IX CONGRESSO BRASILEIRO DE ENGENHARIA E CIÊNCIAS TÉRMICAS

9th BRAZILIAN CONGRESS OF THERMAL ENGINEERING AND SCIENCES

Paper CIT02-0623

COMPUTATIONAL ASPECTS OF METRICS EVALUATION FOR THE FINITE VOLUME METHOD

Edson Gomes Moreira Filho
Department of Mechanical Engineering, EE/COPPE
Universidade Federal do Rio de Janeiro
Cid. Universitária, Cx. Postal:68503
Rio de Janeiro – RJ – 21945-970
Brazil
edson@lmt.coppe.ufrj.br

Marcelo Moreira Mejias
Department of Mechanical Engineering, EE/COPPE
Universidade Federal do Rio de Janeiro
Cid. Universitária, Cx. Postal:68503
Rio de Janeiro – RJ – 21945-970
Brazil
mejias@lmt.coppe.ufrj.br

Helcio Rangel Barreto Orlande
Department of Mechanical Engineering, EE/COPPE
Universidade Federal do Rio de Janeiro
Cid. Universitária, Cx. Postal:68503
Rio de Janeiro – RJ – 21945-970
Brazil
helcio@serv.com.ufrj.br

Albino José Kalab Leiroz
Department of Mechanical and Materials Engineering
Instituto Militar de Engenharia
22290-270 - Rio de Janeiro, RJ
Brazil
leiroz@ime.eb.br

***Abstract**. The numerical solution of partial differential equations within irregular domains using the Finite Volume Method and grid generation techniques requires the evaluation of approximations for the transformation metrics at the volume center and at the center points of the volume faces. However, for highly distorted or stretched volume cells, the average of coordinate approach may not provide an appropriate representation of volume and face center positions. The present work presents an analysis of an alternative technique for the metric evaluation, which makes use, in each coordinate direction, of a grid with twice the number of points used for the governing equation solutions. The proposed approach allows the required transformation metrics and the volume and face center positions to be calculated within the computational domain. The Finite Volume Method is applied to the transformed conservation equations using a regularly spaced grid within the computational domain. In order to analyze the computational performance of the proposed technique, test cases, for which analytical solutions are available, are studied. Initially, analytical grid generation techniques are applied to one-dimensional convection-diffusion model equations. Numerically obtained results are compared with analytical values showing the precision of the proposed approach. Two-dimensional test cases are also studied. Results show that, for a given precision, the proposed double-grid approach allows the usage of coarser discretizing grids. Therefore, a balance between the increase of computational costs associated with numerically generating a finer grid and the solution of the transformed governing equation with more precise transformation metric values is observed.*

***Keywords**. Finite Volume, Double Grid, Grid Generation, Metrics Evaluation.*

1. Introduction

The numerical solution of partial differential equations within irregular domains using the Finite Volume Method and grid generation techniques requires the evaluation of the transformation metrics at the volume center as well as at each volume face center points. For regularly spaced structured grids, the determination of the geometric positions of the points, where metrics evaluation is required, becomes straightforward, since a coincidence with the grid points is observed. For structured grids with irregular spacing and different degrees of volume distortion, the determination of the position of volume and face center points becomes part of the grid generation procedure. For a broad range of applications, numerical grid generation is of particular interest since other approaches, such as the analytical grid generation, are unsuited. During the numerical grid generation, transformation metrics should also be determined.

Transformation metrics and positions of volume and face center points are usually determined during the grid generation procedure by averaging grid point positions (Thompson et all., 1985). Despite satisfying grid metric identities, the traditional approach introduces numerical error, by using grid point position within the physical domain for the averaging procedure, in addition to the domain discretization error. The introduced error presents a close relation to the mesh quality expressed on terms of grid spacing and grid line angle. Improving grid quality, which leads to reduction of the metrics evaluation error, is usually associated with higher computational costs. It is noteworthy mentioning that metrics evaluation and discretization error for the solution of partial differential equations are independent and can be addressed individually.

In order to reduce discretization error and control computational costs, grid optimization techniques, such as clustering of points and grid adaptation, are usually applied. Grid point clustering is usually used within regions of the solution domain where steep gradients of the solution profiles are expected. Adaptive grids use information of the obtained solution profiles in order the control grid parameters based on a established criteria. A different approach is based on the minimization of the discretization error using an optimized grid point distribution (Yamallev, 2001). The different approaches share the focus on discretization error reduction and are usually not concerned with errors associated with numerical metric evaluation and approximate grid geometry determination.

