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WEAK THREE DIMENSIONALITY OF A FLOW AROUND A SLENDER CYLINDER:THE GINZBURG-LANDAU EQUATION

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In this paper a weak three-dimensionality of the flow around a slender cylinder is considered and the related model, the so-called Ginzburg-Landau equation, is here obtained as an asymptotic solution of the 3D (discrete) Navier-Stokes equation. The derivation is in line with existing slender bodies theories, as the Lifting Line Theory, for example, where the basic 2D flow, leading to Landau's equation, is influenced now by a "sidewash" that modifies bi-dimensionally the original flow through mass conservation. The theory is asymptotically consistent and rests on an assumption that holds in the vicinity of the Hopf bifurcation ($Re_{cr} \approx 45$); furthermore, it leads to a well-established way to determine numerically both the Landau's coefficient μ and Ginzburg's coefficient γ. Arguments are given suggesting that this assumption should hold far beyond Hopf bifurcation ($Re >> Re_{cr}$) and, with it, to extend the Ginzburg-Landau equation almost to the border of the transition region $Re \approx 105$. In this work only the theoretical development is addressed; numerical results will be presented in a forthcoming paper.

1. Introduction

Viscous flow around a 2D circular cylinder is known to produce spontaneous harmonic oscillations of the wake for Reynolds number above a critical value $Re_{cr} \approx 45$, the oscillating part of the pressure giving rise to an harmonic transverse force on the cylinder that has importance in several engineering applications. It seems to be now well established that this is essentially a stability problem, see Huerre & Monkewitz (1990), and that the oscillatory wake can be identified with Hopf birfucation in the language of the dynamic systems theory: this idea was first advanced, in a more operational way, by Bishop & Hassan (1964) and it has been verified experimentally, among others, by Provansal et. al. (1987) in their study in the vicinity of the critical Reynolds number.

Empirical evidences show that both the shedding frequency (Strouhal number) and the whole phenomenon of the vortex induced vibration, at least in its more macroscopic appearance, are essentially invariant with Reynolds number up to the transition zone ($Re \approx 10^5$). This observation has led some authors (Iwan & Blevins (1975), for example) to propose (heuristic) *phenomenological models*, based on Van der Pol equation, to predict the hydro-elastic interaction, with results that are impressive given the somewhat loose fluid dynamic background on which they are based; interestingly enough, the predictions from such models are, in some aspects, in much better agreement with experiments than the ones obtained from direct CFD computation. Being heuristic as they are, however, they can be used only as interpolators and hardly to extrapolate results to situations much beyond the empirical data on which they are based; furthermore, some hysteric behavior observed in the experiments are not recovered by these models and, obviously, the direct link with the Navier-Stokes equation is lacking in such approach.

The final purpose of the on-going research is to derive a "fluid-elastic oscillator model" directly from Navier-Stokes equation, rendering it not only whole predictive but also making it possible to be used in different situations from the ones observed in the existing experimental facilities; in particular, the case where the incident current changes both in direction and intensity along the cylinder span is particularly relevant for offshore applications. Notice also that the link with the more fundamental Navier-Stokes equation has here an even greater motivation, since direct computation with CFD did not produce yet a reliable result.

In the present paper only the first step towards this final goal is addressed, namely, to derive the "fluid oscillator model" by considering the cylinder fixed in the flow. The model is represented by the so-called *Ginzburg-Landau Equation*, see (3.1a), first proposed by Albarède et al (1990) in the context of VIV, with a basic difference, however, in relation to the usual approach: now this equation is not fitted externally to the problem but it results from a consistent asymptotic approximation of the 3D (discrete) Navier-Stokes Equation (NSE). In particular, the coefficients of this equation – the *Landau's coefficient* μ and the *Ginzburg's coefficient* γ – are not inferred from the experiments but they are directly computed by well established numerical procedures based on the Finite Element Method (FEM) applied to the 2D cross-flow problem: as usual in "slender bodies theories", one has thus an essentially 2D effort to compute a 3D result.

The discrete FEM model is derived, as always, in a *finite fluid region* R and one has certainly a difficulty to define the "discrete fluid flow operator" in R due to the loosely known form of the proper boundary condition at the *outlet* of R. By considering the flow equation in the wake it has been possible to express the "resistance" offered by the wake on the flow within R, named here the *"wake impedance"*, by an explicit expression that depends solely on the *velocity and* *acceleration* of the flow on the *borderline* that defines the interface between R and the wake. This derivation is elaborated elsewhere and it may have an importance that transcends the specific application aimed in this work.

The final *discrete* NSE emulates the *continuum* NSE with a *local inertia*, a *convective inertia* and a *viscous dissipation* that incorporate the contributions from both the *finite fluid region* R, that is actually discretized, and the *wake*. This discrete set of equations are thus projected into the *solenoidal* and *gradient* sub-spaces and standard results in Linear Algebra are used to show the inner consistency of these projections; in particular, the projection on the *solenoidal sub-space*, that determines the velocity field, leads to a normal quadratic dynamic system to which the usual asymptotic procedure can be applied, to determine first *Landau's Equation* in the 2D context and after the 3D *Ginzburg-Landau Equation*.

As it is known, a myriad of interesting small scale features, some of them uncovered by a detailed numerical analysis, appear concomitantly with the *gross macroscopic ordered* behavior of the wake that really matters in the study of the *hydro-elastic phenomenon* (VIV): the purpose in this work it is not, thus, to present a taxonomy of the chaos but rather to capture the underlying order. To achieve this goal a sort of "blindness" is needed, to avoid a too detailed picture, and the *asymptotic theory* is just a technical filter that provides it. This theory rests on a well defined *assumption*, see (2.21), that can be directly verified by the numerical results; furthermore, although strictly justified only in the vicinity of Hopf bifurcation ($Re \approx Re_{cr}$), it seems to hold in a much broader range of Reynolds numbers, what makes possible to extend Ginzburg-Landau equation to this range. Incidentally, this yet speculative result can furnish a theoretical background for the so-called "phenomenological models" that are in fact applied, with a relative success in the prediction of VIV, in a range of Reynolds numbers far beyond the Hopf bifurcation ($Re >> Re_{cr}$).

The paper is organized as follows: in section 2 the two-dimensional problem is addressed, leading to *Landau's equation*, and in section 3 a *weak three dimensionality* of the flow is considered and *Ginzburg-Landau equation* is obtained. Some more technical results, including the derivation of the "wake impedance", are derived elsewhere and numerical results will be presented in a forthcoming paper.

2. Two-dimensional solution: Landau's Equation

In this section the *two dimensional* cross-flow around a cylinder is considered. Points in the cross section plane are designated by the vector $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$, the fluid velocity by the vector field $\mathbf{u}(\mathbf{x},t) = u(\mathbf{x},t)\mathbf{i} + v(\mathbf{x},t)\mathbf{j}$, the pressure by the function $p(\mathbf{x},t)$ while the differential operator ∇ is defined by the expression $\nabla = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$; these notations will be kept all through the work, even in the next section where the three dimensional correction will be addressed.

Let $d = 1$ be the typical dimension of the cylinder cross section (the cylinder diameter in the case of a circular cylinder) and $U = 1$ be the incident velocity along the x-axis at the infinite; obviously, the non-dimensional frequency $\omega d/U$ coincides numerically with ω and both forms will be used here, depending on the convenience. The fluid density will be also assumed unitary ($\rho = 1$) and thus $\mu_v = \mu_v/\rho U d = 1/Re$, where μ_v is the fluid viscosity and Re the Reynolds number.

It is desirable to work here with a velocity field $\mathbf{u}(\mathbf{x},t)$ that satisfies *homogeneous* boundary condition both at the *infinite* and at the *cross section* contour line ∂B. With this purpose in mind one introduces here an auxiliary vector field $\mathbf{u}_p(\mathbf{x})$ such that

$$\begin{aligned} &\bullet \nabla\cdot\mathbf{u}_p = 0; \\ &\bullet \mathbf{u}_p(\mathbf{x})\Big|_{\mathbf{x}\in\partial B} = \mathbf{0}; \\ &\bullet \lim_{|\mathbf{x}|\to\infty} \mathbf{u}_p(\mathbf{x}) = \mathbf{i}\ , \end{aligned} \tag{2.1a}$$

with the subsidiary condition that $\mathbf{u}_p(\mathbf{x})$ approaches its limit value $\mathbf{i}$ "fast enough", namely: $\mathbf{u}_p(\mathbf{x}) \cong \mathbf{i}$ for $|\mathbf{x}| > 5d$, for example. In the case of a *circular* cross section this field can be determined with the help of the *stream function*

$$\begin{aligned} &\psi_p(\mathbf{x}) = -r\sin\theta + a_1\cdot\frac{\sin\theta}{r^{\alpha-1}} + a_2\cdot\frac{\sin\theta}{r^{\alpha}}; \\ &a_1/r_c^{\alpha} = \alpha+1;\ a_2/r_c^{\alpha+1} = -\alpha;\ (r_c = ½), \end{aligned} \tag{2.1b}$$

where $\mathbf{u}_p(\mathbf{x}) \cong \mathbf{i}$ for $|\mathbf{x}| > 5d$ with an error smaller than 0.025% for $\alpha = 5$. If the cross section is arbitrary this function $\mathbf{u}_p(\mathbf{x})$ can be determined numerically, for instance, but once this is done the actual *velocity field* $\mathbf{u}_T(\mathbf{x},t)$ can be written as

$$\mathbf{u}_T(\mathbf{x},t) = \mathbf{u}_p(\mathbf{x}) + \mathbf{u}(\mathbf{x},t)\,. \tag{2.1c}$$

Introducing now the (volume) force vector

$$\mathbf{f}_p(\mathbf{x}) = -\left(\mathbf{u}_p \cdot \nabla\right)\mathbf{u}_p + \frac{1}{Re}\nabla^2\mathbf{u}_p, \tag{2.2a}$$

the flow problem is reduced to determine the fields {**u**(**x**,t); p(**x**,t)} such that

$$\begin{aligned} &\bullet \frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\nabla^2\mathbf{u} + \left[\left(\mathbf{u}_p \cdot \nabla\right)\mathbf{u} + \left(\mathbf{u} \cdot \nabla\right)\mathbf{u}_p\right] + \left(\mathbf{u} \cdot \nabla\right)\mathbf{u} + \nabla p = \mathbf{f}_p(\mathbf{x}); \\ &\bullet \nabla \cdot \mathbf{u} = 0, \end{aligned} \tag{2.2b}$$

subjected to the *homogeneous boundary conditions*

$$\begin{aligned} &\bullet \mathbf{u}(\mathbf{x},t)\big|_{\mathbf{x}\in\partial B} = \lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x},t) = \mathbf{0}; \\ &\bullet \lim_{|\mathbf{x}|\to\infty} p(\mathbf{x},t) = 0. \end{aligned} \tag{2.2c}$$

The discrete version of (2.2) will be addressed next.

2.1: Boundary conditions and "wake impedance"

In order to deal with (2.2) one must specify, first of all, a *finite fluid region* R, as shown in Fig.(1), where the flow variables will be discretized by Finite Elements. Boundary conditions must be imposed on the border ∂R of R: only with them the *"fluid flow operator"* can be properly defined within R.

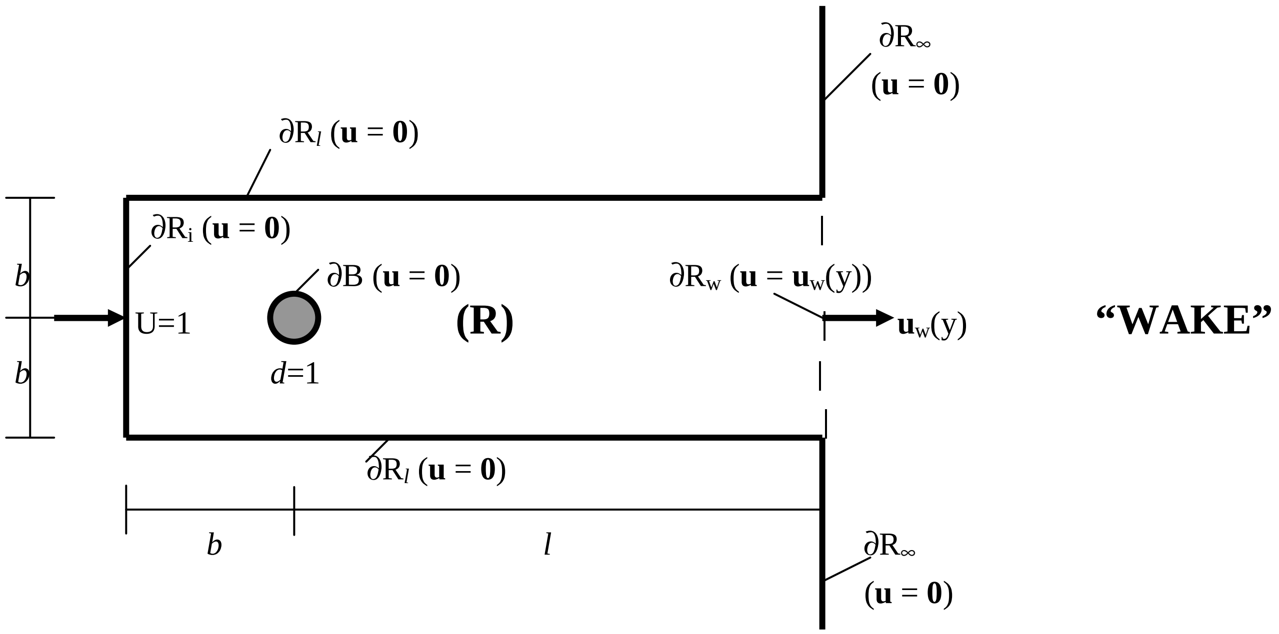

Figure 1. *Finite fluid region* R ($\partial R = \partial B \cup \partial R_i \cup \partial R_l \cup \partial R_w$) and *wake*.
($\mathbf{u}_T(\mathbf{x},t) = \mathbf{u}_p(\mathbf{x}) + \mathbf{u}(\mathbf{x},t)$; $\mathbf{u}_p(\mathbf{x}) \cong \mathbf{i}$ for $|\mathbf{x}| \geq b$).

