\gamma}$, where $r = |\mathbf{x} - \mathbf{x}'|$, devised to mimick the advection produced at inertial range length scales, as it happens in a realistic turbulent flow (Kolmogorov phenomenology gives $\gamma = 1/3$. Alternatively, $\gamma = 0$ corresponds to the advection produced by a smooth velocity field). The external agent $f = f(\mathbf{x}, t)$, on the other hand, is correlated in a way similar to (34).

3.1.3. Stochastic Navier-Stokes Equations

In order to study the inertial range properties of homogeneous isotropic turbulence, which are conjectured to have a universal scaling behaviour, we regard the external force in Eq. (2.1) as a gaussian stochastic field, with zero mean and a large scale two-point correlation function

$$\langle f_\alpha(\mathbf{x}, t) f_\beta(\mathbf{x}', t') \rangle = \delta_{\alpha\beta} D(\mathbf{x} - \mathbf{x}') \delta(t - t'). \tag{36}$$

3.2. The Field Theoretical Framework

Let us consider the following ordinary stochastic differential equation:

$$\dot{x} = -cx + \eta(t) \ , \tag{37}$$

where $c > 0$ and the random function of time $\eta(t)$ is a gaussian noise defined by $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = D\delta(t - t')$. Assume that $x(0) = 0$. The straightforward integration of the above differential equation gives

$$x(T) = \exp(-cT) \int_0^T dt' \exp(ct')\eta(t') \ . \tag{38}$$

It is clear that $x(T)$ is a gaussian random variable, with vanishing mean value, since it is a linear functional of $\eta(t)$. The probability density function (pdf) for $x(T)$ has, as usual, the normal form

$$\rho_T(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-\frac{x^2}{2\sigma}) \ , \tag{39}$$

where $\sigma \equiv \langle [x(T)]^2 \rangle$. We get, from (38),

$$\begin{aligned} \langle [x(T)]^2 \rangle &= \exp(-2cT) \int_0^T dt' \int_0^T dt'' \exp[c(t' + t'')] \langle \eta(t')\eta(t'') \rangle \\ &= \frac{D}{2c}[1 - \exp(-2cT)] \ . \end{aligned} \tag{40}$$

The same results can be obtained through a more laborious method, the Martin-Siggia-Rose functional formalism (Martin *et al.*, 1973), which is in fact worth of consideration when one does not know how to solve exactly the stochastic differential equation, as in turbulence models. We have, introducing the characteristic function $Z_T(\lambda) = \langle \exp[i\lambda x(T)] \rangle$,

$$\rho_T(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda Z_T(\lambda) \exp(-i\lambda x) \ . \tag{41}$$

Let now $x_\eta(t)$ be the solution of (37), with the initial conditon $x_\eta(0) = 0$. We may write (up to a normalization factor which assures that $Z_T(0) = 1$) the path-integral expression

$$\begin{aligned} Z_T(\lambda) &= \langle \int Dx \delta[x(t) - x_\eta(t)] \exp[i\lambda x(T)] \rangle \\ &= \langle \int Dx \delta[\dot{x} + cx - \eta(t)] J[x] \exp[i\lambda x(T)] \rangle \ , \end{aligned} \tag{42}$$

where, in a time-discretized representation,

$$\int DxF[x] \equiv \lim_{N\to\infty} \int dx_1 \int dx_2 ... \int dx_N F[\{x_1, x_2, ..., x_N\}] \ ,$$
$$\delta[x(t) - x_\eta(t)] \equiv \prod_{i=1}^{N} \delta(x_i - x_\eta(t_i)) \ . \quad (43)$$