For Lagrangian dicretizations, for which geometric grid quality becomes important, different approaches for the precise evaluation of volume and face center point positions were developed. The reconstruction of grid lines using straight lines segments and arcs of local circles was introduced in order to construct discretizations for which symmetry is preserved for different coordinate systems (Margolin & Shashkov, 1999). A median grid is constructed using the mid points of each grid cell and defines the cell centers. The computational costs associated with the higher-order reconstruction presents a disadvantage of the proposed approach and the influence of the procedure on the solution precisionj is still to be determined (Margolin & Shashkov, 1999). Numerical solution for Lagrangian hydrodynamics models requires the independence of the obtained results on the distortion of the adopted grid (Hermeline, 2000). The grid distortion independence condition is addressed by the definition of a course and a more refined unstructured grid within the solution domain. The finer grid is constructed with twice the number of points present on the primary grid. The transport equations are integrated over each volume. Applying Green's formula, the volume integrals are reduced to sums of fluxes over volume faces. Values of the unknown function at the primary and the dual grids are used to evaluate the required fluxes. Despite numerical experiments showing the efficiency of the method, for nonlinear problems the conservation principle is violated (Hermeline, 2000). An unstructure dual grid approach was also applied to the numerical solution Eulerian models using the Finite Volume Method (Perrot, 2000). Using staggered grids, results show that conservative schemes for kinetic energy, vorticity and momentum can be constructed. Despite employed in a wide range of applications, the grid generation procedure for unstructured grids can present high complexity and computational costs.

Regularly spaced grids were used to discretize irregular domains for Finite Volume solution of convection-diffusion equations (Calhoun et all, 2000). Volume cells within the solution domain intersect with irregular domain boundaries. Therefore, portions of volume cells close to solid boundaries become blocked to the fluid passage. A capacity function is introduced in order to account for the reduction of passage area. The use of regular grids has the advantage of avoiding the need for grid generation procedures but requires algorithms capable of maintaining stability close to the embedded surfaces (Calhoun at all, 2000).

The present work addresses aspects of grid geometric parameters and metrics evaluation techniques related to the Finite Volume Method. Initially, an auxiliary grid, with twice the number of points used for the solution of the partial differential equation, is introduced in order to allow the evaluation of grid geometrical parameters and transformation metrics within the solution domain. Therefore, volume cell and volume face center positions are evaluated within the transformed domain during the grid generation procedure, avoiding the usage of coordinate averaging within the physical domain (Thompson et all., 1985). The Finite Volume Method is applied to the transformed partial differential equation using the primary grid, leading to a system of algebraic equations. The double-grid approach for metrics evaluation is mentioned in Thompson et all, 1985 and the present work is mainly concerned with evaluating numerical and computational aspects of the procedure. In order to analyze the computational performance of the proposed technique, test cases, for which analytical solutions are available, are considered.

2. Test Case I – One Dimensional Convection-Diffusion Equation.

Initially, an analytical transformation is applied to a steady-state convection-diffusion model equation. Control volume centers are readily obtained from the analytical transformation. The volume face positions are calculated using the double-grid and the coordinate average approach. Despite being analytically available, transformation metric is numerically evaluated, allowing a comparison of the approaches.

The one-dimensional convection-diffusion model problem for a variable ϕ is written in conservative form as

$$\frac{\partial(\rho v\phi)}{\partial y}=\frac{\partial}{\partial y}\left(\frac{k\,\partial\phi}{\partial y}\right), \qquad 0<y<h \tag{1}$$

with boundary conditions

$$\phi=1, \qquad y=0 \tag{2}$$

$$\phi=0, \qquad y=h \tag{3}$$

where ρ, k and h represent , respectively, the medium density, diffusion coefficient and the physical domain length. A constant velocity v is also assumed.

The problem described by Eqs. (1)-(3) allows analytical solution that can be expressed by (Versteeg & Malalasekera, 1995)

$$\phi=1-\frac{exp(\rho v\,y/k)-1}{exp(\rho v\,h/k)-1} \tag{4}$$

The convection-diffusion equation and boundary conditions are transformed to the computational domain using the logarithmic transformation (Anderson et all, 1984)

$$\eta=1-\frac{\ln\left\{\dfrac{\beta+1-y/h}{\beta-1+y/h}\right\}}{\ln\left\{\dfrac{\beta+1}{\beta-1}\right\}} \tag{5}$$

where η is the transformed independent variable and β is a clustering parameter $(1<\beta<\infty)$. Clustering of grid points is intensified around $y=0$ as $\beta\rightarrow 1$

The analytical transformation metric is given by

$$\eta_y=\frac{2\beta}{h\left\{\beta^2-\left[1-(y/h)\right]^2\right\}ln\left[\dfrac{\beta+1}{\beta-1}\right]} \tag{6}$$