The boundary ∂R is made by the cross section contour line ∂B, by the "inlet" ∂R_i at the vertical line $x = -b$, by the "lateral sides" ∂R_l at $y = \pm b$ and by the "outlet" ∂R_w at $x = l$ that defines the interface between R and the *wake region* $x > l$ ($\partial R = \partial B \cup \partial R_i \cup \partial R_l \cup \partial R_w$). The velocity field **u**(**x**,t) is certainly null at ∂B and if *b* is large enough ($b \approx 5d$) and *l* is not much larger than *b* ($l \approx 10d$) it seems reasonable to assume that **u**(**x**,t) is also null at $\partial R_i \cup \partial R_l$: the presence of the cylinder should not perturb the incoming flow at sufficient distance both upwind and laterally. The following *essential boundary condition* is thus assumed on this part of ∂R:

$$\mathbf{u}(\mathbf{x},t)\big|_{\mathbf{x}\in\partial B \cup \partial R_i \cup \partial R_l} = \mathbf{0}. \tag{2.3a}$$

The same homogeneous boundary condition cannot be extended to the "outlet" ∂R_w unless the distance *l* is very large (and so it must be *b*). In fact, the vorticity generated at the cylinder dies out very slowly downstream, typically in a distance of order $l \approx$ Re for the largest wavelength, and since b increases roughly with $l^{1/2}$ the condition (2.3a) could be pushed to ∂R_w only if the finite region R becomes very large[1].

The velocity field at ∂R_w should thus remain unspecified while the proper boundary condition at the *outlet* of R will be defined below in this section. If now the *dynamic equation* in (2.2b) is multiplied by a *virtual velocity* $\delta\mathbf{u}(\mathbf{x})$ that satisfies, as usual, the *same essential boundary condition* (2.3a), and the *continuity equation* in (2.2b) is multiplied by $\delta p(\mathbf{x})$ and both expressions are further integrated in R one obtains, after partial integration, that

$$\begin{aligned}
&\bullet\int_R \frac{\partial\mathbf{u}}{\partial t}\cdot\delta\mathbf{u}\ dR + \int_R \left\{\left[\left(\mathbf{u}_p\cdot\nabla\right)\mathbf{u} + \left(\mathbf{u}\cdot\nabla\right)\mathbf{u}_p\right]\cdot\delta\mathbf{u} + \frac{1}{Re}\left(\nabla u\cdot\nabla(\delta u) + \nabla v\cdot\nabla(\delta v)\right)\right\} dR + \\
&+\int_R \left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\cdot\delta\mathbf{u}\ dR - \int_R p(\nabla\cdot\delta\mathbf{u})dR = \int_R \mathbf{f}_p(\mathbf{x})\cdot\delta\mathbf{u}\ dR + I(\mathbf{u}_w(y);\delta\mathbf{u}_w(y)); \\
&\bullet\int_R \delta p(\nabla\cdot\mathbf{u})dR = 0.
\end{aligned} \tag{2.3b}$$

In (2.3b) the notation $\{(\mathbf{u}_w(y,t);\ \delta\mathbf{u}_w(y)) \equiv (\mathbf{u}(l,y,t);\ \delta\mathbf{u}(l,y));\ |y| \le b\}$ was used to define the velocity and the virtual velocity over ∂R_w and $I(\mathbf{u}_w(y);\ \delta\mathbf{u}_w(y))$ is the *"wake impedance"*, namely,

$$I(\mathbf{u}_w(y);\delta\mathbf{u}_w(y)) = \int_{-b}^{b}\left[\left(-p + \frac{1}{Re}\frac{\partial u}{\partial x}\right)_w\cdot\delta u_w(y) + \left(\frac{1}{Re}\frac{\partial v}{\partial x}\right)_w\cdot\delta v_w(y)\right]dy\,, \tag{2.3c}$$

where, assuming continuity, $(\cdot)_w$ stands for the *stress field in the wake* at $x = l$: the *"wake impedance"* is thus the *virtual power* done by this stress field on the *virtual velocity* $\delta\mathbf{u}_w(y)$ and it represents the "resistance" offered by the wake for the flow within the finite fluid region R.

The *wake region* is bounded on the left by ∂R_w and by two semi-infinite lines ∂R_∞, as indicated in Fig.(1). Since $\mathbf{u}(\mathbf{x},t)$ was assumed null at the lateral sides ∂R_l it is certainly consistent with this assumption to take $\mathbf{u}(\mathbf{x},t) \equiv \mathbf{0}$ on ∂R_∞, since the perturbation caused by the cylinder should be even smaller over ∂R_∞ than over ∂R_l. It turns out then that the flow in the wake is forced solely by the field $\mathbf{u}_w(y,t)$ and thus

$$\begin{aligned}
&\left(-p + \frac{1}{Re}\frac{\partial u}{\partial x}\right)_w = F_X(\mathbf{u}_w); \\
&\left(\frac{1}{Re}\frac{\partial v}{\partial x}\right)_w = F_Y(\mathbf{u}_w),
\end{aligned} \tag{2.4a}$$

where the functionals $\Phi_{X,Y}(\cdot)$ can be determined by solving the flow problem in the wake. If both the velocity and virtual velocity $\{\mathbf{u}_w(y,t);\ \delta\mathbf{u}_w(y)\}$ at the outlet are discretized as

$$\begin{Bmatrix}\mathbf{u}_w(y,t)\\ \delta\mathbf{u}_w(y)\end{Bmatrix} = \sum_{k=1}^{N_w}\left(\begin{Bmatrix}U_{w,k}(t)\\ \delta U_{w,k}\end{Bmatrix}\mathbf{i} + \begin{Bmatrix}U_{w,k+N_w}(t)\\ \delta U_{w,k+N_w}\end{Bmatrix}\mathbf{j}\right)\cdot h_{k,w}(y)\,, \tag{2.4b}$$

where $\{h_{k,w}(y);\ k = 1,2,\ldots,N_w\}$ are the *interpolation functions* for the velocity field restricted to ∂R_w and $\{\mathbf{U}_w(t);\ \delta\mathbf{U}_w\}$ are the nodal values vector, it can be shown that

$$\begin{aligned}
&\bullet I(\mathbf{u}_w(y);\delta\mathbf{u}_w(y)) = \delta\mathbf{U}_w^t\cdot\mathbf{I}(\mathbf{U}_w); \\
&\bullet \mathbf{I}(\mathbf{U}_w) = -\left[\mathbf{M}_w\cdot\dot{\mathbf{U}}_w + \mathbf{K}_w\cdot\mathbf{U}_w + \mathbf{N}_w(\mathbf{U}_w)\cdot\mathbf{U}_w\right],
\end{aligned} \tag{2.4c}$$

the matrices $\{\mathbf{M}_w;\ \mathbf{K}_w;\ \mathbf{N}_w(\mathbf{U}_w)\}$ being computed from explicitly defined *Fourier series*[2].

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[1] This seems to be true even for a "small" Reynolds number: for Re = 41 the wake has already begun to oscillate sinusoidally *far downstream*, see Van Dyke (1982), plate 46. The existing numerical results predict, as a rule, a critical Reynolds number above 40 ($Re_{cr} \approx 45$) although the expected value should be below ($Re_{cr} \approx 35$).

[2] For the *Fourier series* expansion one must impose a finite breadth 2W for the wake, with 2W being arbitrarily large (2W >> b). It can be shown that the number of terms n_L in these series increases both with W and Re; typically, $n_L \cong O(W\cdot(Re)^{1/2})$.

Within R the discrete velocity and pressure fields can be expressed as

$$\mathbf{u}(\mathbf{x},t)=\sum_{k=1}^{N}\left(U_k(t)\mathbf{i}+U_{k+N}(t)\mathbf{j}\right)\cdot h_k(\mathbf{x});$$
$$p(\mathbf{x},t)=\sum_{\alpha=1}^{e}P_\alpha(t)\cdot t_\alpha(\mathbf{x}), \tag{2.5a}$$

with $\{h_k(\mathbf{x}); k = 1,2,...,N\}$ and $\{t_\alpha(\mathbf{x}); \alpha = 1,2,\dots,e\}$ being, respectively, the interpolating functions for the velocity and pressure fields. The functions $\{h_k(\mathbf{x})\}$ are necessarily continuous but the $\{t_\alpha(\mathbf{x})\}$ may or may not be so; in reality, the pressure field does work on $\nabla\cdot\delta\mathbf{u}$ and it seems reasonable to choose the $\{t_\alpha(\mathbf{x})\}$ in conformity with the discrete field $\nabla\cdot\delta\mathbf{u}$ obtained from the $\{h_k(\mathbf{x})\}$. Placing (2.5a) into the integrals that appear in (2.3b) and defining the matrices

$$\int_R \frac{\partial\mathbf{u}}{\partial t}\cdot\delta\mathbf{u}\ dR=\delta\mathbf{U}^t\cdot\mathbf{M}_R\cdot\dot{\mathbf{U}};\quad \int_R(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\delta\mathbf{u}\ dR=\delta\mathbf{U}^t\cdot\mathbf{N}_R(\mathbf{U})\cdot\mathbf{U};$$
$$\int_R\left[\left(\mathbf{u}_p\cdot\nabla\right)\mathbf{u}+\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}_p\right]\cdot\delta\mathbf{u}\ dR+\frac{1}{Re}\int_R\left(\nabla u\cdot\nabla(\delta u)+\nabla v\cdot\nabla(\delta v)\right)dR=\delta\mathbf{U}^t\cdot\mathbf{K}_{p,R}\cdot\mathbf{U};$$
$$\int_R p(\nabla\cdot\delta\mathbf{u})dR=\delta\mathbf{U}^t\cdot\mathbf{R}\cdot\mathbf{P};\quad \int_R \delta p(\nabla\cdot\mathbf{u})dR=\delta\mathbf{P}^t\cdot\mathbf{R}^t\cdot\mathbf{U};$$
$$\int_R \mathbf{f}_p(\mathbf{x})\cdot\delta\mathbf{u}\ dR=\delta\mathbf{U}^t\cdot\mathbf{F}_p, \tag{2.5b}$$

the *discrete form* of the flow equation in weak form reads (see (2.3b) and (2.4c))

$$\bullet\delta\mathbf{U}^t\cdot\left\{\mathbf{M}_R\cdot\dot{\mathbf{U}}+\mathbf{K}_{p,R}\cdot\mathbf{U}+\mathbf{N}_R(\mathbf{U})\cdot\mathbf{U}-\mathbf{R}\cdot\mathbf{P}\right\}=$$
$$=-\delta\mathbf{U}_w^t\cdot\left\{\mathbf{M}_w\cdot\dot{\mathbf{U}}_w+\mathbf{K}_w\cdot\mathbf{U}_w+\mathbf{N}_w(\mathbf{U}_w)\cdot\mathbf{U}_w\right\}+\delta\mathbf{U}^t\cdot\mathbf{F}_p;$$
$$\bullet\mathbf{R}^t\cdot\mathbf{U}=\mathbf{0}. \tag{2.5c}$$

Observing that $\{\mathbf{U}_w; \delta\mathbf{U}_w\}$ are, in fact, the part of $\{\mathbf{U}; \delta\mathbf{U}\}$ defined in ∂R_w, one can take the "wake impedance" on the left side

$$\bullet\delta\mathbf{U}^t\cdot\left\{\mathbf{M}\cdot\dot{\mathbf{U}}+\mathbf{K}_p\cdot\mathbf{U}+\mathbf{N}(\mathbf{U})\cdot\mathbf{U}-\mathbf{R}\cdot\mathbf{P}\right\}=\delta\mathbf{U}^t\cdot\mathbf{F}_p;$$
$$\bullet\mathbf{R}^t\cdot\mathbf{U}=\mathbf{0}. \tag{2.6}$$

The *dynamic parcels* of (2.6), proportional to $\{\mathbf{M}; \mathbf{K}_p; \mathbf{N}(\mathbf{U})\}$, come both from the *finite fluid region* R, where the flow variables are discretized, and the *wake region* downstream: **M** is the *local inertia matrix*, $\mathbf{K}_p$ represents, as indicated in (2.5b), the influence of the *viscous stress* and the *convective acceleration* due to the *auxiliary field* $\mathbf{u}_p(\mathbf{x})$ and $\mathbf{N}(\mathbf{U})\cdot\mathbf{U}$ is the *convective inertia force*. The parcels $\{\mathbf{R}^t\cdot\mathbf{U}; \mathbf{R}\cdot\mathbf{P}\}$ are due to *mass conservation* and the related *constraint force* (pressure) defined in the finite region R: the matrix – **R** represents the (discrete) *gradient operator* and $\mathbf{R}^t$ the (discrete) *divergence operator*. Obviously, mass is already conserved in the "wake solution" (2.4c) and this constraint should not appear again at (2.6).

The (discrete) *Navier-Stokes equation* (2.6) emulates thus the structure of the original equation (2.2b) and it will be analyzed next: in this discrete form the *mathematical analysis* is much simpler, once is based only on some general results in Linear Algebra, and, furthermore, its outcome has an operational appeal given its direct link to the final numerical results.

2.2: The solenoidal and gradient sub-spaces

Let W_N be the 2N-dimensional linear space of the *discrete velocity vector* **U** and L_e be the e-dimensional linear space of the *discrete pressure vector* **P**; in both spaces it will be assumed the standard *inner product* $<\mathbf{U};\mathbf{V}> = \mathbf{U}^t\cdot\mathbf{V}$ and the related *norm* $\|\mathbf{U}\|^2 = <\mathbf{U};\mathbf{U}> = \mathbf{U}^t\cdot\mathbf{U}$.

Let also $\{J_s; G_r\}$ be the s-dimensional and r-dimensional sub-spaces of W_N defined by the relations

$$J_s = \left\{ \mathbf{V} \in W_N : \mathbf{R}^t \cdot \mathbf{V} = \mathbf{0} \in L_e \right\};$$
$$G_r = \left\{ \mathbf{V} \in W_N : \mathbf{V} = \mathbf{R} \cdot \boldsymbol{\varphi}; \boldsymbol{\varphi} \in L_e \right\}. \tag{2.7a}$$

Elements of G_r are "gradients" of "scalar fields" $\boldsymbol{\varphi} \in L_e$ and for this reason G_r is called the *"gradient sub-space"* of W_N; notice that G_r is generated by the linear combinations of the column vectors of **R**. Elements of J_s have null divergence and so J_s is called the *"solenoidal sub-space"*; by definition, their elements are orthogonal to the column vectors of **R** and thus J_s is the *orthogonal complement* of G_r or (r + s = 2N)

$$W_N = J_s \oplus G_r , \tag{2.7b}$$

see Ladyzenskaja (1969).