In our particular problem, there is no need to worry with the jacobian $J[x]$, since it is independent of $x(t)$. Actually, it is possible to prove that $J = 1$ holds if time is discretized, for general stochastic differential equations of the form $\dot{x} + L(x) = \eta$ (Zinn-Justin, 1996). To proceed, we rewrite the Dirac delta functional as an integration over an auxiliary field $\hat{x} = \hat{x}(t)$:

$$Z_T(\lambda) = \langle \int Dx D\hat{x} \exp[i \int_0^T dt \hat{x}(\dot{x} + cx - \eta) + i\lambda x(T)] \rangle$$
$$= \int Dx D\hat{x} \exp[i \int_0^T dt \hat{x}(\dot{x} + cx) + i\lambda x(T)] \langle \exp(-i \int_0^T dt \hat{x}\eta) \rangle$$
$$= \int Dx D\hat{x} \exp\{i \int_0^T dt[\hat{x}(\dot{x} + cx) + i\frac{D}{2}\hat{x}^2] + i\lambda x(T)\} \ . \quad (44)$$

Following the traditional notation, inspired by quantum physics concepts, we write $Z_T(\lambda) = \int Dx D\hat{x} \exp[iS + i\lambda x(T)]$, where $S = \int_0^T dt[\hat{x}(\dot{x} + cx) + i\frac{D}{2}\hat{x}^2]$ is the so-called Martin-Siggia-Rose (MSR) action. Since the MSR action turned to be a simple quadratic functional, we may exactly integrate over $\hat{x}(t)$ to obtain

$$Z_T(\lambda) = \int Dx \exp[-\frac{1}{2D} \int_0^T dt(\dot{x} + cx)^2 + i\lambda x(T)]$$
$$= \int Dx \exp[-\frac{1}{2D} \int_0^T dt(\dot{x}^2 + c^2x^2) - \frac{c}{2D}x(T)^2 + i\lambda x(T)]$$
$$= \int dx G(x,T) \exp[-\frac{c}{2D}x^2 + i\lambda x] \ , \quad (45)$$

where

$$G(x,T) \equiv \int_{x(0)=0}^{x(T)=x} Dx \exp[-\frac{1}{2D} \int_0^T dt(\dot{x}^2 + c^2x^2)] \ . \quad (46)$$

Taking the Fourier transform of $Z_T(\lambda)$ given in (3.13), we find

$$\rho_T(x) = \mathcal{N} G(x,T) \exp(-\frac{c}{2D}x^2) \ , \quad (47)$$

where $\mathcal{N}$ is the normalization factor which guarantees that $\int dx \rho_T(x) = 1$. The good news is that $G(x,T)$ is nothing more than the imaginary-time propagator for the quantum harmonic oscillator (Feynman and Hibbs, 1963, Moriconi, 2004b). Up to an unimportant normalization factor, we have

$$G(x,T) = \exp[-\frac{c}{2D} \coth(cT) x^2] \ . \quad (48)$$

We find, thus,

$$\rho_T(x) = \mathcal{N} \exp\{-\frac{c}{2D}[1 + \coth(cT)]x^2\} = \mathcal{N} \exp(-\frac{x^2}{2\sigma}) \ , \quad (49)$$

with $\sigma = (D/c)[1 + \coth(cT)]^{-1} = (D/2c)[1 - \exp(-2cT)]$, precisely as given by Eq. (3.8).

The previous computational steps, from equation (3.5) to (3.12), may be generalized to deal with stochastic turbulence models. It is instructive to focus our attention on the Burgers model, which is a prototypical example for the application of the MSR field theory formalism to the turbulence problem. Let $\mathcal{O}(x,t,\delta)$ be an arbitary observable defined at position x and time t, and which depends on a length scale δ. It could be given, for instance, by the velocity differences $\mathcal{O}(x,t,\delta) = u(x+\delta,t) - u(x,t) \equiv \Delta u$. In order to compute the pdf for Δu, denoted $\rho(\Delta u)$, in the asymptotic stationary statistical state (assumed to be homogeneous in time and space variables), we define the characteristic function

$$Z(\lambda) = \langle \exp[i\lambda \Delta u] \rangle \ . \quad (50)$$

Using now (3.1) and (3.2), we will have, for (3.18),

$$Z(\lambda) = \int D\hat{u}Du \exp\{i \int dtdx\hat{u}[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}]$$
$$- \int dtdxdx'\hat{u}(x,t)D(x-x')\hat{u}(x',t) + i\lambda\Delta u\} \ . \tag{51}$$