Applying the transformation defined by Eq. (4), the conservative form of Eq. (1) is written as

$$\partial_\eta(\eta_y\rho v\phi)=\partial_\eta\left[\eta_y k\,\partial_\eta(\eta_y\phi)\right], \qquad 0<\eta<1 \tag{7}$$

with boundary conditions

$$\phi=1, \qquad \eta=0 \tag{8}$$

$$\phi=0, \qquad \eta=1 \tag{9}$$

Applying the Finite Volume Method to the model equation transformed form – Eq. (7) – and WUDS discretization scheme to the convective terms (Maliska, 1995), the resulting algebraic equation for the internal volumes can be written as

$$a_P\phi_P-a_E\phi_E-a_W\phi_W=0 \tag{10}$$

where the coefficients are defined as

$$a_E=-(\rho v)_e(0.5-\alpha_e)+\frac{\beta_e(k\eta_y)_e}{\Delta\eta} \tag{11}$$

$$a_W = (\rho v)_w (0.5 - \alpha_w) + \frac{\beta_w (k\eta_y)_w}{\Delta\eta} \quad (12)$$

$$a_P = a_E + a_W \quad (13)$$

The system of algebraic equations is solved leading to ϕ-profiles within the solution domain. Tables (1a)-(1c) present pure diffusion results obtained for different values of the clustering parameter β, which appears in the analytical transformation defined by Eq.(4). Tables (1a)-(1c) show the numerical solutions obtained using the double-grid (DG) and the coordinate average (CA) approaches in order to obtain the volumes center points. The analytical solution and the relative error obtained with each numerical approach are also included on the tables. Results shown in Tables (1a)-(1c) were obtained using discretizing grids with *5* control volumes in the. Numerical and analytical solutions were evaluated at the center of each control volume. Positions where the solutions are computed are displaced towards $x = 0$ as β approaches unity.

Table 1a. Steady-state one-dimensional pure-diffusion $(v = 0)$ problem – $\beta \to \infty$ (no grid distortion).

Position	ϕ			**Relative Error**	
	DG Approach	**CA Approach**	**Analytical**	**DG Approach**	**CA Approach**
0.100000	*0.900000*	*0.900000*	*0.900000*	*0.00000*	*0.00000*
0.300000	*0.700000*	*0.700000*	*0.700000*	*0.00000*	*0.00000*
0.500000	*0.500000*	*0.500000*	*0.500000*	*0.00000*	*0.00000*
0.700000	*0.300000*	*0.300000*	*0.300000*	*0.00000*	*0.00000*
0.900000	*0.100000*	*0.100000*	*0.100000*	*0.00000*	*0.00000*

Table 1b. Steady-state one-dimensional pure-diffusion $(v = 0)$ problem – $\beta = 2.0$.

Position	ϕ			**Relative Error**	
	DG Approach	**CA Approach**	**Analytical**	**DG Approach**	**CA Approach**
0.084635	*0.917479*	*0.939504*	*0.915365*	*0.002305*	*0.025693*
0.266757	*0.734993*	*0.78106*	*0.733243*	*0.002381*	*0.061221*
0.464102	*0.537211*	*0.586086*	*0.535898*	*0.002444*	*0.085632*
0.673368	*0.327447*	*0.361603*	*0.326632*	*0.002489*	*0.096711*
0.890249	*0.110027*	*0.120534*	*0.109751*	*0.002513*	*0.089464*

Table 1c. Steady-state one-dimensional pure-diffusion $(v = 0)$ problem – $\beta = 1.01$.

Position	ϕ			**Relative Error**	
	DG Approach	**CA Approach**	**Analytical**	**DG Approach**	**CA Approach**
0.006936	*0.994718*	*0.999976*	*0.993064*	*0.001663*	*0.006913*
0.038154	*0.96477*	*0.998982*	*0.961846*	*0.003031*	*0.037175*
0.123092	*0.882719*	*0.982792*	*0.876908*	*0.006583*	*0.107738*
0.331874	*0.677742*	*0.829724*	*0.668126*	*0.014187*	*0.194761*
0.738288	*0.26806*	*0.276575*	*0.261712*	*0.023683*	*0.05374*

The two techniques used for the metrics evaluation are equivalent for regular meshes $(\beta \to \infty)$, since, due to the absence of distortion on the grid, the center point of each volume coincides with the center point obtained with the double-grid approach. It should be noted that the numerical results shown in Tables (1a)-(1c) present a deviation from the analytically obtained values of less than *11%* with either of metrics evaluation procedures, even for the small number of control volumes used in the considered grids. An analysis of Tables (1a)-(1c) reveals a general increase in the relative errors of the two numerical solutions when the mesh distortion is increased, i.e., $\beta \to 1$. However, it is worth mentioning that the relative errors in Tables (1b) and (1c) related to the double-grid approach are, for most of the examined points, one order of magnitude smaller than those obtained with the averaging technique of metrics evaluation. The solution improvement observed for the same mesh is related to the usage of the double-grid approach, which takes into account more accurately the transformation under consideration. Therefore, the numerical representation of solution gradients is improved by the double-grid approach, leading to a reduction of the overall discretization error.

In order to verify the convective effects influence on the behavior observed for the pure-diffusion case, results for $v = 0.1$ and different clustering parameter values are also analyzed in Tables (2a)-(2c). For the depicted results, *20* control volumes were used for the spatial discretization. Similarly to the pure-diffusion problem, the results obtained with both numerical approaches are identical for $v = 0.1$, when a regular mesh ($\beta \to \infty$) is used - Table (2a).