One introduces here the operators

$$\bullet \boldsymbol{\Delta} = \mathbf{R}^t \cdot \mathbf{R} : L_e \to L_e;$$
$$\bullet \boldsymbol{\nabla} = \mathbf{R} \cdot \mathbf{R}^t : W_N \to W_N, \tag{2.8}$$

where $\boldsymbol{\Delta}$ is the (discrete) *Laplacian operator* and $\boldsymbol{\nabla}$ will be named the *"conjugated Laplacian"*. Both $\boldsymbol{\Delta}$ and $\boldsymbol{\nabla}$ are represented by *symmetric, positive semi-definite sparse matrices*, the "sparseness" being a consequence of the "local character" of the Finite Element discretization.

By definition $\mathbf{R}^t \cdot \mathbf{V} = \mathbf{0}$ if $\mathbf{V} \in J_s$ and so $\boldsymbol{\nabla} \cdot \mathbf{V} = \mathbf{0}$ or $J_s \subseteq$ *Null* ($\boldsymbol{\nabla}$). In reality, it can be shown that $J_s \equiv$ *Null* ($\boldsymbol{\nabla}$). Let $\{\mathbf{T}_\alpha;\ \alpha = 1,2,\ldots,s\}$ be an *othonormal basis* of *Null* ($\boldsymbol{\nabla}$) $\equiv J_s$ and $\{\mathbf{G}_j;\ j = 1,2,\ldots,r\}$ be the *orthonormal eigenvectors* corresponding to the *positive spectrum* $\{\kappa_j > 0;\ j = 1,2,\ldots,r\}$ of $\boldsymbol{\nabla}$, namely:

$$\bullet \boldsymbol{\nabla} \cdot \mathbf{T}_\alpha = \mathbf{0};\ \ \alpha = 1,2,...,s;$$
$$\bullet \boldsymbol{\nabla} \cdot \mathbf{G}_j = \kappa_j \mathbf{G}_j;\ \ j = 1,2,...,r. \tag{2.9a}$$

Obviously $\{\mathbf{G}_1; \mathbf{G}_2; \ldots; \mathbf{G}_r\}$ is a basis of G_r while $\{\mathbf{T}_1; \mathbf{T}_2; \ldots;\mathbf{T}_s\}$ is a basis of J_s; assuming that $0 < \kappa_1 \le \kappa_2 \le \ldots \le \kappa_r$ consider the matrix

$$\boldsymbol{\nabla}_1 = \mathbf{I} - \frac{1}{\kappa_r} \boldsymbol{\nabla} . \tag{2.9b}$$

Certainly $\boldsymbol{\nabla}_1$ is a *symmetric, positive semi-definite sparse matrix* with a spectrum in the interval [0;1]; furthermore

$$\bullet \boldsymbol{\nabla}_1 \cdot \mathbf{T}_\alpha = \mathbf{T}_\alpha , \tag{2.9c}$$

a relation that can be used to determine an *orthonormal basis* of the *solenoidal sub-space*[3] .

The *Laplacian operator* $\boldsymbol{\Delta}$ may have a non-empty null sub-space (*Null* ($\boldsymbol{\Delta}$) $\neq \emptyset$) but it certainly has a *positive spectrum*; in fact, if

$$\bullet \hat{\mathbf{G}}_j = \frac{1}{\sqrt{\kappa_j}} \mathbf{R}^t \cdot \mathbf{G}_j , \tag{2.10a}$$

then one can easily check that $\| \hat{\mathbf{G}}_j \| = 1$ with

$$\bullet \boldsymbol{\Delta} \cdot \hat{\mathbf{G}}_j = \kappa_j \hat{\mathbf{G}}_j . \tag{2.10b}$$

Observing now the conjugated relation

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[3] The ARPACK algorithm is specially suited to deal with eigenvalue problems of a large *sparse* matrices, see Lehoucq & Sorensen & Yang(1997).

$$\bullet \mathbf{G}_j = \frac{1}{\sqrt{\kappa_j}} \mathbf{R} \cdot \hat{\mathbf{G}}_j, \tag{2.10c}$$

the following result can be derived: the positive spectrum of $\mathbf{\Delta}$ must coincide, necessarily, with the positive spectrum $\{\kappa_j > 0;\ j = 1,2,\ldots,r\}$ of $\mathbf{\nabla}$. In fact, if $\kappa \neq \kappa_j$ were a *positive* eigenvalue of $\mathbf{\Delta}$ with eigenvector $\hat{\mathbf{G}}$ then $\mathbf{G} = \mathbf{R} \cdot \hat{\mathbf{G}}$ should be an eigenvector of $\mathbf{\nabla}$ with the same eigenvalue κ and so $\kappa \in \{\kappa_j > 0;\ j = 1,2,\ldots,r\}$ once, by definition, this is the set of *all* positive eigenvalues of $\mathbf{\nabla}$. The operators $\{\mathbf{\Delta};\mathbf{\nabla}\}$ establish, thus, a duality between the sub-space $\hat{\mathrm{G}}_r \subset \mathrm{L}_e$, generated by the vectors $\{\hat{\mathbf{G}}_1;\hat{\mathbf{G}}_2;\ldots;\hat{\mathbf{G}}_r\}$, and the gradient sub-space $\mathrm{G}_r \subset \mathrm{W}_N$: if $\boldsymbol{\varphi} \in \hat{\mathrm{G}}_r$ then $\mathbf{R}\cdot\boldsymbol{\varphi} \in \mathrm{G}_r$ and if $\mathbf{V} \in \mathrm{G}_r$ then $\mathbf{R}^t\cdot\mathbf{V} \in \hat{\mathrm{G}}_r$; the sub-space *Null* ($\mathbf{\Delta}$) is the orthogonal complement of $\hat{\mathrm{G}}_r$ and so

$$\begin{aligned} \mathrm{L}_e &= Null(\mathbf{\Delta}) \oplus \hat{\mathrm{G}}_r; \\ \mathrm{W}_N &= Null(\mathbf{\nabla}) \oplus \mathrm{G}_r. \end{aligned} \tag{2.11a}$$

Let $\{\mathbf{S}_\alpha;\ \alpha = 1,2,\ldots,e\text{-}r\}$ be an orthonormal basis of *Null* ($\mathbf{\Delta}$), named the *"spurious pressure modes"* in the specialized literature, see Gunzburger (1985); they satisfy the relations

$$\bullet \mathbf{\Delta} \cdot \mathbf{S}_\alpha = \mathbf{0};\ \alpha = 1,2,\ldots,e-r \tag{2.11b}$$

and, as it will be seen in the next item, these modes play in the discrete problem the same role played by the *constant pressure field* in the continuum problem, namely: they do not interfere with the dynamics of the flow. In accordance with (2.9b) one introduces here the matrix

$$\bullet \mathbf{\Delta}_1 = \mathbf{I} - \frac{1}{\kappa_r} \mathbf{\Delta}, \tag{2.11c}$$

where, again, $\mathbf{\Delta}_1$ is a *symmetric, positive semi-definite sparse matrix* with spectrum in the interval [0;1]; this matrix will be used in the next item in the context of the *Poisson's equation* for the (discrete) pressure field.

It seems worthwhile to finish this section with a more technical remark about the Finite Element discretization, related to the so called "div-stability condition" (Ladyzhenskaya – Babuska – Brezzi condition). The point is that for some classes of Finite Elements (FE) the smallest eigenvalue κ_1 becomes "too small" as the mesh size h goes to zero, indicating that elements of the *gradient sub-space* G_r tend to "slip" into the *solenoidal sub-space* J_s as $h \to 0$. In this case the solenoidal sub-space J_s becomes "too rarefied" once at least some of the solenoidal fields are, in fact, "slipping modes" of G_r; this problem is particularly acute for the simplest FE discretization, where the velocity field is linear piecewise continuous and the pressure is constant in each element, see Gunzburger (1985) and Bathe (1996), for example. However, as shown in Aranha (2003), if the actual solenoidal sub-space is *enlarged* by these "slipping modes" in a way dictated by the "div-stability condition" this problem can be overcome without impairing the standard Finite Element convergence rate. In the present context some of the eigenvalues-eigenmodes in G_r are naturally computed in the effort to determine the basis $\{\mathbf{T}_\alpha;\ \alpha = 1,2,\ldots,s\}$ of J_s and this "enlarging" process can then be worked out easily; or, in short, questions related to the "div-stability condition" are of no special concern here.

2.3: The solenoidal velocity field and poisson's equation

The solution of the (discrete) *Navier-Stokes equation* (2.6) will be dealt in two stages: first, the projection of (2.6) into J_s will result in a standard nonlinear differential equation for the *velocity*; second, Poisson's equation for the *pressure* will be derived by projecting (2.6) into G_r. The simple structure of the dynamic equation in J_s allows one to develop *standard asymptotic analysis* for the underlying nonlinear system and to obtain, in this way, *Landau's equation* in the vicinity of the *Hopf bifurcation*; in the other side, it is possible to show, with the help of item (2.2), that *Poisson's equation* has a solution and that this solution is *"unique"*. With this purpose in mind one introduces the matrices

$$\begin{aligned}
&\bullet \mathbf{T} = [\mathbf{T}_1; \mathbf{T}_2; ...; \mathbf{T}_s];\\
&\bullet \mathbf{M}_s = \mathbf{T}^t \cdot \mathbf{M} \cdot \mathbf{T};\\
&\bullet \mathbf{K}_{p,s} = \mathbf{T}^t \cdot \mathbf{K}_p \cdot \mathbf{T};\\
&\bullet \mathbf{N}_s(\mathbf{q}) = \mathbf{T}^t \cdot \mathbf{N}(\mathbf{T}\cdot\mathbf{q}) \cdot \mathbf{T};\\
&\bullet \mathbf{F}_{p,s} = \mathbf{T}^t \cdot \mathbf{F}_p,
\end{aligned} \tag{2.12a}$$

where $\mathbf{R}^t\cdot\mathbf{T} = \mathbf{0}$ since $\mathbf{T}_\alpha \in J_s$; also, given arbitrary s-dimensional vectors $\{\mathbf{q}(t);\delta\mathbf{q}\}$ one has

$$\begin{aligned}
&\mathbf{U}(t) \in J_s \Leftrightarrow \mathbf{U}(t) = \mathbf{T}\cdot\mathbf{q}(t);\\
&\delta\mathbf{U} \in J_s \Leftrightarrow \delta\mathbf{U} = \mathbf{T}\cdot\delta\mathbf{q}.
\end{aligned} \tag{2.12b}$$

The virtual velocity $\delta\mathbf{U} \in W_N$ in (2.6) belongs either to J_s or to G_r; assuming first $\delta\mathbf{U} = \mathbf{T}\cdot\delta\mathbf{q} \in J_s$ and recalling that $\mathbf{T}^t\cdot\mathbf{R} = (\mathbf{R}^t\cdot\mathbf{T})^t = \mathbf{0}^t$ one obtains, with the help of (2.12b), the following equation for the variable $\mathbf{q}(t)$ (see (2.12a)):

$$\mathbf{M}_s \cdot \dot{\mathbf{q}} + \mathbf{K}_{p,s}\cdot\mathbf{q} + \mathbf{N}_s(\mathbf{q})\cdot\mathbf{q} = \mathbf{F}_{p,s}\,. \tag{2.13}$$

The asymptotic solution of (2.13), leading to *Landau's equation*, will be addressed in item (2.5). Introducing the *dynamic force vector* (see (2.6))

$$\bullet \mathbf{F}_D(t) = \mathbf{M}\cdot\dot{\mathbf{U}} + \mathbf{K}_p\cdot\mathbf{U} + \mathbf{N}(\mathbf{U})\cdot\mathbf{U} - \mathbf{F}_p \in W_N\,, \tag{2.14a}$$

one can write $\mathbf{\Phi}_D(t)$ in the form ($\{\mathbf{T}_\alpha\}\cup\{\mathbf{G}_j\}$ is an *orthonormal basis* of W_N)

$$\begin{aligned}
&\mathbf{F}_D(t) = \sum_{\alpha=1}^{s} f_{d,\alpha}(t)\cdot\mathbf{T}_\alpha + \sum_{j=1}^{r} f_{d,j}(t)\cdot\mathbf{G}_j;\\
&f_{d,\alpha}(t) = \left\langle \mathbf{F}_D(t);\mathbf{T}_\alpha\right\rangle; f_{d,j}(t) = \left\langle \mathbf{F}_D(t);\mathbf{G}_j\right\rangle.
\end{aligned} \tag{2.14b}$$

By placing now $\delta\mathbf{U} = \mathbf{R}\cdot\delta\boldsymbol{\varphi} \in G_r$ in (2.6) one obtains, with the help of (2.8) and (2.10a), the following *Poisson's equation* for the pressure $\mathbf{P}$

$$\bullet \mathbf{\Delta}\cdot\mathbf{P} = \mathbf{R}^t\cdot\mathbf{F}_D(t) = \sum_{j=1}^{r} f_{d,j}(t)\sqrt{\kappa_j}\cdot\hat{\mathbf{G}}_j \in \hat{G}_r\,, \tag{2.14c}$$

since $\mathbf{R}^t\cdot\mathbf{T}_\alpha \equiv \mathbf{0}$ ($\mathbf{T}_\alpha \in J_s$). But $\mathbf{P} \in L_e$ and $\{\mathbf{S}_\alpha;\ \alpha = 1,2,..,e\text{-}r\}\cup\{\hat{\mathbf{G}}_j;\ j = 1,2,..,r\}$ is an *orthonormal basis* of L_e: expressing $\mathbf{P}$ in this basis and using (2.10b) it is easy to check that the general solution of (2.14c) is given by (recall that $\mathbf{\Delta}\cdot\mathbf{S}_\alpha = \mathbf{0}$)

$$\mathbf{P} = \underbrace{\sum_{\alpha=1}^{e-r} a_\alpha\cdot\mathbf{S}_\alpha}_{\mathbf{P}_s} + \sum_{j=1}^{r}\frac{f_{d,j}(t)}{\sqrt{\kappa_j}}\cdot\hat{\mathbf{G}}_j\,,$$

where $\{a_\alpha;\ \alpha = 1,2,\ldots,e\text{-}r\}$ are arbitrary coefficients. The *"spurious mode"* $\mathbf{P}_s = \Sigma a_\alpha\cdot\mathbf{S}_\alpha$ plays here the same role played by the *constant pressure field* in the continuum problem: in fact, *its (discrete) gradient* $\mathbf{R}\cdot\mathbf{P}_s$ – and it is in this way that the pressure appears in the discrete flow equation (2.6) – *is null* since $\mathbf{\Delta}\cdot\mathbf{P}_s \equiv \mathbf{0}$ (recall that $\mathbf{\Delta}\cdot\mathbf{S}_\alpha = \mathbf{0}$) and

$$\|\mathbf{R}\cdot\mathbf{P}_s\|^2 = \mathbf{P}_s^t\cdot\mathbf{R}^t\cdot\mathbf{R}\cdot\mathbf{P}_s = \mathbf{P}_s^t\cdot\mathbf{\Delta}\cdot\mathbf{P}_s \equiv 0\,. \tag{2.15a}$$

The solution $\mathbf{P}$ of the *Poisson's equation* is thus *"uniquely"* defined by the expression

$$\bullet \mathbf{P}(t) = \sum_{j=1}^{r} \frac{f_{d,j}(t)}{\sqrt{\kappa_j}} \cdot \hat{\mathbf{G}}_j \,, \tag{2.15b}$$

and it is important to point out that **P**(t) can be also obtained by *repeated multiplications* of the *sparse matrix* $\mathbf{\Delta}_1$; in fact, from (2.11c) it follows that

$$\mathbf{\Delta}_1 \cdot \hat{\mathbf{G}}_j = \varepsilon_j \hat{\mathbf{G}}_j ; \; \varepsilon_j = 1 - \kappa_j / \kappa_r \,,$$

and from the convergence of the *geometric series* $\Sigma(\varepsilon_j)^n = 1/(1-\varepsilon_j)$ one obtains

$$\bullet \mathbf{P}(t) = \frac{1}{\kappa_r} \sum_{n=0}^{\infty} \mathbf{\Delta}_1^n \cdot \mathbf{R}^t \cdot \mathbf{F}_D(t) \,. \tag{2.15c}$$

The series (2.15c) has an “almost geometric structure” and its convergence can be thus accelerated by *Shanks Transformation*, see Bender & Orszag (1978).