Above, the characteristic function is computed at time $t = 0$, which is a physically meaningful average, if we assume that the flow's initial state is defined at $t \to -\infty$. The MSR action is then written as

$$S = \int dtdx\hat{u}[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}] + i\int dtdxdx'\hat{u}(x,t)D(x-x')\hat{u}(x',t) \ . \tag{52}$$

3.3. Instantons and Intermittency

We are aware, since the fundamental work of Batchelor and Townsed (1949) , of the existence of intermittent, non-gaussian fluctuations revealed by some physical observables in turbulence. While the velocity field is not an intermittent variable in general (Tabeling *et al.*, 1996), velocity differences at inertial and viscous scales are. The choice of the observables $\mathcal{O}(x,t,\delta)$ depends on the details of the model under consideration. Laboratory experiments and numerical simulations are of crucial importance here, in order to guide our choices, which, hopefully could bring some light in the understanding of the underlying mechanisms of turbulence generation. In Kraichnan's model of passive advection it is usual to take $\mathcal{O}(\mathbf{x},t,\delta) = \theta(\mathbf{x}+\delta\hat{n},t) - \theta(\mathbf{x},t)$, i.e, scalar differences along some direction $\hat{n}$. In three-dimensional, realistic flows, several works have been devoted to the analysis of statistical fluctuations of powers of transverse/longitudinal velocity differences (or powers of gradients) [see Frisch's book (1995) for a review].

In Burgers turbulence, the flow can be depicted as a gas of dilute shock waves (Saffman, 1971), which explicitly break the parity symmetry $x \to -x$ of the flow configurations (as it could be guessed from the form of the static shock solution $u(x) = -U\tanh(Ux/\nu)$ of Eq. (3.1) for the case of a vanishing external force). It turns that the velocity structure functions for δ small enough (within inertial range scales) are given by (Saffman, 1971)

$$S_q \equiv \langle|\Delta u|^q\rangle \sim \delta \ , \tag{53}$$

while the usual K41 phenomenology would predict $S_q \sim \delta^{q/3}$. Such a strong deviation from the standard K41 theory renders the Burgers model one of the most attractive systems for theoretical study. In particular, one would be interested to describe the pdf tails of δu, supposedly asymmetric. There is a compelling evidence that the left and right tails are given by $\rho \sim \Delta u^{-7/2}$ and $\rho \sim \exp(-c\Delta u^3)$, respectively. These results have been rigorously proved (under some plausibility assumptions) by E and Eijnden (. The instanton strategy, a general method to approach the intermittency problem in turbulence models, which we describe now, works well for the burgers right tail pdf, while the left tail is still an open problem within this line of research. The essential idea of the method is to extend the definition of the characteristic function to the complex plane, through the mapping $\lambda \to -i\lambda$. If the pdf decay at the right tail is faster than any simple exponential, we could get its precise form from the knowledge of $Z(-i\lambda)$ for large values of λ. On the other hand, the functional integration (3.20) could be computed in the large λ limit with the help of the saddle point technique. The saddle-point equations are straightforwardly obtained from the functional derivatives of the MSR action with respect to the fields $\hat{u}$ and u. The solutions of this set of equations are dubbed "instantons" (a name identical to the one for analogous gauge theory saddle-point configurations). Gurarie and Migdal (1996) were able to find the Burgers instanton, which led in fact to the correct asymptotic form of the velocity difference pdf at the right tail. It is interesting to note that there is a scaling symmetry in the saddle-point equations which necessarily leads to the right tail pdf $\sim \exp(-c\Delta u^3)$ in the limit of vanishing viscosity.