Table 2a- Steady-state one-dimensional convection-diffusion problem with $v = 0.1$ and $\beta \to \infty$ (no grid distortion).

Position	ϕ			**Relative Error**	
	DG Approach	**CA Approach**	**Analytical**	**DG Approach**	**CA Approach**
0.025	*0.985837*	*0.985837*	*0.985267*	*0.000579*	*0.000579*
0.075	*0.955276*	*0.955276*	*0.954673*	*0.000632*	*0.000632*
0.125	*0.923147*	*0.923147*	*0.922511*	*0.000690*	*0.000690*
0.175	*0.889371*	*0.889371*	*0.888699*	*0.000756*	*0.000756*
0.225	*0.853862*	*0.853862*	*0.853154*	*0.000830*	*0.000830*
0.275	*0.816532*	*0.816532*	*0.815787*	*0.000914*	*0.000914*
0.325	*0.777288*	*0.777288*	*0.776503*	*0.001011*	*0.001011*
0.375	*0.736031*	*0.736031*	*0.735206*	*0.001122*	*0.001122*
0.425	*0.692657*	*0.692657*	*0.691791*	*0.001253*	*0.001253*
0.475	*0.647060*	*0.647060*	*0.646150*	*0.001408*	*0.001408*
0.525	*0.599123*	*0.599123*	*0.598169*	*0.001596*	*0.001596*
0.575	*0.548729*	*0.548729*	*0.547728*	*0.001827*	*0.001827*
0.625	*0.495749*	*0.495749*	*0.494701*	*0.002119*	*0.002119*
0.675	*0.440053*	*0.440053*	*0.438955*	*0.002500*	*0.002500*
0.725	*0.381500*	*0.381500*	*0.380351*	*0.003019*	*0.003019*
0.775	*0.319944*	*0.319944*	*0.318743*	*0.003767*	*0.003767*
0.825	*0.255230*	*0.255230*	*0.253975*	*0.004942*	*0.004942*
0.875	*0.187198*	*0.187198*	*0.185887*	*0.007053*	*0.007053*
0.925	*0.115677*	*0.115677*	*0.114308*	*0.011974*	*0.011974*
0.975	*0.040487*	*0.040487*	*0.039059*	*0.036567*	*0.036567*

Table 2b- Steady-state one-dimensional convection-diffusion problem with $v = 0.1$ and $\beta = 2.0$.

Position	ϕ			**Relative Error**	
	DG Approach	**CA Approach**	**Analytical**	**DG Approach**	**CA Approach**
0.020740	*0.988287*	*0.992376*	*0.987804*	*0.000490*	*0.004629*
0.063060	*0.962643*	*0.974768*	*0.962119*	*0.000545*	*0.013147*
0.106483	*0.935176*	*0.954937*	*0.934609*	*0.000607*	*0.021750*
0.150985	*0.905762*	*0.932662*	*0.905150*	*0.000676*	*0.030395*
0.196536	*0.874266*	*0.907707*	*0.873606*	*0.000755*	*0.039034*
0.243104	*0.840549*	*0.879827*	*0.839839*	*0.000845*	*0.047613*
0.290650	*0.804464*	*0.848769*	*0.803702*	*0.000949*	*0.056075*
0.339130	*0.765858*	*0.814277*	*0.765041*	*0.001069*	*0.064358*
0.388499	*0.724573*	*0.776091*	*0.723698*	*0.001210*	*0.072397*
0.438703	*0.680445*	*0.733953*	*0.679508*	*0.001378*	*0.080123*
0.489688	*0.633304*	*0.687611*	*0.632303*	*0.001582*	*0.087470*
0.541391	*0.582977*	*0.636825*	*0.581910*	*0.001833*	*0.094370*
0.593750	*0.529289*	*0.581372*	*0.528153*	*0.002151*	*0.100764*
0.646696	*0.472062*	*0.521048*	*0.470853*	*0.002567*	*0.106603*
0.700159	*0.411118*	*0.455679*	*0.409833*	*0.003134*	*0.111865*
0.754063	*0.346279*	*0.385127*	*0.344916*	*0.003952*	*0.116582*
0.808333	*0.277371*	*0.309289*	*0.275926*	*0.005237*	*0.120914*
0.862889	*0.204223*	*0.228113*	*0.202693*	*0.007548*	*0.125407*
0.917651	*0.126673*	*0.141594*	*0.125055*	*0.012939*	*0.132256*
0.972536	*0.044565*	*0.049787*	*0.042856*	*0.039889*	*0.161735*

As the grid is distorted by tuning the value of β, errors associated with the coordinate averaging approach are shown to substantially increase, as can be observed from the results on Tables (2b) and (2c). On the other hand, the relative error between the analytical solution and the numerical results obtained from the double-grid approach are practically unaffected by the introduction of mesh distortion. In fact, it is observed from Tables (2b) and (2c) that errors obtained with the double-grid approach are at least one order of magnitude smaller than the errors obtained with the traditional averaging technique. Besides, for many positions in the solution domain, the relative errors associated with the double-grid technique are two orders of magnitude smaller than those for the coordinate average.