2.4: The steady state solution ($\mathbf{U}_e$)

Let $\mathbf{q}_{e,p}$ be the *steady solution* of (2.13), namely

$$\bullet \mathbf{K}_{p,s} \cdot \mathbf{q}_{e,p} + \mathbf{N}_s(\mathbf{q}_{e,p}) \cdot \mathbf{q}_{e,p} = \mathbf{F}_{p,s} \,, \tag{2.16a}$$

where both $\mathbf{K}_{p,s}$ and $\mathbf{F}_{p,s}$ are functions of the Reynolds number ($\mathbf{K}_{p,s} = \mathbf{K}_{p,s}(Re)$; $\mathbf{F}_{p,s} = \mathbf{F}_{p,s}(Re)$) and so it is the steady solution: $\mathbf{q}_{e,p} = \mathbf{q}_{e,p}(Re)$. If (2.16a) is differentiated with respect to Re and the matrix $\mathbf{K}_s = \mathbf{K}_s(\mathbf{q}_{e,p})$ is defined by the identity (f' stands for the derivative with respect to Re)

$$\bullet \mathbf{K}_s(\mathbf{q}_{e,p}) \cdot \mathbf{q}'_{e,p} = \mathbf{K}_{p,s} \cdot \mathbf{q}'_{e,p} + \mathbf{N}_s(\mathbf{q}'_{e,p}) \cdot \mathbf{q}_{e,p} + \mathbf{N}_s(\mathbf{q}_{e,p}) \cdot \mathbf{q}'_{e,p} \,, \tag{2.16b}$$

one obtains for $\mathbf{q}'_{e,p}$ the following *nonlinear* **(**$\mathbf{K}_s = \mathbf{K}_s(\mathbf{q}_{e,p})$**)** differential equation:

$$\bullet \mathbf{K}_s \cdot \mathbf{q}'_{e,p} + \mathbf{K}'_{p,s} \cdot \mathbf{q}_{e,p} = \mathbf{F}'_{p,s} \,. \tag{2.16c}$$

The solution of (2.16c) obviously *exists* and it is *unique* as long as

$$det \; \mathbf{K}_s(\mathbf{q}_{e,p}) \neq 0 \,, \tag{2.16d}$$

and if this latter condition is fulfilled one is able, by integrating (2.16c), to march the root of (2.16a) as the Reynolds number increases: equations (2.16c,a) define a *predictor-corrector method* to determine the steady solution. In the other hand, if *det* $\mathbf{K}_s(\mathbf{q}_{e,p}) = 0$ one would have a classic *bifurcation of the equilibrium*: the *steady solution* could then be marched out in Re by defining a proper (“statically stable”) branch after the bifurcation. As it will be seen in the next item, the *underlying assumption* behind the *asymptotic theory* to be developed in this work implies in a condition *stronger* than (2.16d): within this context it can be taken here that a *steady solution* exists and it is uniquely defined.

If $\{\mathbf{x}_k;\, k = 1,2,\ldots,N\}$ are the nodes of the Finite Element mesh then, given any continuous field $\mathbf{u}(\mathbf{x},t)$, the *“nodal interpolate”* $\mathbf{u}_h(x,t)$ is defined by the expression

$$\mathbf{u}_h(\mathbf{x},t) = \sum_{k=1}^{N} \left(u(\mathbf{x}_k,t)\mathbf{i} + v(\mathbf{x}_k,t)\mathbf{j} \right) \cdot h_k(\mathbf{x}) = \sum_{k=1}^{N} \left(U_k(t)\mathbf{i} + U_{k+N}(t)\mathbf{j} \right) \cdot h_k(\mathbf{x}) \,, \tag{2.17a}$$

and, within the context of the discrete model, one can ignore the difference between $\mathbf{u}(\mathbf{x},t)$ and its “nodal interpolate” $\mathbf{u}_h(\mathbf{x},t)$: there is thus a one-to-one relation between the field $\mathbf{u}(\mathbf{x},t)$ and the *“nodal values vector”* $\mathbf{U}(t)$ ($\mathbf{u}(\mathbf{x},t) \Leftrightarrow \mathbf{U}(t)$). If now $\mathbf{U}_p$ is the “nodal values vector” of the auxiliary field $\mathbf{u}_p(\mathbf{x})$ ($\mathbf{u}_p(\mathbf{x}) \Leftrightarrow \mathbf{U}_p$), the *steady solution* $\mathbf{u}_e(\mathbf{x}) \Leftrightarrow \mathbf{U}_e$ is defined by the expression

$$\bullet \mathbf{U}_e = \mathbf{U}_p + \mathbf{T}\cdot\mathbf{q}_{e,p}, \tag{2.17b}$$

and the *global solution* $\mathbf{u}_T(\mathbf{x},t)\Leftrightarrow\mathbf{U}_T(t)$ can be thus written as

$$\bullet \mathbf{U}_T(t) = \mathbf{U}_e + \mathbf{U}_o(t)\ ;\ \mathbf{U}_o(t) = \mathbf{T}\cdot\mathbf{q}_o(t). \tag{2.17c}$$

Keeping in mind these definitions and introducing also the field $\mathbf{u}_o(\mathbf{x},t)\Leftrightarrow\mathbf{U}_o(t)$, it is an easy task to show that the matrix $\mathbf{K}_s$ can be expressed in the form (see (2.5b))

$$\begin{aligned}&\bullet\int_R\left[(\mathbf{u}_e\cdot\nabla)\mathbf{u}_o+(\mathbf{u}_o\cdot\nabla)\mathbf{u}_e\right]\cdot\delta\mathbf{u}\ dR+\frac{1}{Re}\int_R\left(\nabla u_o\cdot\nabla(\delta u)+\nabla v_o\cdot\nabla(\delta v)\right)dR=\delta\mathbf{U}^t\cdot\mathbf{K}\cdot\mathbf{U}_o;\\&\bullet\mathbf{K}_s=\mathbf{T}^t\cdot\mathbf{K}\cdot\mathbf{T}.\end{aligned} \tag{2.17d}$$

The parcel $[(\mathbf{u}_e\cdot\nabla)\mathbf{u}_o + (\mathbf{u}_o\cdot\nabla)\mathbf{u}_e]$ of the *convective acceleration* introduces, as usual, a *non-symmetry* in $\mathbf{K}_s$ that plays a role in *stability theory* to be addressed next.

2.5: Hopf bifurcation and asymptotic solution of fluid equation

The differential equation for the *perturbation* $\mathbf{U}_o(t) = \mathbf{T}\cdot\mathbf{q}_o(t)$ on the *steady solution* $\mathbf{U}_e$ can be easily derived by using the definition $\mathbf{q}(t) = \mathbf{q}_{e,p} + \mathbf{q}_o(t)$ in (2.13); one obtains then, with the help of (2.16a), the *homogeneous nonlinear equation*

$$\mathbf{M}_s\cdot\dot{\mathbf{q}}_o+\mathbf{K}_s\cdot\mathbf{q}_o+\mathbf{N}_s(\mathbf{q}_o)\cdot\mathbf{q}_o=\mathbf{0}, \tag{2.18a}$$

the *non-symmetric* matrix $\mathbf{K}_s$ being defined in item (2.4). The eigenvalues of $\mathbf{K}_s$ are thus complex, in general, and they will be defined as follows:

$$\begin{aligned}&\left\{\lambda_1=\sigma+i\omega\ ;\lambda_2=\sigma-i\omega;\lambda_3=\sigma_3+i\omega_3;...;\lambda_s=\sigma_s+i\omega_s\right\};\\&\sigma\ge\sigma_3\ge...\ge\sigma_s.\end{aligned} \tag{2.18c}$$

The first mode, the one that becomes first unstable since $\sigma \ge \sigma_j$, is of the form

$$\bullet\mathbf{q}_o(t)=e^{\lambda_1 t}\cdot\mathbf{E}, \tag{2.19a}$$

where the mode $\mathbf{E}$ is such that ((*) stands for the complex conjugate)

$$\begin{aligned}&\bullet\left(\lambda_1\mathbf{M}_s+\mathbf{K}_s\right)\cdot\mathbf{E}=\mathbf{0};\\&\bullet(\mathbf{E}^*)^t\cdot\mathbf{M}_s\cdot\mathbf{E}=1.\end{aligned} \tag{2.19b}$$

Notice that $\lambda_1(Re) = \sigma(Re) + i\omega(Re)$ where, from (2.19b), it follows that

$$\begin{aligned}&\bullet\sigma(Re)=-\tfrac{1}{2}(\mathbf{E}^*)^t\cdot\left(\mathbf{K}_s+\mathbf{K}_s^t\right)\cdot\mathbf{E}=-\tfrac{1}{2}(\mathbf{T}\cdot\mathbf{E}^*)^t\cdot\left(\mathbf{K}+\mathbf{K}^t\right)\cdot(\mathbf{T}\cdot\mathbf{E});\\&\bullet i\omega(Re)=-\tfrac{1}{2}(\mathbf{E}^*)^t\cdot\left(\mathbf{K}_s-\mathbf{K}_s^t\right)\cdot\mathbf{E}=-\tfrac{1}{2}(\mathbf{T}\cdot\mathbf{E}^*)^t\cdot\left(\mathbf{K}-\mathbf{K}^t\right)\cdot(\mathbf{T}\cdot\mathbf{E}).\end{aligned} \tag{2.19c}$$

Relation (2.19c) can be used in conjunction to (2.17d) to provide more explicit expression for $\{\sigma(Re);\ \omega(Re)\}$; indeed, if $\mathbf{e}(\mathbf{x})\Leftrightarrow\mathbf{T}\cdot\mathbf{E}$ one obtains

$$\bullet\sigma(\mathrm{Re}) = -\tfrac{1}{2}\int_R \left[u_e(\mathbf{x})\left(\frac{\partial \mathbf{e}}{\partial x}\cdot\mathbf{e}^* + \frac{\partial \mathbf{e}^*}{\partial x}\cdot\mathbf{e}\right) + v_e(\mathbf{x})\left(\frac{\partial \mathbf{e}}{\partial y}\cdot\mathbf{e}^* + \frac{\partial \mathbf{e}^*}{\partial y}\cdot\mathbf{e}\right) + 2\gamma_e(\mathbf{x})\left(e_x e_y^* + e_x^* e_y\right)\right] dR -$$

$$-\tfrac{1}{2}\int_R \left(\frac{\partial u_e}{\partial x} - \frac{\partial v_e}{\partial y}\right)\cdot\left(|e_x|^2 - |e_y|^2\right) dR - \frac{1}{\mathrm{Re}}\overbrace{\int_R \left(|\nabla e_x|^2 + |\nabla e_y|^2\right) dR}^{b_1(\mathrm{Re})}; \tag{2.20a}$$

$$\bullet i\omega(\mathrm{Re}) = -\tfrac{1}{2}\int_R \left[u_e(\mathbf{x})\left(\frac{\partial \mathbf{e}}{\partial x}\cdot\mathbf{e}^* - \frac{\partial \mathbf{e}^*}{\partial x}\cdot\mathbf{e}\right) + v_e(\mathbf{x})\left(\frac{\partial \mathbf{e}}{\partial y}\cdot\mathbf{e}^* - \frac{\partial \mathbf{e}^*}{\partial y}\cdot\mathbf{e}\right) + 2\Omega_e(\mathbf{x})\left(e_x e_y^* - e_x^* e_y\right)\right] dR,$$

where $\gamma_e(\mathbf{x}) = ½(\partial v_e/\partial x + \partial u_e/\partial y)$ is the *shear rate of deformation* of the *steady solution* $\mathbf{u}_e(\mathbf{x})$ and $\Omega_e(\mathbf{x}) = ½(\partial v_e/\partial x - \partial u_e/\partial y)$ its *vorticity*.