The existence of a scaling symmetry for the saddle-point equations, which is an advantage for the Burgers model, becomes the source of great difficulties in other models, like the ones defined in Sec. (3), where faster-than-gaussiandecaying pdfs are not observed for the intermittent variables. Nevertheless, it is worth mentioning that the instaton method can be applied to these models as well (Balkovski and Lebedev, 1998 and Moriconi, 2004), in reasonable agreement with numerical, experimental, and alternative analytical studies.

3.4. Conformal Turbulence

In the real world, approximately two-dimensional motion may be observed in many systems, like soap films, stratified flows, or rotating fluids. The later, in particular, have been receiving much atention due to their relevance to oceanic and atmospheric sciences. One of the advantages of lower-dimensional turbulence is that higher Reynolds numbers may be achieved in numerical simulations. Also, as a general rule, intermittency effects are more pronounced here than in the three-dimensional case, making it easier, in principle, to study their generation mechanisms.

Defining the stream function ψ by the relation $v_\alpha = \epsilon_{\beta\alpha}\partial_\beta\psi$ and the vorticity $\omega = \partial^2\psi$, the two-dimensional N-S. equations for ω may be written as

$$\partial_t\omega + \epsilon_{\alpha\beta}\partial_\alpha\psi\partial^2\partial_\beta\psi = \epsilon_{\alpha\beta}\partial_\alpha f_\beta + \nu\partial^2\omega \ . \tag{54}$$

In the inviscid case ($\nu = 0$) and in the absence of external forces, the above equation implies that there is, besides energy, an infinite number of conserved quantities, given by

$$I_n = \int d^2\vec{x}\omega^n \ , \tag{55}$$

where n is a positive integer. I_2 is known as "enstrophy," having an important role in the cascade picture of two-dimensional turbulence. Kraichnan (1967) advanced the hypothesis that not only energy, but also these additional conserved quantities would flow across the inertial range. A careful analysis of the energy and enstrophy fluxes leads to a surprising result. Energy is transported now to larger length scales, while enstrophy flows towards smaller ones, in such a way that both fluxes cannot coexist in the same range of wave numbers. Regarding the energy spectrum, if the system is forced at wave number k_0, the energy transport to wave numbers $k < k_0$ is characterized by $E(k) \sim k^{-5/3}$, as in the Kolmogorov's theory, and the constant enstrophy flux towards $k > k_0$ is associated with $E(k) \sim k^{-3}$. It is believed that Kraichnan's idea of the enstrophy cascade is physically correct, but numerical simulations (Legras *et al.*, 1998) show that the energy spectrum decay is given by exponents close to -3.5, varying according to the nature of the large scale external forcing.

Polyakov has suggested a conformal field theory approach to two-dimensional turbulence (Polyakov, 1993), from which the exponents describing the energy spectrum decay may be found exactly. Conformal methods have been very important in the understanding of critical phenomena in two dimensions, where specific models were seen to correspond to different realizations of the Virasoro algebra. Among the conformal theories, the "minimal models" play a special role, since they have a finite number of scaling operators. These models (Belavin *et al.*, 1984) are generically defined by a pair of relatively prime numbers, (p, q), with $p < q$. They contain a subset of $(p-1)(q-1)/2$ scalar primary operators, $\psi_{(m,n)}$, labelled by $1 \le m < p$ and $1 \le n \le (q-1)/2$, if p is even, or $1 \le m \le (p-1)/2$ and $1 \le n < q$, otherwise, having dimensions $\Delta_{(m,n)} = \left((pn-qm)^2 - (p-q)^2\right)/4pq$. The reason for the choice of scalar operators is that we deal with isotropic correlation functions in the turbulence problem. The operator product expansion (OPE) of two primary operators $\psi_{(r_1,s_1)}(z)$ and $\psi_{(r_2,s_2)}(z')$, with $|z - z'| \to 0$ is written as