Table 2c- Steady-state one-dimensional convection-diffusion problem with $v = 0.1$ and $\beta = 1.01$.

Position	ϕ			**Relative Error**	
	DG Approach	**CA Approach**	**Analytical**	**DG Approach**	**CA Approach**
0.001410	*0.999242*	*0.999999*	*0.999179*	*0.000063*	*0.000821*
0.004849	*0.997263*	*0.999997*	*0.997171*	*0.000092*	*0.002833*
0.009314	*0.994682*	*0.999991*	*0.994554*	*0.000129*	*0.005467*
0.015106	*0.991318*	*0.999978*	*0.991142*	*0.000178*	*0.008915*
0.022607	*0.986932*	*0.999950*	*0.986694*	*0.000242*	*0.013435*
0.032300	*0.981214*	*0.999889*	*0.980895*	*0.000325*	*0.019364*
0.044796	*0.973761*	*0.999758*	*0.973337*	*0.000436*	*0.027145*
0.060851	*0.964046*	*0.999477*	*0.963486*	*0.000581*	*0.037355*
0.081392	*0.951386*	*0.998881*	*0.950651*	*0.000773*	*0.050733*
0.107528	*0.934894*	*0.997632*	*0.933933*	*0.001029*	*0.068205*
0.140555	*0.913423*	*0.995060*	*0.912173*	*0.001371*	*0.090868*
0.181927	*0.885500*	*0.989874*	*0.883880*	*0.001832*	*0.119918*
0.233188	*0.849247*	*0.979695*	*0.847162*	*0.002462*	*0.156444*
0.295851	*0.802311*	*0.960382*	*0.799643*	*0.003336*	*0.201014*
0.371196	*0.741806*	*0.925261*	*0.738421*	*0.004584*	*0.253027*
0.460012	*0.664323*	*0.864628*	*0.660071*	*0.006442*	*0.309901*
0.562296	*0.566060*	*0.766313*	*0.560784*	*0.009407*	*0.366502*
0.676973	*0.443152*	*0.618256*	*0.436698*	*0.014780*	*0.415751*
0.801744	*0.292281*	*0.413415*	*0.284503*	*0.027340*	*0.453116*
0.933144	*0.111541*	*0.155529*	*0.102307*	*0.090262*	*0.520214*

3. Test Case II – Two-Dimensional Spherical Diffusion Equation in Cylindrical Coordinates

A two-dimensional transient diffusion equation in cylindrical coordinates is now considered. In order to explore the double-grid approach characteristics when applied of irregular domain treatment, a transient heat conduction problem within an axisymmetric spherical shell is solved with the model diffusion equation. Grids of different distortion levels are analyzed. The conservative form of the equation being considered is written within the spherical shell as

$$\rho\, c_p\, \frac{\partial \phi}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{k\,\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{k\,\partial \phi}{\partial z}\right), \qquad R_i < r < R_e, \quad 0 < z < \sqrt{R_e^2 - r^2}, \quad t > 0 \tag{13}$$

with boundary conditions

$$\phi = 0\,, \qquad r = R_i\,, \quad 0 < z < \sqrt{R^2 - r^2}\,, \quad t > 0 \tag{14}$$

$$\phi = 0\,, \qquad r = R_e\,, \quad 0 < z < \sqrt{R^2 - r^2}\,, \quad t > 0 \tag{15}$$

$$\frac{\partial \phi}{\partial z} = 0\,, \qquad R_i < r < R_e\,, \quad z = 0, \quad t > 0 \tag{16}$$

and initial condition

$$\phi = 10\,, \qquad R_i < r < R_e\,, \quad 0 < z < \sqrt{R_e^2 - r^2}\,, \quad t > 0 \tag{17}$$

where R_i and R_e are the internal and external radius of the spherical shell, respectively. For the shown results, the dimensions of the sphere are taken as $R_i = 0.1$ and $R_e = 1$ and the physical properties of Ni Steel (Özi ik, 1993) are considered.

During the numerical grid generation procedure, clustering of points near the domain boundaries and grid distortion are also introduced, in order to control computational costs and further test the double-grid approach. Poisson equations are used for the numerical grid generation procedure. Second-order schemes are used are used to discretize the Poisson equation into algebraic form. The resulting system of algebraic equation is solved by an iterative procedure with 10^{-8} tolerance.

In order to evaluate the efficiency of the proposed approach, numerical results are compared with the analytical solution of the physical problem obtained by classical integral transform technique (Özi ik, 1993). The number of eigenvalues used in the analytical solution is automatically determined by applying a convergence criteria that guarantees 6 significant digits on the final solution.