The above expression suggests to write σ(Re) as $\sigma(\mathrm{Re}) = a_1(\mathrm{Re}) - b_1(\mathrm{Re})/\mathrm{Re}$, with $b_1(\mathrm{Re}) > 0$. The *Strouhal number* S(Re)[4] $\cong \omega(\mathrm{Re})/2\pi$ is known to change weakly with Re and from the structure of (2.20a) one should expect that both $\{a_1(\mathrm{Re}); b_1(\mathrm{Re})\}$ also do; if now one writes $\{a_1(\mathrm{Re}) = a(\mathrm{Re})\mathrm{S}(\mathrm{Re}); b_1(\mathrm{Re}) = b(\mathrm{Re})\cdot a(\mathrm{Re})\mathrm{S}(\mathrm{Re})\}$ and assumes that $\{a_o; \mathrm{S}_o; b_o\}$ are typical values of the *slowing varying* functions $\{a(\mathrm{Re}); \mathrm{S}(\mathrm{Re}); b(\mathrm{Re})\}$ one obtains

$$\bullet\sigma(\mathrm{Re}) = a(\mathrm{Re})\cdot\mathrm{S}(\mathrm{Re})\left[1 - \frac{b(\mathrm{Re})}{\mathrm{Re}}\right] \cong a_o\cdot\mathrm{S}_o\left[1 - \frac{b_o}{\mathrm{Re}}\right], \tag{2.20b}$$

an expression that has some empirical support, as discussed in item (2.6) below. It follows that σ(Re) is (roughly) monotonically increasing with Re, where $\sigma(\mathrm{Re}) < 0$ for $\mathrm{Re} < b_o = \mathrm{Re}_{cr}$ and $\sigma(\mathrm{Re}) > 0$ for $\mathrm{Re} > b_o = \mathrm{Re}_{cr}$. The value of Re_{cr} inferred from numerical simulations seems to coalesce around 45 ($\mathrm{Re}_{cr} \approx 45$) although there are experimental evidences showing that this threshold value is a bit smaller ($\mathrm{Re}_{cr} \approx 35$). Obviously, for Re below Re_{cr} the *steady solution* is *stable* ($\sigma(\mathrm{Re}) < 0$) while it becomes *unstable* for $\mathrm{Re} > \mathrm{Re}_{cr}$ ($\sigma(\mathrm{Re}) < 0$); furthermore, $\omega(\mathrm{Re}_{cr}) \neq 0$ and one has thus a typical *Hopf bifurcation*.

For Re above Re_{cr} but close to it one has $0 < \sigma(\mathrm{Re}) << 1$, since $\sigma(\mathrm{Re}_{cr}) = 0$; furthermore, the experiments suggest – and the numerical results confirm – that *only one mode* is *unstable* in this range of Reynolds numbers ($\sigma_j < 0$; $j \geq 3$). The *asymptotic solution* to be developed is based on the following *assumption*

$$\begin{aligned} &\mathrm{i)}\ 0 < \sigma(\mathrm{Re}) << 1;\ \ \omega(\mathrm{Re}) \cong O(1); \\ &\mathrm{ii)}\ \sigma_j(\mathrm{Re}) < 0\ \text{ for } j = 3, \ldots, s, \end{aligned} \tag{2.21}$$

that should be strictly satisfied in the vicinity of *Hopf bifurcation*; notice that (2.21) implies, necessarily, that $\{\lambda_1(\mathrm{Re}) \neq 0; \lambda_j(\mathrm{Re}) \neq 0\}$, since $\{\omega(\mathrm{Re}) \neq 0; \sigma_j(\mathrm{Re}) < 0\}$, and so *det* $\mathbf{K}_s(\mathrm{Re}) \neq 0$.

The *adjoint eigenvalue problem* ($\mathbf{K}_s \rightarrow \mathbf{K}_s^t$) plays a role, as it will be seen, in the derivation of the asymptotic theory. This problem has the same eigenvalues (2.18c) but the eigenvectors are distinct; in particular, to the "unstable" eigenvalue λ_1 it is associated the *adjoint unstable mode* $\mathbf{E}_a$ where

$$\bullet\left(\lambda_1\mathbf{M}_s + \mathbf{K}_s^t\right)\cdot\mathbf{E}_a = \mathbf{0}. \tag{2.22a}$$

From (2.21) it follows that the "unstable" eigenvalue $\lambda_1 = \sigma + i\omega$ is a *single root* of the related *characteristic equation*: if it were not, some of the λ_j would be equal to λ_1 and the condition (ii) in (2.21) would be not fulfilled. Under the condition that λ_1 is a *single root* it is possible to show that

$$\bullet\mathbf{E}_a^t\cdot\mathbf{M}_s\cdot\mathbf{E} \neq 0, \tag{2.22b}$$

and thus $\mathbf{E}_a$ can be normalized by the condition

$$\bullet\mathbf{E}_a^t\cdot\mathbf{M}_s\cdot\mathbf{E} = 1, \tag{2.22c}$$

a relation that it will be used below in this section.

The argument now is classic and it will be just sketched here: for $0 < \sigma(\mathrm{Re}) << 1$ the amplitude $A(t)\cdot e^{i\omega t}$ of the unstable mode $\mathbf{E} = \mathbf{E}_R + i\mathbf{E}_I$ increases (initially) exponentially with time ($A(t) \propto e^{\sigma t}$; $dA/dt = \sigma A$) and the solution of the dynamic system (2.18a) is attracted, since $\sigma_j < 0$ for $j \geq 3$, to the (unstable) two dimensional manifold tangent to the

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[4] The actual *Strouhal frequency* $\omega_s(\mathrm{Re})$ differs slightly from ω(Re), see (2.23b).

plane generated by $\{\mathbf{E}_R; \mathbf{E}_I\}$; the exponential growth in this manifold is halted by the *nonlinear term* and expanding $\mathbf{N}(\mathbf{q}_o)\cdot\mathbf{q}_o$ in power series in the amplitude only the cubic term $-\mu|A(t)|^2A(t)\cdot e^{i\omega t}$ can match the term $\sigma A(t)\cdot e^{i\omega t}$ that causes the exponential growth. The equation for the amplitude A(t) – namely, *Landau's equation* – is thus given by[5]

$$\bullet\frac{dA}{dt}-\sigma A+\mu\,|\,A\,|^2\;A=0;\;\mu=\mu_R+i\mu_I,\tag{2.23a}$$

the steady solution (*limit cycle*) $A_c\cdot\exp(i\omega_s t)$ being given by

$$\begin{aligned}&\bullet A_c=\sqrt{\frac{\sigma}{\mu_R}};\\&\bullet\omega_s=\omega-\frac{\mu_I}{\mu_R}\sigma.\end{aligned}\tag{2.23b}$$

The formal asymptotic solution of the (discrete) *Navier-Stokes* equation will be derived next. In fact, writing the solution of (2.13) in the form

$$\bullet\mathbf{q}(t)=\mathbf{q}_{e,p}+\mathbf{q}_o(t),\tag{2.24a}$$

with the *perturbation* $\mathbf{q}_o(t)$ satisfying (2.18a), one must have, to leading order, that $\mathbf{q}_o(t)\cong[A(t)\cdot\mathbf{E}+(*)]$, since the solution of (2.18a) should follow, at least initially, the unstable mode $\mathbf{E}$. The amplitude A(t), however, is such that (see (2.23))

$$\begin{aligned}&\bullet A(t)\cong O(\sigma^{1/2})<<1;\\&\bullet\frac{dA}{dt}\cong O(\sigma A),\end{aligned}\tag{2.24b}$$

and expanding $\mathbf{q}_o(t)$ in the *small parameter* $A(t)\cong O(\sigma^{1/2})$ one obtains, with an error in (2.24a) of the form $[1+O(\sigma^2)]$, that

$$\begin{aligned}\bullet\mathbf{q}_o(t)=&\underbrace{\left[A(t)\cdot\mathbf{E}\cdot e^{i\omega t}+(*)\right]}_{O(\sigma^{1/2})}+\underbrace{\left[|\,A(t)\,|^2\cdot\mathbf{\Lambda}_{20}+\left(A^2(t)\cdot\mathbf{\Lambda}_{22}\cdot e^{2i\omega t}+(*)\right)\right]}_{O(\sigma)}+\\&+\underbrace{\left[|A(t)|^2A(t)\cdot\mathbf{\Lambda}_{31}\cdot e^{i\omega t}+A^3(t)\cdot\mathbf{\Lambda}_{33}\cdot e^{3i\omega t}+(*)\right]}_{O(\sigma^{3/2})}+O(\sigma^2).\end{aligned}\tag{2.25a}$$

The nonlinear term in (2.18a) can, accordingly, be written as

$$\bullet\mathbf{N}_s(\mathbf{q}_o)\cdot\mathbf{q}_o=\left[\mathbf{N}_{20}+\left(\mathbf{N}_{22}\cdot e^{2i\omega t}+(*)\right)\right]+\left[\mathbf{N}_{31}\cdot e^{i\omega t}+\mathbf{N}_{33}\cdot e^{3i\omega t}+(*)\right]+O(\sigma^2),\tag{2.25b}$$

with

$$\begin{aligned}&\bullet\mathbf{N}_{20}=\mathbf{N}_s\left(\mathbf{E}\right)\cdot\mathbf{E}^*+\mathbf{N}_s(\mathbf{E}^*)\cdot\mathbf{E};\\&\bullet\mathbf{N}_{22}=\mathbf{N}_s(\mathbf{E})\cdot\mathbf{E};\\&\bullet\mathbf{N}_{31}=\mathbf{N}_s(\mathbf{E})\cdot\mathbf{\Lambda}_{20}+\mathbf{N}_s(\mathbf{\Lambda}_{20})\cdot\mathbf{E}+\mathbf{N}_s(\mathbf{\Lambda}_{22})\cdot\mathbf{E}^*+\mathbf{N}_s(\mathbf{E}^*)\cdot\mathbf{\Lambda}_{22};\\&\bullet\mathbf{N}_{33}=\mathbf{N}_s(\mathbf{E})\cdot\mathbf{\Lambda}_{22}+\mathbf{N}_s(\mathbf{\Lambda}_{22})\cdot\mathbf{E}\;.\end{aligned}\tag{2.25c}$$

Placing (2.25) into (2.18a) one obtains ($\lambda_1=\sigma+i\omega$)

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[5] It has been implicitly assumed here that one has a *super-critical Hopf bifurcation* with μ_R = *Real* $\mu > 0$. This assumption has an *experimental* support, see Provansal & Mathis & Boyer (1987) and Leweke & Provansal (1994) among others, and it has been also verified *numerically* by Noack & Eckelmann (1994). Some preliminary numerical computation, to be published soon, corroborates this result.

$$\left[\left(\frac{dA}{dt}-\sigma A\right)\mathbf{M}_s\cdot\mathbf{E}+|A|^2\,A\left(\mathbf{N}_{31}+(\lambda_1\mathbf{M}_s+\mathbf{K}_s)\cdot\mathbf{\Lambda}_{31}\right)\right]e^{i\omega t}+$$
$$+|A|^2\left[\mathbf{K}_s\cdot\mathbf{\Lambda}_{20}+\mathbf{N}_{20}\right]+A^2\left[\left(2i\omega\mathbf{M}_s+\mathbf{K}_s\right)\mathbf{\Lambda}_{22}+\mathbf{N}_{22}\right]e^{2i\omega t}+ \tag{2.26a}$$
$$+A^3\left[\left(3i\omega\mathbf{M}_s+\mathbf{K}_s\right)\mathbf{\Lambda}_{33}+\mathbf{N}_{33}\right]e^{3i\omega t}+O(\sigma^2)=0,$$

where the term $|A|^2A(\sigma\mathbf{M}_s\cdot\mathbf{\Lambda}_{31})e^{i\omega t}\cong O(\sigma^{5/2})$ has been added just for convenience; if now $\{\mathbf{\Lambda}_{20};\mathbf{\Lambda}_{22};\mathbf{\Lambda}_{33}\}$ are solutions of the (*non-singular*[6]) linear systems

$$\bullet\mathbf{K}_s\cdot\mathbf{\Lambda}_{20}+\mathbf{N}_{20}=\mathbf{0};$$
$$\bullet\left(2i\omega\mathbf{M}_s+\mathbf{K}_s\right)\cdot\mathbf{\Lambda}_{22}+\mathbf{N}_{22}=\mathbf{0}; \tag{2.26b}$$
$$\bullet\left(3i\omega\mathbf{M}_s+\mathbf{K}_s\right)\cdot\mathbf{\Lambda}_{33}+\mathbf{N}_{33}=\mathbf{0},$$

(2.26a) reduces to

$$\left(\frac{dA}{dt}-\sigma A\right)\mathbf{M}_s\cdot\mathbf{E}+|A|^2\,A\left(\mathbf{N}_{31}+(\lambda_1\mathbf{M}_s+\mathbf{K}_s)\cdot\mathbf{\Lambda}_{31}\right)=0\,. \tag{2.26c}$$

Multiplying (2.26c) on the left by the *adjoint unstable mode* $\mathbf{E}_a$ and using (2.22a,c) *Landau's equation* (2.23a) is obtained with

$$\bullet\mu=\mathbf{E}_a^t\cdot\mathbf{N}_{31}\,. \tag{2.27}$$

Summarizing: solving the eigenvalues problems (2.19b) and (2.22a,c) the vectors $\{\mathbf{N}_{20};\mathbf{N}_{22}\}$ can be computed from (2.25c) and the solutions $\{\mathbf{\Lambda}_{20};\mathbf{\Lambda}_{22}\}$ of the linear systems (2.26b) can be determined; with them the vector $\mathbf{N}_{31}$ can be obtained from (2.25c) and *Landau's coefficient* μ is thus given by (2.27).