$$\psi_{(r_1,s_1)}(z)\psi_{(r_2,s_2)}(z') = \sum_{(r_3,s_3)} (a\bar{a})^{\left(\Delta_{(r_3,s_3)} - \Delta_{(r_1,s_1)} - \Delta_{(r_2,s_2)}\right)} \sum_{(n,m)} C^{(r_3,s_3)}_{\{(n_1,\ldots,n_k);(m_1,\ldots,m_l)\}}$$
$$\times L_{-n_1}\ldots L_{-n_k}\bar{L}_{-m_1}\ldots\bar{L}_{-m_l}a^{\sum n}\bar{a}^{\sum m}\psi_{(r_3,s_3)}(z) \ , \tag{56}$$

where $|r_1 - r_2| + 1 \le r_3 \le \min(r_1 + r_2 - 1, 2p - r_1 - r_2 - 1)$, $|s_1 - s_2| + 1 \le s_3 \le \min(s_1 + s_2 - 1, 2q - s_1 - s_2 - 1)$ and we have introduced, in (56), the Virasoro generators of conformal transformations, L_{-n} and $\bar{L}_{-n}$. The interest in these models is related not only to their finite number of primary operators, but also to the fact that their dimensions and the form of short distance products are completely known.

Let us now apply the above operator structures in the problem of two-dimensional turbulence. We may write Hopf's equations for the vorticity correlation functions,

$$\partial_t[< \omega(x_1,t)\omega(x_2,t)\ldots\omega(x_n,t) >] = 0 \ , \tag{57}$$

where time derivatives are expressed through equations (54). In the inertial range, as discussed in the previous section, both forcing and viscosity terms may be neglected in order to formulate a simplified set of Hopf equations. Considering, furthermore, the convection term in (54) as a vanishing point-split product of fields, that is, $\oint_{|z-z'|=|a|}(dz'/a)\epsilon_{\alpha\beta}\partial_\alpha\psi(z)\partial^2\partial_\beta\psi(z') \to 0$, when $|z - z'| \to 0$, we would have, then, an exact solution of (57). A concrete realization of this possibility may be achieved if we regard the stream function ψ as a primary operator of some conformal minimal model. In this case, we may use all the available information on operator dimensions and OPE's to obtain physical results. According to this assumption, let ϕ be the primary operator which has the lowest dimension, $\Delta\phi$, appearing in the OPE $\psi\psi$, between fields with the same dimension $\Delta\psi$.

Taking $a \equiv |a| \exp(i\theta)$, we will have, thus,

$$\lim_{|a|\to 0} \oint_{|z-z'|=|a|} \frac{dz'}{a} \epsilon_{\alpha\beta}\partial_\alpha\psi(z)\partial^2\partial_\beta\psi(z') \sim \int d\theta \left[\partial_{\bar{a}}^2\partial_a\partial_z - \partial_a^2\partial_{\bar{a}}\partial_{\bar{z}}\right](a\bar{a})^{(\Delta\phi-2\Delta\psi)}$$
$$\times \sum C_{\{n;m\}} L_{-n_1}...L_{-n_k}\bar{L}_{-m_1}...\bar{L}_{-m_l} a^{\sum n}\bar{a}^{\sum m}\phi(z,\bar{z})$$
$$\sim (a\bar{a})^{(\Delta\phi-2\Delta\psi)}\left[L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2\right]\phi \ , \quad (58)$$

as the dominant contribution in this short distance product. It is important to note that in order to get (58) it was necessary to set $C_{\{1;2\}} = C_{\{2;1\}}$ and $C_{\{1;(1,1)\}} = C_{\{(1,1);1\}}$, as it follows from the pseudoscalar nature of the ϵ factor above. We see, then, that (58) vanishes with $|a| \to 0$ if

$$\Delta\phi > 2\Delta\psi \ , \quad (59)$$

which is one of the constraints that the chosen minimal model has to satisfy. An additional constraint comes from the condition of a constant enstrophy or energy flux through the inertial range. In the energy cascade case this means (Frisch, 1995) that $< \dot{v}_\alpha(x)v_\alpha(0) >\sim x^0$. Analogously, it may be proved that the condition for a constant enstrophy flux is $< \dot{\omega}(x)\omega(0) >\sim x^0$, which gives

$$< \dot{\omega}(x)\omega(0) >\sim (a\bar{a})^{(\Delta\phi-2\Delta\psi)} < \left[\left(L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2\right)\phi(x)\right]\partial^2\psi(0) > \ . \quad (60)$$