The cylindrical diffusion model equation – Eq.(16) – is written in conservative and generalized coordinate form as

$$\rho\, c_p\, \frac{\partial \phi}{\partial t} = \frac{1}{J\, r}\left\{\frac{\partial}{\partial \eta}\left[\frac{k\, r}{J}\left(\frac{\partial}{\partial \eta}(\phi\, z_\xi) - \frac{\partial}{\partial \xi}(\phi\, z_\eta)\right) z_\xi\right] - \frac{\partial}{\partial \xi}\left[\frac{k\, r}{J}\left(\frac{\partial}{\partial \eta}(\phi\, z_\xi) - \frac{\partial}{\partial \xi}(\phi\, z_\eta)\right) z_\eta\right] + \right.$$
$$\left. + r\,\frac{\partial}{\partial \xi}\left[\frac{k}{J}\left(\frac{\partial}{\partial \xi}(\phi\, r_\eta) - \frac{\partial}{\partial \eta}(\phi\, r_\xi)\right) r_\eta\right] - r\,\frac{\partial}{\partial \eta}\left[\frac{k}{J}\left(\frac{\partial}{\partial \xi}(\phi\, r_\eta) - \frac{\partial}{\partial \eta}(\phi\, r_\xi)\right) r_\xi\right]\right\}, \quad 0 < \eta < 1, 0 < \xi < 1, t > 0 \tag{18}$$

with boundary conditions

$$\phi = 0\,, \qquad 0 < \eta < 1, \xi = 0, t > 0 \tag{19}$$

$$\phi = 0\,, \qquad 0 < \eta < 1, \xi = 1, t > 0 \tag{20}$$

$$r_\xi\left[\frac{\partial}{\partial \eta}(\phi\, r_\xi) - \frac{\partial}{\partial \xi}(\phi\, r_\eta)\right] + z_\xi\left[\frac{\partial}{\partial \eta}(\phi\, z_\xi) - \frac{\partial}{\partial \xi}(\phi\, z_\eta)\right] = 0\,, \qquad \eta = 0, 0 < \xi < 1, t > 0 \tag{21}$$

$$r_\xi\left[\frac{\partial}{\partial \eta}(\phi\, r_\xi) - \frac{\partial}{\partial \xi}(\phi\, r_\eta)\right] + z_\xi\left[\frac{\partial}{\partial \eta}(\phi\, z_\xi) - \frac{\partial}{\partial \xi}(\phi\, z_\eta)\right] = 0\,, \qquad \eta = 1, 0 < \xi < 1, t > 0 \tag{22}$$

and initial condition

$$\phi = 10\,, \qquad 0 < \eta < 1, 0 < \xi < 1, t > 0 \tag{23}$$

where J is the Jacobian of the transformation.

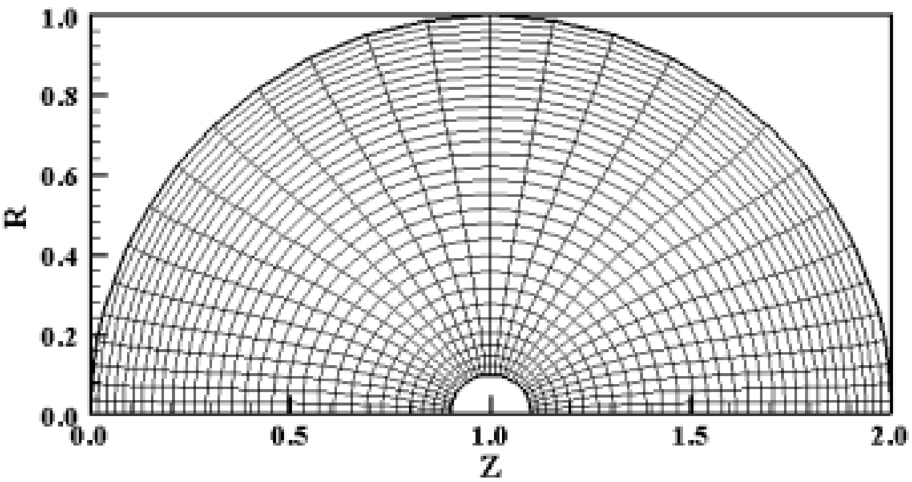

Figure 1. Typical low distortion mesh for a hollow sphere - Test Case II – 30x30 control volumes

Figure (1) illustrates a typical mesh used for the test case, involving *30x30* volumes in the *r* and *z* directions, respectively. It should be noted from Fig.(1) that attraction of grid lines towards the domain boundaries was used during grid generation procedure in order to generate a distorted mesh, where the effects of the double-grid approach are to be analyzed. Figure (2) shows the transient variation of the maximum relative errors for the numerical solutions obtained with the double-grid and coordinate averaging techniques, for meshes with different number of control volumes. The results depicted in Fig. (2) show that the double-grid approach leads to a maximum relative error reduction for all the grids studied. It should also be emphasized that results shown in Fig. (2) indicate slightly smaller relative errors for the double-grid approach with *50x50* control volumes than the ones for *80x80* grids when the coordinate average approach is used. Therefore, for an specified tolerance, results show that the computational costs associated with the solution of the diffusion model equation - Eq. (13) – can be significantly reduced by the use of the double-grid approach.