Writing now $\mathbf{N}_{31}$ in the form

$$\mathbf{N}_{31}=\mu\mathbf{M}_s\cdot\mathbf{E}+\mathbf{N}_{31,\perp}^{(a)};$$
$$\mathbf{E}_a^t\cdot\mathbf{N}_{31,\perp}^{(a)}=0, \tag{2.28a}$$

and placing (2.28a) into (2.26c) one obtains, with the help of (2.23a), the equality

$$\delta\mathbf{q}^t\cdot\left[\left(\lambda_1\mathbf{M}_s+\mathbf{K}_s\right)\cdot\mathbf{\Lambda}_{31}+\mathbf{N}_{31,\perp}^{(a)}\right]=0\ \textit{all}\ \delta\mathbf{q}\in J_s\,. \tag{2.28b}$$

Introducing the (s – 2)-dimensional sub-spaces $\{J_\perp; J_{a,\perp}\}$ of J_s by the definitions

$$\bullet J_\perp=\left\{\mathbf{q}_\perp:\mathbf{q}_\perp^t\cdot\mathbf{E}=0\right\}\Rightarrow\mathbf{q}_\perp=\mathbf{S}_\perp\cdot\mathbf{x};$$
$$\bullet J_{a,\perp}=\left\{\mathbf{q}_{a,\perp}:\mathbf{q}_{a,\perp}^t\cdot\mathbf{E}_a=0\right\}\Rightarrow\mathbf{q}_{a,\perp}=\mathbf{S}_{a,\perp}\cdot\mathbf{x}, \tag{2.28c}$$

with $\mathbf{x}$ being an arbitrary (s – 2)-dimensional vector, and the matrices

$$\bullet\mathbf{M}_{s,\perp}=\mathbf{S}_{a,\perp}^t\cdot\mathbf{M}_s\cdot\mathbf{S}_\perp;$$
$$\bullet\mathbf{K}_{s,\perp}=\mathbf{S}_{a,\perp}^t\cdot\mathbf{K}_s\cdot\mathbf{S}_\perp, \tag{2.28d}$$

the vector $\mathbf{\Lambda}_{31}$ can be obtained from the solution of the (*non singular*[7]) linear system

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[6] Notice that (2.21) rules out {0; 2iω; 3iω} as possible eigenvalues of $\mathbf{K}_s$ once $\sigma_j < 0$ for $j \geq 3$ and $\omega \neq 0$; the matrices $\{\mathbf{K}_s; (2i\omega\mathbf{M}_s + \mathbf{K}_s); (3i\omega\mathbf{M}_s + \mathbf{K}_s)\}$in (2.26b) are thus non-singular.

$$\bullet\left(\lambda_1\mathbf{M}_{s,\perp}+\mathbf{K}_{s,\perp}\right)\cdot\mathbf{x}_{31}+\mathbf{S}^{t}_{a,\perp}\cdot\mathbf{N}^{(a)}_{31,\perp}=\mathbf{0};\qquad \bullet\mathbf{\Lambda}_{31}=\mathbf{S}_{\perp}\cdot\mathbf{x}_{31}. \tag{2.28e}$$

Reverting to the "nodal values vector" $\mathbf{U}(t) = \mathbf{T}\cdot(\mathbf{q}_{e,p} + \mathbf{q}_o(t))$, see (2.1c), (2.17), one has

$$\bullet\mathbf{U}(t)=\mathbf{U}_{e,p}+\underbrace{\left[A(t)\cdot\mathbf{E}_U\cdot e^{i\omega t}+(*)\right]}_{O(\sigma^{1/2})}+\underbrace{\left[|A(t)|^2\cdot\mathbf{\Lambda}_{20,U}+\left(A^2(t)\cdot\mathbf{\Lambda}_{22,U}\cdot e^{2i\omega t}+(*)\right)\right]}_{O(\sigma)}+ \\ +\underbrace{\left[|A(t)|^2A(t)\cdot\mathbf{\Lambda}_{31,U}\cdot e^{i\omega t}+A^3(t)\cdot\mathbf{\Lambda}_{33,U}\cdot e^{3i\omega t}+(*)\right]}_{O(\sigma^{3/2})}+O(\sigma^2), \tag{2.29a}$$

with

$$\bullet\left\{\mathbf{U}_{e,p};\mathbf{E}_U;\mathbf{\Lambda}_{20,U};\mathbf{\Lambda}_{22,U};\mathbf{\Lambda}_{31,U};\mathbf{\Lambda}_{33,U}\right\}=\mathbf{T}\cdot\left\{\mathbf{q}_{e,p};\mathbf{E};\mathbf{\Lambda}_{20};\mathbf{\Lambda}_{22};\mathbf{\Lambda}_{31};\mathbf{\Lambda}_{33}\right\}, \tag{2.29b}$$

and placing (2.29a) into (2.14a) one obtains

$$\bullet\mathbf{F}_D(t)=\mathbf{F}_{00}+\underbrace{\left[A(t)\cdot\mathbf{F}_{11}\cdot e^{i\omega t}+(*)\right]}_{O(\sigma^{1/2})}+\underbrace{\left[|A(t)|^2\cdot\mathbf{F}_{20}+\left(A^2(t)\cdot\mathbf{F}_{22}\cdot e^{2i\omega t}+(*)\right)\right]}_{O(\sigma)}+ \\ +\underbrace{\left[|A(t)|^2A(t)\cdot\mathbf{F}_{31}\cdot e^{i\omega t}+A^3(t)\cdot\mathbf{F}_{33}\cdot e^{3i\omega t}+(*)\right]}_{O(\sigma^{3/2})}+O(\sigma^2), \tag{2.30a}$$

with (see (2.17d) and (2.23a) with $\lambda_1 = \sigma + i\omega$)

$$\begin{aligned}&\bullet\mathbf{F}_{00}=\mathbf{K}_p\cdot\mathbf{U}_{e,p}+\mathbf{N}(\mathbf{U}_{e,p})\cdot\mathbf{U}_{e,p}-\mathbf{F}_p;\\&\bullet\mathbf{F}_{11}=\left(\lambda_1\mathbf{M}+\mathbf{K}\right)\cdot\mathbf{E}_U;\\&\bullet\mathbf{F}_{20}=\mathbf{K}\cdot\mathbf{\Lambda}_{20,U}+\mathbf{N}_{20,U};\\&\bullet\mathbf{F}_{22}=\left(2i\omega\mathbf{M}+\mathbf{K}\right)\cdot\mathbf{\Lambda}_{22,U}+\mathbf{N}_{22,U};\\&\bullet\mathbf{F}_{31}=\left(\lambda_1\mathbf{M}+\mathbf{K}\right)\cdot\mathbf{\Lambda}_{31,U}-\mu\mathbf{M}\cdot\mathbf{E}_U+\mathbf{N}_{31,U};\\&\bullet\mathbf{F}_{33}=\left(3i\omega\mathbf{M}+\mathbf{K}\right)\cdot\mathbf{\Lambda}_{33,U}+\mathbf{N}_{33,U},\end{aligned} \tag{2.30b}$$

where $\{\mathbf{N}_{20,U};\ \mathbf{N}_{22,U};\ \mathbf{N}_{31,U};\ \mathbf{N}_{33,U}\}$ are defined as in (2.25c) with $\{\mathbf{N}_{jk}\to\mathbf{N}_{jk,U};\ \mathbf{E}\to\mathbf{E}_U;\ \mathbf{\Lambda}_{20}\to\mathbf{\Lambda}_{20,U};\ \mathbf{\Lambda}_{22}\to\mathbf{\Lambda}_{22,U}\}$. Considering now the solutions of the *Poisson's equations* (see (2.14c))

$$\bullet\mathbf{\Delta}\cdot\mathbf{P}_{jk}=\mathbf{R}^t\cdot\mathbf{F}_{jk};(jk)=\{(00);(11);(20);(22);(31);(33)\}, \tag{2.30c}$$

the (discrete) *pressure field* is given by ($\mathbf{P}_e = \mathbf{P}_{00}$ is the pressure related to $\mathbf{u}_e(\mathbf{x})\Leftrightarrow\mathbf{U}_e$)

$$\bullet\mathbf{P}(t)=\mathbf{P}_e+\underbrace{\left[A(t)\cdot\mathbf{P}_{11}\cdot e^{i\omega t}+(*)\right]}_{O(\sigma^{1/2})}+\underbrace{\left[|A(t)|^2\cdot\mathbf{P}_{20}+\left(A^2(t)\cdot\mathbf{P}_{22}\cdot e^{2i\omega t}+(*)\right)\right]}_{O(\sigma)}+ \\ +\underbrace{\left[|A(t)|^2A(t)\cdot\mathbf{P}_{31}\cdot e^{i\omega t}+A^3(t)\cdot\mathbf{P}_{33}\cdot e^{3i\omega t}+(*)\right]}_{O(\sigma^{3/2})}+O(\sigma^2), \tag{2.30d}$$

with the same error factor $[1 + O(\sigma^2)]$ of the velocity field approximation.

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[7] Recall that (2.21) implies that λ_1 is a *single root* of the characteristic polynomial of $\mathbf{K}_s$ and so the matrix $\lambda_1\mathbf{M}_s + \mathbf{K}_s$, restricted to the sub-space orthogonal to the eigenvectors $\{\mathbf{E};\ \mathbf{E}_a\}$, must be non-singular.

Expressions (2.29a) and (2.30d) synthesize the asymptotic solution of the (discrete) *Navier-Stokes equation* in the vicinity of *Hopf bifurcation* ($Re \approx Re_{cr}$); in the next item, the possibility to extend this solution to the range $Re >> Re_{cr}$ is discussed.

2.6: Extension of Landau's Equation beyond hopf bifurcation

The *asymptotic solution* of the *(discrete) Navier-Stokes* was derived assuming (2.21), two conditions that hold in the vicinity of the *Hopf bifurcation* but not necessarily only there. The purpose here is to give arguments that suggest that (2.21) – and, with it, the given asymptotic solution – can be extended far beyond *Hopf bifurcation*, that is, to the range $Re >> Re_{cr}$.

In fact, numerical results by Noack & Eckelmann (1994) indicate that (2.21) remains essentially correct in the range $Re_{cr} \cong 45 < Re < 300$ and Henderson (1997), after a detailed numerical work, affirms that up to Re = 1000 no other bifurcation, besides the one at Re_{cr}, could be observed in the *two dimensional model*; or, in other words, that $\{\sigma_j < 0;\ j \geq 3\}$ in this range.

From the experimental side, Provansal & Mathis & Boyer (1987) have observed that the inferred value of σ(Re) could be fitted to the expression

$$\bullet\sigma(\mathrm{Re}) = \frac{\sigma(\mathrm{Re})d}{\mathrm{U}} \cong 0.20\left(1 - \frac{\mathrm{Re}_{cr}}{\mathrm{Re}}\right), \qquad (2.31)$$

while Leweke & Provansal assumed (2.31) in the range $Re_{cr} \cong 45 < Re < 300$.

As already discussed, the empirical relation (2.31) seems to have a foot on the more basic set of equations that describe the fluid flow, since a similar expression can be derived exactly, see (2.20b); furthermore, it indicates that σ(Re) can be considered a "small parameter" of order 0.20 (or less). All together, these evidences suggest that (2.21) could be pushed at least up to $Re \approx 1000$ since, following Henderson, no other bifurcation (in 2D) could be found in this range while (2.31), together with (2.20b), seems to indicate that σ(Re) remains in fact "small" as asserted in (2.21).

Experimental results on VIV are mostly in the range $10^3 < Re < 10^4$, see Khalak & Wiliamson (1996), and they do not seem to depend very much on Re. The observed harmonic pattern is very neat and this, undoubtedly, was the motivation behind a bold assumption introduced by Bishop & Hassan (1964) to describe the flow around a circular cylinder: they proposed to represent the flow by an *one degree of freedom* "wake oscillator model" based on Van der Pol equation that leads, for a small enough σ, to *Landau's equation* (2.23a); this idea was further developed and it is the basis of the so-called *phenomenological models* used to predict VIV, see Iwan & Blevins (1975). In despite of the loose link with the more basic flow equation the predictions from these models have some accuracy, showing that the *ordered oscillatory behavior* in the wake can be apparently described by means of an *one degree of freedom system* related to the *unstable mode* of the problem. The *smallness* of σ, a common feature in all "wake oscillator models", coupled to the flow representation by *only one unstable mode*, can be translated in the following words: the basic assumption (2.21) is, apparently, correct in the range $10^3 < Re < 10^4$ of Reynolds numbers of the VIV experiments. However, from $Re \approx 10^4$ until the *transition region* $Re \approx 10^5$ nothing very much different occurs and one can possibly push (2.21) up to the *transition region* $Re \approx 10^5$. For $Re > 10^6$ the boundary layer is fully turbulent but a (relatively) well defined Strouhal frequency can be detected again: in line with the overall view taken here, one speculates that the same conditions (2.21) can hold if one searches for the stability of the *time averaged* (turbulent[8]) *symmetric solution* of the flow around the circular cylinder.

These extensions should be obviously confirmed numerically but certainly the present theory has a range of application much broader than foreseen a priori; more than that, (2.21) can be used to determine precisely this range.

3. Weak three-dimensionality: Ginzburg-Landau Equation

The *perturbation* $\mathbf{u}_o(\mathbf{x},t)$ on the 2D steady solution $\mathbf{u}_e(\mathbf{x})$ is, to leading order, given by $\mathbf{u}_o(\mathbf{x},t) \cong A(t)\cdot\mathbf{e}(\mathbf{x})$ where $\mathbf{e}(\mathbf{x}) \Leftrightarrow \mathbf{T}\cdot\mathbf{E}$ is the *unstable mode* and A(t) its *complex amplitude*: $A(t) = |A(t)|\cdot\exp(\varphi(t))$. It is known for a long time – see, for instance, Toebes (1969) – that the vortices *are not* shed in phase along the span of a fixed cylinder, and in fact the *correlation* among the vortices emitted in distinct sections tends to zero "very fast"[9], in a length of order of 10*d*. The phase φ should then change in the *longitudinal* z-direction and so it does the amplitude A and the perturbation $\mathbf{u}_o(\mathbf{x},t)$; or $\{A = A(z,t);\ \mathbf{u}_o = \mathbf{u}_o(\mathbf{x},z,t)\}$, where $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ continues to represent the position vector in the cross section plane and

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[8] *Boundary layer turbulence* is likely due to the *concomitant* instabilities of several *symmetric modes*, as elaborated in a forthcoming paper; it will be also discussed there a possible scenario for the *transition region* $10^5 < Re < 10^6$.

[9] *"Very fast"* in the relation to the longitudinal length of the slender cylinder; in fact, *"very slowly"* in the natural length scale *d* of the cross flow problem.

$\mathbf{u}_o(\mathbf{x},z,t) = u_o(\mathbf{x},z,t)\mathbf{i} + v_o(\mathbf{x},z,t)\mathbf{j}$ the perturbed velocity field in this plane at the z-level. The *longitudinal component* of the *perturbed* velocity field will be designated by $w_o(\mathbf{x},z,t)$.

The variation of A(z,t) in the *longitudinal* z-direction should be expressed by an *even derivative* with respect to *z*, since there is no preferred direction, and observing that the *viscous diffusion* in this direction, given by $(1/Re)\partial^2\mathbf{u}_o/\partial z^2$, implies to leading order in a term proportional to $\partial^2 A/\partial z^2$, the following 3D correction is proposed for Landau's equation (2.23a):

$$\bullet \frac{\partial A}{\partial t} - \sigma A - \gamma \frac{\partial^2 A}{\partial z^2} + \mu |A|^2 A = 0 \cdot \qquad (3.1a)$$

This is the *Ginzburg-Landau Equation* (GLE), first proposed by Ginzburg more than fifty years ago in his study on superconductivity, see Ginzburg & Landau (1950); notice that, in general, both *Landau's coefficient* μ and *Ginzburg's coefficient* γ are complex numbers: $\{\mu = \mu_R + i\mu_I;\ \gamma = \gamma_R + i\gamma_I\}$. Assuming, as before, $\sigma << 1$ (see (2.21)) and recalling that $A \cong O(\sigma^{1/2})$, a proper balance of the terms in (3.1a) indicates that the *length scale* l_z for the longitudinal variation of A(z,t) must be such that

$$\bullet l_z \cong O\left(\frac{d}{\sqrt{\sigma}}\right) >> d \quad , \qquad (3.1b)$$

or, in short: (3.1a) deals with a *weak three-dimensional variation* of the flow field. As it is discussed in the last item of the present section, GLE can be easily extended to the case where both the *geometry* and the *incident flow* change in the longitudinal direction if the rate of change is weaker than (3.1b).