The correlation function at the RHS of (60) is now evaluated by means of a purely dimensional argument, as $L^{-2(\Delta\phi+\Delta\psi+3)}$, which makes sense if one thinks that there is an effective infrared cutoff in the theory at the length scales given by L, where random forces act. Imposing (60) to be independent of L, we get

$$\Delta\phi + \Delta\psi + 3 = 0 \ . \quad (61)$$

In the case of an energy cascade, the argument is the same and the constraint turns out to be

$$\Delta\phi + \Delta\psi + 2 = 0 \ . \quad (62)$$

It is known that there is an infinite number of minimal models compatible with (59) and (61) or (62) (Lowe, 1993). The general belief, and still an open problem, is that there may be universality classes, associated to the statistical properties of the forcing terms, which would single out one or another of the possible solutions. Let us note that the minimal models found in this way are non-unitary, since the short-distance product $\psi(z)\psi(z')$ goes to zero when $z \to z'$.

The connection of the conformal approach with real experiments or numerical simulations is made through the computation of inertial range exponents, which describe the decrease of energy in the region of higher Fourier modes. In the situation where VEV's of single operators vanish, the inertial range exponents are given by $4\Delta\psi + 1$ and, in the opposite case, by $4\Delta\psi - 2\Delta\phi + 1$. A good agreement has been reached between the former possibility, for the direct enstrophy cascade case, and numerical simulations (Legras *et al.*, 1998, Babiano *et al.*, 1995 and Benzi *et al.*, 1995) of the two-dimensional Navier-Stokes equations.

3.5. Conclusions

The field theory approach, in the path-integral version, provides a general framework for the investigation of the intermittency phenomenon. Probability density functions and expectation values of intermittent observables can be computed in asymptotic regimes, clearly exhibiting deviations from gaussian statistics and the Kolmogorov phenomenological predictions. Alternatively, the existence of closed operatorial structures, as the two-dimensional conformal field theories, suggest interesting clues in the study of non-perturbative solutions of the Hopf equations. The concept of the "operator product expansion", for instance, which has been historically restricted to the realm of high-energy physics and the statistical mechanics of second order phase transitions, is likely to have a crucial place in the future theoretical investigations of turbulence.

4. References

C. Foias, O. P. Manley, R. Rosa, and R. Temam, Comptes Rendus Acad. Sci. Paris, Série I, **333**, 499 (2001).

C. Foias, M. S. Jolly, O. P. Manley, R. Rosa, and R. Temam, Kolmogorov theory via finite time averages, *to appear.*

J.M. Burgers, Adv. in Appl. Mech. **1**, 171 (1948).

A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, Nucl. Phys. **B241**, 33 (1984).

A.M. Polyakov, Nucl. Phys. **B396**, 367 (1993).