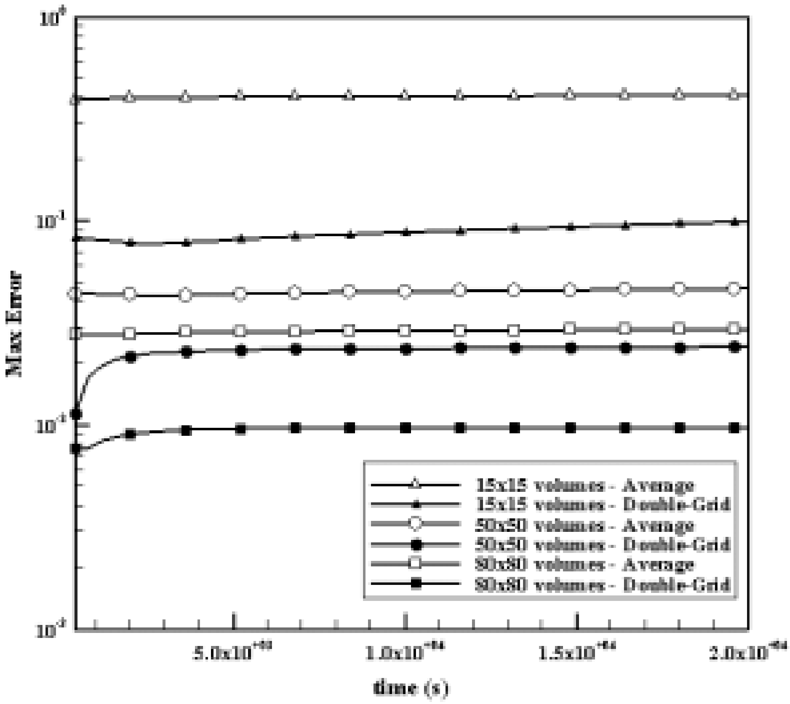

Figure 2. Maximum relative error for different grid. Test Case II - Diffusion Equation in Cylindrical Coordinates.

In order to further test the double-grid approach, the two-dimensional spherical diffusion problem described by Eqs. (13) - (17) is revisited using grids with enhanced distortion. Figure (3) shows a typical high-distortion grid used in the calculations. Results for the maximum relative error are shown in Fig. (4) for different grids. For the high-distortion grid, results shown in Fig. (4) indicate an improvement on the maximum relative error when the double-grid approach is used. Relative errors results also indicate, for the high-distortion grids, that computation costs can be reduced for a specified tolerance. It is interesting to note from Fig. (4) that using *15x15* volume grids and the double-grid approach leads to similar maximum errors as grids with *50x50* volumes when metrics are evaluated using the coordinate average approach. The same equivalence of maximum error is observed for *50x50* double-grid and *80x80* coordinate-average results. Therefore, the use of the double-grid approach for distorted grids can result on substantial reduction of the number of control volumes required to reach an *a priori* established accuracy level. As a result, the computational costs can also be significantly reduced with accurate solutions being computed on coarser grids.

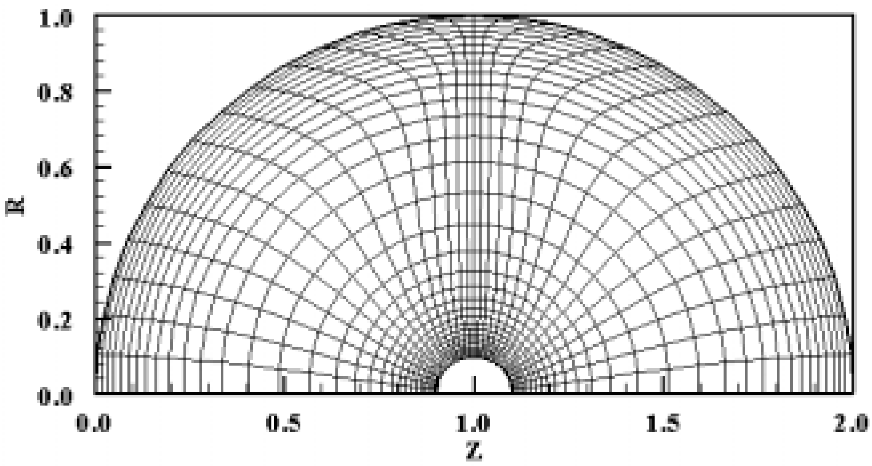

Figure 3. Typical high distortion mesh for a hollow sphere - Test Case II – *30x30* control volumes