In what follows, (3.1a) will be obtained as a consistent asymptotic approximation of the NSE and, in deriving it, a procedure to determine *Ginzburg's coefficient* γ will be defined.

3.1: Asymptotic approximation for 3D field equation

Let $\mathbf{u}_{(3d)}(\mathbf{x},z,t) = \mathbf{u}_e(\mathbf{x}) + [\mathbf{u}_o(\mathbf{x},z,t) + w_o(\mathbf{x},z,t)\mathbf{k}]$ be the 3D velocity field for the flow around a slender cylinder, with $\mathbf{u}_e(\mathbf{x})$ being the 2D *steady solution* and $[\mathbf{u}_o(\mathbf{x},z,t) + w_o(\mathbf{x},z,t)\mathbf{k}]$ the *perturbation* on it; if, as defined in section (2), $\nabla = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$, the 3D NSE for the *perturbation* $[\mathbf{u}_o(\mathbf{x},z,t) + w_o(\mathbf{x},z,t)\mathbf{k}]$ is given by

$$\bullet \frac{\partial \mathbf{u}_o}{\partial t} + (\mathbf{u}_e\cdot\nabla)\mathbf{u}_o + (\mathbf{u}_o\cdot\nabla)\mathbf{u}_e + (\mathbf{u}_o\cdot\nabla)\mathbf{u}_o - \frac{1}{Re}\nabla^2\mathbf{u}_o + \nabla p_o = \underbrace{\frac{1}{Re}\frac{\partial^2\mathbf{u}_o}{\partial z^2}}_{O(\sigma^{3/2})} - \underbrace{w_o\frac{\partial \mathbf{u}_o}{\partial z}}_{O(\sigma w_o)};$$

$$\bullet \underbrace{\frac{\partial w_o}{\partial t} + (\mathbf{u}_e\cdot\nabla)w_o - \frac{1}{Re}\nabla^2 w_o}_{O(w_o)} = -\underbrace{\frac{\partial p_o}{\partial z}}_{O(\sigma)} - \underbrace{(\mathbf{u}_o\cdot\nabla)w_o}_{O(\sigma^{1/2}w_o)} - \underbrace{w_o\frac{\partial w_o}{\partial z}}_{O(\sigma^{1/2}w_o^2)} + \underbrace{\frac{1}{Re}\frac{\partial^2 w_o}{\partial z^2}}_{O(\sigma w_o)}; \qquad (3.2a)$$

$$\bullet \nabla\cdot\mathbf{u}_o = -\underbrace{\frac{\partial w_o}{\partial z}}_{O(\sigma^{1/2}w_o)} \ ,$$

where (3.1b) was used to estimate the order of magnitude of the z-derivative and $\{\mathbf{u}_o;\ p_o\} \cong O(\sigma^{1/2})$, see section (2). From the w_o-equation one obtains, at once, that

$$\bullet w_o \cong O(\sigma), \qquad (3.2b)$$

showing that the *longitudinal perturbed velocity* w_o is, as expected, of smaller order than the *perturbed cross-flow* $\mathbf{u}_o$. Furthermore, if terms of order σ^2 are disregarded, as before, it is possible to check from (3.2a) that the longitudinal velocity w_o *does not* affect the (dynamic) $\mathbf{u}_o$-equation: the longitudinal flow affects the 2D solution $\mathbf{u}_o$ only through the *mass conservation equation*, with a term of order $O(\sigma^{3/2})$, the same order of magnitude of the longitudinal diffusion in the $\mathbf{u}_o$-equation. One is left, thus, with the equation

$$\bullet \frac{\partial \mathbf{u}_o}{\partial t} + (\mathbf{u}_e \cdot \nabla)\mathbf{u}_o + (\mathbf{u}_o \cdot \nabla)\mathbf{u}_e + (\mathbf{u}_o \cdot \nabla)\mathbf{u}_o - \frac{1}{Re}\nabla^2 \mathbf{u}_o + \nabla p_o = \underbrace{\frac{1}{Re}\frac{\partial^2 \mathbf{u}_o}{\partial z^2}}_{O(\sigma^{3/2})};$$

$$\bullet \underbrace{\frac{\partial w_o}{\partial t} + (\mathbf{u}_e \cdot \nabla) w_o - \frac{1}{Re}\nabla^2 w_o}_{O(\sigma)} = -\underbrace{\frac{\partial p_o}{\partial z}}_{O(\sigma)}; \tag{3.2c}$$

$$\bullet \nabla \cdot \mathbf{u}_o = -\underbrace{\frac{\partial w_o}{\partial z}}_{O(\sigma^{3/2})},$$

that defines, with an error factor $[1 + O(\sigma^2)]$, an *asymptotic approximation* for $\mathbf{u}_{(3d)}(\mathbf{x},z,t)$.

Equation (3.2c) is, in some sense, standard in existing "slender bodies theories": the 2D structure, represented by the *dynamic* $\mathbf{u}_o$-equation, is not spoiled by the *longitudinal velocity* w_o, the influence of this parcel appearing only in an oblique way in the problem. In fact, a "spontaneous" 3D perturbation on the cross-flow introduces a pressure gradient that forces a longitudinal flow w_o and only then, through mass conservation, the 3D perturbation feeds back the 2D original equation: as in the well known *Lifting Line Theory*, the three-dimensionality, represented by the *"sidewash"* w_o, affects essentially the *kinematics* of the 2D flow. Furthermore, the pressure gradient, and so the "sidewash", is proportional to $\partial A/\partial z$ and then $\nabla\cdot\mathbf{u}_o \propto \partial^2 A/\partial z^2$: this correction on the cross-flow is added to the straight *diffusion term*, proportional to $\partial^2\mathbf{u}_o/\partial z^2$, to produce the *Ginzburg coefficient* $\gamma = \gamma_R + i\gamma_I$. The influence of the *"sidewash"* on the final equation is thus twofold: first, it gives rise to a *longitudinal diffusion*, proportional to $\nabla^2 w_o$, that together with the *cross-flow diffusion* $\partial^2\mathbf{u}_o/\partial z^2$ determines γ_R; second, the longitudinal flow w_o introduces a kind of *"compressibility"* for the cross-flow $\mathbf{u}_o$ ($\nabla\cdot\mathbf{u}_o \propto \partial^2 A/\partial z^2$) and an *"acoustic wave equation"* must be expected then, described here by the *"longitudinal wave operator"* $\partial A/\partial t - i\gamma_I\partial^2 A/\partial z^2 = 0$ with a dispersion relation $\omega + \gamma_I k^2 = 0$. If the nonlinear term $i\mu_I|A|^2A$ is added to this wave operator one obtains the *cubic Schrödinger equation* $i\partial A/\partial t + \gamma_I\partial^2 A/\partial z^2 - \mu_I|A|^2A = 0$, a conspicuous presence in the study of *nonlinear dispersive wave systems*, see Whitham (1974). In what follows the discrete solution of (3.2c) will be defined and discussed.

3.2: The "sidewash" and mass conservation

One starts by considering the w_o-equation, forced by the term $-\partial p_o/\partial z$; to leading order one has (see (2.30d) using $p_{11}(\mathbf{x})\Leftrightarrow\mathbf{P}_{11}$)

$$-\frac{\partial p_o}{\partial z} = -\left[p_{11}(\mathbf{x})\frac{\partial A}{\partial z}e^{i\omega t} + (*)\right] + O(\sigma^{3/2}),$$

and thus writing $w_o(\mathbf{x},z,t)$ in the form

$$\bullet w_o(\mathbf{x},z,t) = \left[w_{11}(\mathbf{x})\cdot\frac{\partial A}{\partial z}e^{i\omega t} + (*)\right], \tag{3.3a}$$

the following equation for $w_{11}(\mathbf{x})$ can be obtained[10]:

$$\bullet i\omega w_{11}(\mathbf{x}) + (\mathbf{u}_e\cdot\nabla)w_{11}(\mathbf{x}) - \frac{1}{Re}\nabla^2 w_{11}(\mathbf{x}) = -p_{11}(\mathbf{x}). \tag{3.3b}$$

Taking the same mesh used to discretize both the velocity and pressure fields in section (2), namely, assuming

$$w_{11}(\mathbf{x}) = \sum_{j=1}^{N} W_{11,j}\cdot h_j(\mathbf{x});\ \mathbf{W}_{11} = \{W_{11,j}\};$$
$$p_{11}(\mathbf{x}) = \sum_{\alpha=1}^{e} P_{11,\alpha}\cdot t_\alpha(\mathbf{x});\ \mathbf{P}_{11} = \{P_{11,\alpha}\}, \tag{3.3c}$$

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[10] For simplicity, *homogeneous boundary condition* is assumed on ∂R. Others boundary conditions could be used instead but this simple one is reasonable and easier from a more technical point of view.

and introducing the matrices {**m**; **k**; $\mathbf{R}_w$} by the expressions

$$\int_R w_{11}(\mathbf{x})\cdot\delta w(\mathbf{x})dR = \delta\mathbf{W}^t\cdot\mathbf{m}\cdot\mathbf{W}_{11};\ \int_R p_{11}(\mathbf{x})\cdot\delta w(\mathbf{x})dR = \delta\mathbf{W}^t\cdot\mathbf{R}_w\cdot\mathbf{P}_{11};$$
$$\int_R\left[\left(\mathbf{u}_e\cdot\nabla\right)w_{11}(\mathbf{x})\cdot\delta w(\mathbf{x}) + \frac{1}{Re}\nabla w_{11}(\mathbf{x})\cdot\nabla(\delta w(\mathbf{x}))\right]dR = \delta\mathbf{W}^t\cdot\mathbf{k}\cdot\mathbf{W}_{11}, \tag{3.3d}$$

the following algebraic equation is obtained for the “nodal values vector” $\mathbf{W}_{11}$,

$$\bullet\left(i\omega\,\mathbf{m}+\mathbf{k}\right)\cdot\mathbf{W}_{11} = -\mathbf{R}_w\cdot\mathbf{P}_{11}, \tag{3.4a}$$

the non-singularity of (3.4a) being granted by the fact that all eigenvalues {χ_j; j = 1,2,..,N} of the matrix **k** have necessarily[11] *negative real parts* and so{$\chi_j \neq i\omega$ *all* j}. The *continuity equation* in its weak form reads

$$\int_R \delta p(\mathbf{x})\left[\nabla\cdot\mathbf{u}_o + \left(w_{11}(\mathbf{x})\frac{\partial^2 A}{\partial z^2}e^{i\omega t} + (*)\right)\right]dR = 0,$$

and the discrete form of this equation is given by

$$\bullet\mathbf{R}^t\cdot\mathbf{U}_o = -\left(\mathbf{R}_w^t\cdot\mathbf{W}_{11}\cdot\frac{\partial^2 A}{\partial z^2}e^{i\omega t} + (*)\right). \tag{3.4b}$$

If (3.4b) is multiplied on the left by **R** and the definition of the *“conjugated Laplacian”* $\boldsymbol{\nabla} = \mathbf{R}\cdot\mathbf{R}^t$ is used, one obtains for $\mathbf{U}_o(t)$ the equation

$$\bullet\boldsymbol{\nabla}\cdot\mathbf{U}_o = -\left(\mathbf{R}\cdot(\mathbf{R}_w^t\cdot\mathbf{W}_{11})\cdot\frac{\partial^2 A}{\partial z^2}e^{i\omega t} + (*)\right), \tag{3.5a}$$

whose general solution can be written as (recall that *Null*($\boldsymbol{\nabla}$) = J_s)

$$\bullet\mathbf{U}_o(t) = \mathbf{T}\cdot\mathbf{q}_o(z,t) - \left[\mathbf{C}\cdot\frac{\partial^2 A}{\partial z^2}e^{i\omega t} + (*)\right], \tag{3.5b}$$

with $\mathbf{C}\in G_r$ being the *unique* (within G_r) *solution* of

$$\bullet\boldsymbol{\nabla}\cdot\mathbf{C} = \mathbf{R}\cdot(\mathbf{R}_w^t\cdot\mathbf{W}_{11})\in G_r. \tag{3.5c}$$

Notice that **C** can be defined by the sum of the series (see (2.9b))

$$\bullet\mathbf{C} = \frac{1}{\kappa_r}\sum_{n=0}^{\infty}\boldsymbol{\nabla}_1^n\cdot\mathbf{R}\cdot(\mathbf{R}_w^t\cdot\mathbf{W}_{11}), \tag{3.5d}$$

where (3.5d) has again an “almost geometric structure” and its convergence can be thus accelerated by Shanks Transformation, see (2.15c).

3.3: The Ginzburg-Landau Equation (GLE)

The discrete form of the $\mathbf{u}_o$-equation in (3.2c) is given by

$$\delta\mathbf{U}^t\cdot\left\{\mathbf{M}\cdot\dot{\mathbf{U}}_o + \mathbf{K}\cdot\mathbf{U}_o + \mathbf{N}(\mathbf{U}_o)\cdot\mathbf{U}_o - \mathbf{R}\cdot\mathbf{P}_o\right\} = \delta\mathbf{U}^t\cdot\left(\frac{1}{Re}\mathbf{M}\cdot\mathbf{T}\cdot\mathbf{E}\cdot\frac{\partial^2 A}{\partial z^2}e^{i\omega t} + (*)\right) \tag{3.6a}$$

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[11] The parcel proportional to $\mathbf{u}_e$ in (3.3d) leads to an anti-symmetric matrix while the one proportional to 1/Re leads to a symmetric positive definite matrix. From this it follows at once that *Real* $\chi < 0$ if ($\chi\mathbf{m} + \mathbf{k})\cdot\mathbf{X} = \mathbf{0}$.

where the leading order term

$$\mathbf{U}_o(z,t) = \left[\mathbf{T}\cdot\mathbf{E}\ A(z,t)e^{i\omega t} + (*)\right] \cong O(\sigma^{1/2}) \tag{3.6b}$$

was used in the right side of (3.2c). Placing now (3.5b) into (3.6a) one obtains, after projecting into the solenoidal sub-space, the equation

$$\begin{aligned} &\mathbf{M}_s\cdot\dot{\mathbf{q}}_o + \mathbf{K}_s\cdot\mathbf{q}_o - \left[\left(\mathbf{C}_s + \frac{1}{Re}\mathbf{M}_s\cdot\mathbf{E}\right)\frac{\partial^2 A}{\partial z^2}e^{i\omega t} + (*)\right] + \mathbf{N}_s(\mathbf{q}_o)\cdot\mathbf{q}_o = \mathbf{0}; \\ &\mathbf{C}_s = \mathbf{T}^t\cdot\left(i\omega\,\mathbf{M} + \mathbf{K}\right)\cdot\mathbf{C}, \end{aligned} \tag{3.6c}$$

whose *asymptotic solution* (2.25a) has an amplitude A(z,t) that satisfies the GLE (3.1a) with

$$\bullet\gamma = \gamma_R + i\gamma_I = \mathbf{E}_a^t\cdot\mathbf{C}_s + \frac{1}{Re}, \tag{3.7}$$

since $\mathbf{E}_a^t\cdot\mathbf{M}\cdot\mathbf{E} = 1$. The remaining terms for the velocity and pressure fields are given by (2.29a);(2.30b,c,d), adding the pressure parcels proportional to $\partial^2A/\partial z^2$. The attention will be turned next to a more detailed analysis of GLE.