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, Oxford (1996).
C. Foias, Rend. Sem. Mat. Univ. Padova **48**, 219 (1972).
C. Foias, O. P. Manley, R. Rosa, and R. Temam, *Navier-Stokes Equations and Turbulence,* Encyclopedia of Mathematics and its Applications, Vol. **83**, Cambridge University Press, Cambridge, 2001.
C. Foias, Rend. Sem. Mat. Univ. Padova **49**, 9 (1973).
R. Rosa, Applications of Mathematics, **47**, 485 (2002).
C. Foias, M. S. Jolly, O. P. Manley, and R. Rosa, J. Stat. Phys. **108**, 591 (2002).
C. Foias, M. S. Jolly, and O. P. Manley, Kraichnan turbulence via finite time averages, *to appear.*
P. Constantin and C. R. Doering, Phys. Rev. E **49**, 4087 (1994).
P. Constantin and C. R. Doering, Phys. Rev. E **51**, 3192 (1995).
H. Bercovici, P. Constantin, C. Foias, and O. P. Manley, J. Stat. Phys. **80**, 579 (1995).
C. R. Doering and C. Foias, J. Fluid Mech. **467**, 289 (2002).
C. Foias, What do the Navier-Stokes equations tell us about turbulence?, in Harmonic Analysis and Nonlinear Differential Equations (Riverside, CA, 1995), Contemp. Math. **208**, 151 (1997).
C. Foias, O. P. Manley, R. Rosa, and R. Temam, Comptes Rendus Acad. Sci. Paris, Série I, **332**, 509 (2001).
K. Huang, *Statistical Mechanics*, John Wiley & Sons, Inc., New York (1987).
A.M. Polyakov, Phys. rev E **52**, 6183 (1995).
V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).
A. Chekhlov and V. Yakhot, Phys. Rev. Lett. **77**, 3118 (1996).
W. E, K. Khanin, A. Mazel, and Ya.G. Sinai, Phys. Rev. Lett. **78**, 1904 (1997).
T. Gotoh and R.H. Kraichnan, Phys. Fluids **10**, 2859 (1998).
W. E and E.V. Eijnden, Comm. Pure and Appl. Math. Vol. **LIII**, 0852 (2000).
L. Moriconi and G.S. Dias, Phys. Lett. A **287**, 356 (2001).
R.H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
K. Gawedzki and A. Kupianen, Phys. Rev. Lett. **75**, 3608 (1995).
B.I. Shraiman and E.D. Siggia, Phys. Rev. Lett. **77**, 2463 (1996).
G. Falkovich, I. Kolokov, V. Lebedev and A. Migdal, Phys. Rev. E **54**, 4896 (1996).
E. Balkovski and V. Lebedev, Phys. Rev. E **58**, 5776 (1998).
H.W. Wyld, Ann. Phys. **14**, 143 (1961).
S. Orszag, I. Staroselsky, W.S. Flannery, and Y. Zhang in *Simulation and Modeling of Turbulent Flows*, Edited by T.B. Gatski, M.Y. Hussaini, and J.L. Lumley, Oxford University Press, Oxford (1996).
L. Moriconi and F.A.S. Nobre, Phys. Rev. E **65**, 036302 (2002).
L. Moriconi, Phys. Rev. E **70**, 025302(R) (2004).
L. Moriconi and F.A.S. Nobre, Phys. Rev. E **70**, xxx (2004).
D. Lowe, Mod. Phys. Lett. A **8**, 923 (1993).
L. Moriconi, Phys. Rev. E **54**, 1550 (1996).
E.A. Novikov, Zh. Exper. Teor. Fiz. **47**, 1919 (1964).
P.C. Martin, E.D. Siggia and H.A. Rose, Phys. Rev. A **8**, 423 (1973).
R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), pp. 71–73 and 273–279.
L. Moriconi, Am. J. Phys, **72**, 1258 (2004b).
G.K. Batchelor and A.A. Townsed, Proc. R. Soc. Lond. A **199**, 238 (1949).
P. Tabeling, G. Zocchi, F. Belin, J. Maurer, and H. Willaime, Phys. Rev. E **53**, 1613 (1996).
U. Frisch, *Turbulence – The Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge (1995).
P.G. Saffmann, Stud. Appl. Math. **50**, 277 (1971).
R.H. Kraichnan, Phys. Fluids **10**, 14177 (1967).
B. Legras, P. Santangelo and R. Benzi, Europhys. Lett. **5**, 37 (1988).
A. Babiano, B. Dubrulle, and P. Frick, Phys. Rev. E **52**, 3719 (1995).
R. Benzi, B. Legras, G. Parisi, R. Scardovelli, Europhys. Lett. **29**, 203 (1995).