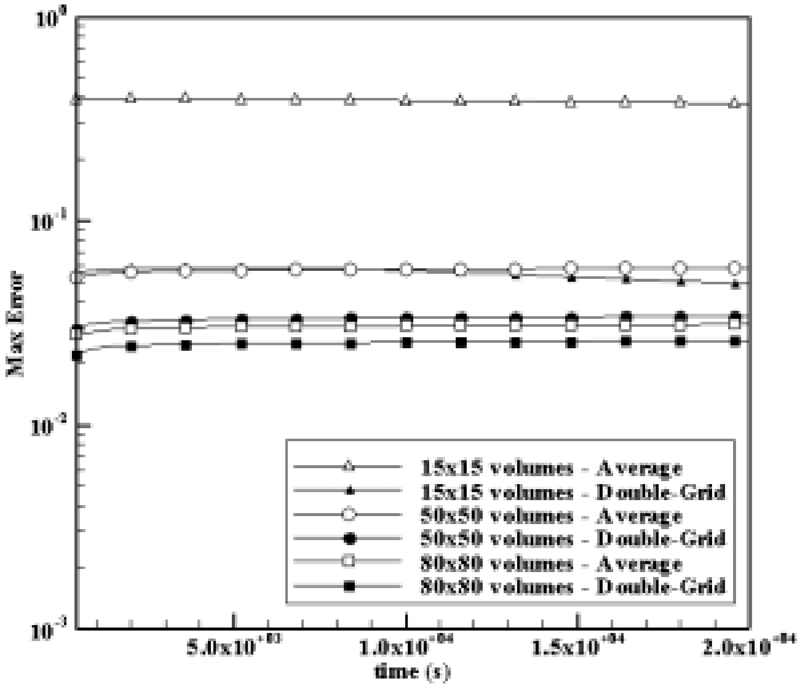

Figure 4. Maximum relative error for different grid. Test Case II - Diffusion Equation in Cylindrical Coordinates.

5. Conclusion

The present work presents a comparison of two different techniques for the computation of transformation metric. The metrics of the transformation, which involves the first derivatives of the independent variables in the physical domain, with respect to the independent variables in the computational domain, appear in the transformed governing equations. Therefore, the metrics play an important role in the accuracy of the method of solution of the governing equations in the computational domain. The two methods for the computation of the metrics examined in the present work are the use of the double-grid and of averaging of node coordinates.

Generally, the use of the double-grid approach resulted on more accurate solutions for meshes with fixed number of volumes. Alternatively, with the use of the double-grid approach, solutions of accuracy similar to those of the averaging technique could be obtained on a much coarser grid, resulting on substantial reduction of computational costs. Since the discretization errors present on the partial derivative representations are similar for both metrics evaluation approaches being considered, improvements on the obtained solutions can be associated to the double-grid approach.

5. Acknowledgment

The authors would like to acknowledge the financial support provided by the CNPq , CAPES & FAPERJ. Computer resources where allocated by the *Thermal Engine Laboratory* of the COPPE/UFRJ and by the *Aerodynamics and Thermosciences Laboratory* of IME.

6. References

Anderson, D.A., Tannehill, J.C., Pletcher, R.H., 1984,“Computational Fluid Mechanics and Heat Transfer”, McGraw Hill, New York.

Calhoun D., LeVeque R. J., 2000, “A Cartesian Grid Finite-Volume Method for Advection-Difusion Equation in Irregular Geometries”, J. Comput. Phys., Vol. 157, pp. 143-180.

Gnoffo, P. A., 1982, “A Vectorized, Finite-Volume, Adaptive-Grid Algorithm for Navier-Stokes Calculations”, Numerical Grid Generation, pp. 819-835.

Hermeline F., 2000, “A Finite Volume Method for the Approximation of Diffusion Operators on Distorted Meshes” ”, J. Comput. Phys., Vol. 160, pp. 481-499

Margolin, L. and Shashkov, M., 1999, “Using a Curvilinear Grid to Construct Symmetry-Preserving Discretizations for Lagrangian Gas Dynamics”, J. Comput. Phys., Vol. 149, pp. 389-417.

Mastin, C.W., 1982, “Error Induced by Coordinate Systems”, Numerical Grid Generation, pp. 31-40.

Özi ik, M.N., “Heat Conduction”, John Wiley, New York.

Patankar, S. V., 1980, “Numerical Heat Transfer and Fluid Flow”, Hemisphere Publishing, New York.

Perot B., 2000, “Conservation Properties of Unstructured Staggered Mesh Schemes” ”, J. Comput. Phys., Vol. 159, pp. 58-59.

Thompson, J. F., Warsi Z. U. A. and Mastin, C., 1985, “Numerical Grid Generation”, Elsevier Science Publishing Co., Inc., 483 p.

Thompson, J. F., 1982, “General Curvilinear Coordinate Systems”, Numerical Grid Generation, pp. 1-29.

Versteeg, H. K. and Malalasekera, W., 1995, “An Introduction to Computational Fluid Dynamics”, Prentice Hall, New York.

Yamaleev, N. K., 2001, “Minimization of the Truncation Error by Grid Adaptation”, J. Comput. Phys., Vol. 170, pp. 459-497.