3.4: GLE: boundary conditions and wave-like limit cycles

If the cylinder's span is defined in the interval $-l \le z \le l$ boundary conditions must be imposed at the cylinder ends $z = \pm l$, one in each extremity. Using again the notation $\{\mathbf{e}(\mathbf{x}) \Leftrightarrow \mathbf{T}\cdot\mathbf{E};\ w_{11}(\mathbf{x}) \Leftrightarrow \mathbf{W}_{11}\}$ the velocity field can be written, to leading order, in the form

$$\mathbf{u}_{(3D)}(\mathbf{x},z,t) = \mathbf{u}_e(\mathbf{x}) + \left[\left(\underbrace{A(z,t)\cdot\mathbf{e}(\mathbf{x})}_{\mathbf{u}_o(\mathbf{x},z,t)} + \underbrace{\frac{\partial A}{\partial z}(z,t)\cdot w_{11}(\mathbf{x})\mathbf{k}}_{w_o(\mathbf{x},z,t)}\right)e^{i\omega t} + (*)\right], \tag{3.8a}$$

and two conditions can be naturally imposed on A(z,t), namely:

$$\begin{aligned} &\text{i)}\ \frac{\partial A}{\partial z}(\pm l,t) = 0 \Leftrightarrow w_o(\mathbf{x};\pm l,t) = 0; \\ &\text{ii)}\ \ A(\pm l,t) = 0 \Leftrightarrow \mathbf{u}_o(\mathbf{x};\pm l,t) = \mathbf{0}\ . \end{aligned} \tag{3.8b}$$

Boundary condition (i) is apparently more appropriated for the case where the cylinder ends either at the *free-surface* in a water channel or else if the *"end-cylinder technique"*[12] is used at its bottom end: in these situations one expects that the perturbation on the 2D steady solution should be, by far, dominated by the cross-flow $\mathbf{u}_o(\mathbf{x},z,t)$. Boundary condition (ii) is more awkward to be interpreted though it seems to be adequate to represent a cylinder ending in the *interior of the fluid*, where then the flow perturbation $w_o(\mathbf{x},z,t)$ in the longitudinal direction should be stronger than the perturbed cross-flow $\mathbf{u}_o(\mathbf{x},z,t)$ near this "free end". Possibly linear combinations of (3.8b), including *periodic boundary conditions*, see (3.13b) below, could also be imposed.

The GLE (3.1a) depends on three coefficients, $\{\sigma;\ \mu = \mu_R + i\mu_I;\ \gamma = \gamma_R + i\gamma_I\}$, and it is important to understand how the qualitative behavior of them affects the solution. The real parameters $\{\sigma;\ \mu_R;\ \gamma_R\}$ should be all positive: $\sigma > 0$ is a *negative dissipation* that causes the instability, $\mu_R > 0$ is a *non-linear diffusion* due to the 2D cross flow and $\gamma_R > 0$ is essentially a *linear viscous diffusion* caused both by the viscous stress $1/Re(\nabla^2 w_o)$ of the longitudinal velocity in the cross-section plane and by the cross-flow viscous stress $1/Re(\partial^2\mathbf{u}_o/\partial z^2)$; notice that the inequality $\mu_R > 0$ – the super-criticality of the Hopf bifurcation – was discussed in section (2) while the relation $\gamma_R > 0$ can be inferred from the Principle of the Virtual Power.

The GLE (3.1a) has wave-like solutions of the form

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[12] Namely, a larger cylinder is smoothly fitted to the *bottom end*, increasing locally the *cross-flow* and creating a bottom end condition similar to the one found at the free surface; see Khalak & Williamson (1996)

$$A_0(z,t) = R_0 \cdot e^{i(k_0 z - \Delta\omega_0 t)};$$
$$R_0 = \left(\frac{\sigma - k_0^2\gamma_R}{\mu_R}\right)^{1/2};\Delta\omega_0 = k_0^2\gamma_I + \mu_I R_0^2\ , \tag{3.9a}$$

the stability condition of these wave-like solutions being given by the condition

$$\{(\mu_I\gamma_I + \mu_R\gamma_R) > 0; R_0 > R_{0,L}(\sigma,\gamma,\mu)\}\ . \tag{3.9b}$$

Notice that besides a restriction on the coupled effect of *dispersion* ($\mu_I\gamma_I$) and *diffusion* ($\mu_R\gamma_R$), condition (3.13) offers also a restriction on the wave amplitude R_0: this amplitude should be, in general, *larger* than a lower bound $R_{0,L} = R_{0,L}(\sigma,\gamma,\mu)$ for, otherwise, the wave solution (3.2) is possibly within the *repulsion basin* of the *unstable* null solution $A(z,t) \equiv 0$. In the other hand, the existence of a *continuum* of *stable solutions* (3.9c) in the range $R_{0,L} \le R_0 \le (\sigma/\mu_R)^{1/2}$ (or in the range $0 \le |k_0| \le k_{0,L}$) seems to be in line with the diversity of shedding modes (oblique, parallel, etc) found in the experiments with fixed cylinders, see Khalak & Williamson (1996). Notice, in particular, that (3.9c) satisfies the following boundary conditions at the cylinder ends $z = \pm l$,

$$\frac{\partial A}{\partial z}(\pm l, t) - ik_0 A(\pm l, t) = 0\ , \tag{3.9c}$$

that reduces to the condition (i) in (3.8b) when $k_0 = 0$ (2D solution): as a matter of fact, the *"end-cylinder technique"* was introduced just to create conditions to favor the *parallel shedding* on a cylinder in a water channel, see again Khalak & Williamson (1996), since it forces, apparently, the condition $\partial A/\partial z \approx 0$ at the bottom end. Obviously, more assertive statements about some features of the solutions can only be done by direct numerical simulation of (3.1a) but it is felt that this simple stability analysis helps to focus some relevant issues.

3.5: Spanwise variation of geometry and current

So far the analyzes was restricted to the simple *uniform flow* along a *cylinder* – being more precise, all empirical and numerical evidences commented here are related specifically to the flow around *circular* cylinders – although one should be concerned, from a more practical point of view, with problems where the cross section geometry and the incident current changes along the span, both in intensity and in direction. Relevant examples are the flow around a *tapered cylinder*, used to emulate a current variation along the span, the flow around a *circular cylinder with strakes*, very important from a practical point of view, or else the change of the *of the incident flow direction* along depth, a situation usually encountered in Ocean Engineering. In all these examples the *unstable mode* $\mathbf{E}_U$ *changes* in the z-direction and so it does the sectional pressure field $\mathbf{P}_{11}$ that forces the longitudinal flow $w_o(\mathbf{x},z,t)$, see (2.30b,c,d); writing, as before, $p_{11}(\mathbf{x},z) \Leftrightarrow \mathbf{P}_{11}(z)$, one has that $\partial p_{11}/\partial z \cong O(p_{11}/l_{g,c})$, where $l_{g,c}$ is the length scale for the longitudinal variation of both the geometry and the current, and the question one intends to answer is the following: how small can be $l_{g,c}$ in order the GLE (3.1a) remains valid, with the same error factor of the form $[1 + O(\sigma^2)]$, even in the presence of these variations? Obviously, the basic parameters should then change (weakly) with z – namely, $\{\omega = \omega(z); \sigma = \sigma(z); \mu = \mu(z); \gamma = \gamma(z)\}$ – but the structure of (3.1a) would remain the same and so the 2D expressions used to compute them.

To answer such question one must recall that the three-dimensionality was forced by the pressure gradient that, to leading order, is given by

$$\frac{\partial p}{\partial z}(\mathbf{x}, z, t) = \left[\left(\frac{\partial A}{\partial z}p_{11} + A\frac{\partial p_{11}}{\partial z}\right)e^{i\omega t} + (*)\right] + O(\sigma^{3/2})\ , \tag{3.14a}$$

and observing that to retain the *final* error $O(\sigma^2)$ only the term of order $O(\sigma)$ must be kept in (3.14a), the variation $\partial p_{11}/\partial z$ could be ignored if it is of order $O(\sigma)$ or smaller: in this case the term $A \cdot \partial p_{11}/\partial z$ would be of order $O(\sigma^{3/2})$ or smaller, since $A \cong O(\sigma^{1/2})$, and the variation of the *geometry* and/or the *incident flow direction* would appear at most at the order $O(\sigma^2)$ and can, thus, be disregarded. It turns out that (3.1a) remains correct, with the sectional parameters $\{\omega(z); \sigma(z); \mu(z); \gamma(z)\}$, whenever the *length scale* $l_{g,c}$ is so large that $(\partial p_{11}/\partial z \cong O(p_{11}/l_{g,c}))$

$$\frac{\partial p_{11}}{\partial z} \le O(\sigma) \Leftrightarrow l_{g,c} \ge O\left(\frac{d}{\sigma}\right). \tag{3.14b}$$

As seen in (2.31), empirical evidences suggest that $\sigma \approx 0.20$ and so $l_{g,c} \geq O(5d)$, the fastest change in z-direction within the context of GLE being defined by the relation $l_{g,c} \approx 5d$. It is a matter of curiosity to observe that $5d$ is a typical length scale for most of the "suppressions devices" used to mitigate (or eliminate) VIV; for instance, this is a typical value for the *helicoidal pitch of the strakes* or for the *wavelength of the wavy cylinder* analyzed by Bearman (2000).

4. Conclusion

In the present paper a consistent asymptotic approximation for the flow around a slender cylinder was developed, leading to the Ginzburg-Landau equation. The theory is based on an *assumption* concerning the *behavior of the eigenvalues* related to a 2D perturbation on the steady 2D solution $\mathbf{u}_e(\mathbf{x})$; it states that there exists *only one* unstable mode with eigenvalue $\lambda_1 = \sigma + i\omega$ and, furthermore, that $\{\omega \neq 0; 0 < \sigma << 1\}$, see (2.21).

Both conditions are satisfied in the vicinity of the Hopf bifurcation at $Re \approx Re_{cr} \cong 45$ but to obtain the desired approximation some more technical results were needed. First, the *"wake impedance"* was introduced, by considering properly the flow in the wake and determining then how the wake "resists" to the flow within the *finite fluid region* that is actually discretized; second, by projecting the discrete flow equations into the *solenoidal* and *gradient* sub-spaces it has been possible, using some standard results in Linear Algebra, to show not only the inner consistency of the model but also to obtain the coefficients of the Ginzburg-Landau equation. In particular, the *Ginzburg coefficient* $\gamma = \gamma_R + i\gamma_I$ was analyzed, where the *"diffusive feature"* of the real part γ_R was elaborated and also the *"wave feature"* of γ_I was established, once it is related to the *"compressibility"* of the cross-flow, namely, to the *work done* by the *cross-flow pressure field* on the *divergence of the cross-flow velocity field*.

The final goal of the on-going research is to address the VIV problem, of considerable importance in some Ocean Engineering applications, mainly in the analysis of the "risers" of a floating production system. This problem has been tentatively addressed, with a relative success, by the so-called *"phenomenological models"*, where the flow is simply described by a *Van der Pol oscillator* with coefficients inferred from some experimental results. The situation here is not very much different, at least from the operational point of view, to the usual approach related to Ginzburg-Landau equation (GLE), once this "model" is fitted externally to the problem and the coefficients are then inferred also by some experimental (or numerical) results. But there is a conceptual difference, at least in its origin, in both approaches: GLE was thought to be valid only in the vicinity of Re_{cr} (although it has been used far beyond it, at least up to the range $Re \approx 300$) while the Van der Pol model was aimed, from its very first motivation, to deal with the experiments on VIV, where $10^3 < Re < 10^4$ roughly speaking. Observing that the Van der Pol model assumes, implicitly, that *only one mode is unstable* with a "negative damping" $\sigma << 1$, one may be tempted to conclude, based on the relatively good predictive ability of these *"phenomenological models"* and also on the already proposed extensions of GLE to the range $Re \approx 300$, that the underlying assumption of the present asymptotic theory holds, in fact, in a much broader range of Reynolds numbers than foreseen a priori.

This conjecture can be raised to the status of a *hypothesis* in the mathematical development and, with it, to extend consistently the GLE to the "whole" range of Re; afterwards, by looking to the actual numerical results one can confirm (or not) this hypothesis. The present theory does not seem to be at odds, thus, with tradition in physical science: indeed, given a set of a somewhat disperse results and observations, all of them related however to the conspicuous *oscillatory behavior* of the phenomenon, they can be gathered by means of an assumption, synthesized by (2.21), that places all them in an unique framework, namely, the GLE in a *"wide range"* of Reynolds numbers. Furthermore, as stated above, this theory brings in its lay out the possibility to be *refuted*, once the basic assumption can be checked directly by means of (numerical) experiments; or, from a more practical point of view, it allows one to determine precisely how "wide" can be the range of Reynolds numbers covered by it.

The route to be followed next, in this on-going research, is thus clear: first, to obtain numerical results that could possibly confirm assumption (2.21) in a certain range of Re; second, to obtain an extension of the GLE equation that can deal also with an oscillating cylinder; third, to compare the VIV predictions obtained from this extended model with the existing experimental results. The hope is that they will provide consistent results but, anyway, one has here at least a consistent theory in a relatively large vicinity of Re_{cr}, what may have an interest in itself.

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