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HYBRID SOLUTIONS THROUGH THE GENERALIZED INTEGRAL TRANSFORM TECHNIQUE FOR HYDRODYNAMICALLY DEVELOPING NON-NEWTONIAN FLOWS IN CIRCULAR TUBES

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Abstract. *The Generalized Integral Transform Technique (GITT) is employed in the solution of the momentum equations in hydrodynamically developing laminar flow of non-Newtonian power-law fluids inside circular ducts. Results for the velocity field and friction factor-Reynolds number product are computed for different power-law indices, which are tabulated and graphically presented as functions of the dimensionless coordinates. Critical comparisons with previous results in the literature are also performed, in order to validate the numerical codes developed in the present work and to demonstrate the consistency of the final results.*

Keywords*. generalized integral transform technique, power-law model, hydrodynamically developing laminar flow.*

1. Introduction

The analysis of the hydrodynamically developing laminar flow inside ducts of cylindrical geometry has been subject of great interest as demonstrated by the available literature, mainly to the case of flows inside a concentric annular region which is more represented by concentric circular ducts. Therefore, a correct prevision related to the heat transfer between the channel wall and the fluid studied is extremely important in equipment design and thermal devices in general. Heat transfer to purely viscous non-Newtonian fluids is frequently encountered in various industries (e.g., chemical, petrochemical and food processing). These fluids are commonly processed under laminar flow conditions because of their high apparent viscosities and also the small hydraulic diameters employed in compact heat exchangers. An important feature of most purely viscous non-Newtonian fluids is that some of their rheological and thermophysical properties are very sensitive to temperature. This variation can have a large effect on the development of the velocity and temperature profiles, consequently on the pressure drop and heat transfer rates.

A brief literature survey indicates that Lin and Shah (1978) have studied the heat transfer problem of power-law fluids with yield stress, flowing in the entrance region of a circular duct and parallel plates channel, they used a forward marching procedure to solve the related momentum and energy equations. Cuccurullo and Berardi (1998) investigated the simultaneously developing of velocity and temperature profiles in the entrance pipe flow. The flow was assumed to be steady-state for a non-Newtonian fluid in incompressible laminar pipe flow. The fluid behavior was assumed to follow the Ostwald-de Waele power-law model. The developing velocity and temperature profiles were solved by the integral method. Results were presented and discussed in terms of axial and radial velocity profiles, Fanning friction factors and Nusselt numbers for different fluid properties and thermal boundary conditions.

In this context, the present study is motivated to employ the Generalized Integral Transform Technique (GITT) in the solution of the momentum equations for non-Newtonian power-law fluids flowing in the entrance region of circular tubes, and for this purpose, the boundary-layer formulation in terms of primitive variables is adopted. Numerical results for the velocity field and local Fanning friction factor are obtained considering the effect of the power-law index, and the results for velocity profile are compared with those reported in the literature. The GITT approach is an extension of the Classical Integral Transform Technique, which is based on eigenfunction expansions yielding to solutions where the most features are the automatic and straightforward global error control and an only mild cost increase in overall computational effort for multidimensional situations. The Most recent contributions are aimed at the accurate solution of non-linear heat and fluid flow problems, which include problems with variable properties, moving boundaries, irregular geometries, non-linear source terms, non-linear boundary conditions, Navier-Stokes equations and boundary layer equations (Machado and Cotta, 1991). In recent works, Quaresma (1997), Magno (1998), Nascimento (2000) and Chaves (2001) have been carried out important studies related to the GITT application in different kind of problems. A detailed compilation dealing with the advances of this technique on diffusion-convection problems can be find out in Cotta (1993, 1998) and Cotta and Mikhailov (1997).

2. Analysis

Within the range of validity of the boundary layer hypothesis, the continuity and momentum equations in the primitive-variables formulation for this problem are written in dimensionless form, respectively as:

$$\frac{\partial U}{\partial Z}+\frac{1}{R}\frac{\partial}{\partial R}\left(RV\right)=0\,;\quad 0<R<1\,;\quad Z>0 \tag{1}$$

$$U\frac{\partial U}{\partial Z}+V\frac{\partial U}{\partial R}=-\frac{\partial P}{\partial Z}+\frac{1}{Re}\frac{1}{R}\frac{\partial}{\partial R}\left(\eta\,R\frac{\partial U}{\partial R}\right);\quad 0<R<1\,;\quad Z>0 \tag{2}$$

where

$$\eta=\left[\left(\frac{\partial U}{\partial R}\right)^{2}\right]^{\frac{n-1}{2}} \tag{3}$$

with the following inlet and boundary conditions:

$$Z=0:\quad U(R,Z)=1\,;\quad V(R,Z)=0\,; \tag{4a,b}$$

$$R=0:\quad \frac{\partial U(R,Z)}{\partial R}=0\,;\quad V(R,Z)=0\,; \tag{4c,d}$$

$$R=1:\quad U(R,Z)=0\,;\quad V(R,Z)=0\,; \tag{4e,f}$$

The dimensionless groups employed in equations above are defined as:

$$R=\frac{r}{r_w}\,;\quad U=\frac{u}{u_0}\,;\quad V=\frac{v}{u_0}\,;\quad Z=\frac{z}{r_w}\,;\quad P=\frac{p}{\rho u_0^2}\,;\quad Re=\frac{\rho u_0^{2-n} r_w^n}{K} \tag{5a,f}$$

For improving the computational performance in the solution of the velocity field, with respect to the direct procedure (Cotta and Carvalho, 1991), the fully developed flow situation is separated from the complete potential, in the form:

$$U(R,Z)=U_{fd}(R)+U_F(R,Z) \tag{6}$$

$$U_{fd}(R)=\frac{3n+1}{n+1}\left(1-R^{\frac{n+1}{n}}\right) \tag{7}$$

This is a commonly used strategy in the integral transform approach (Cotta and Serfaty, 1991; Cotta, 1993), equivalent to the separation of the steady-state solution in a transient problem, which acts by filtering the equation source terms responsible for the slower convergence rates in non-homogeneous problems. Then, after the substitution of the splitting-up scheme, Eq. (6), the problem formulation is rewritten as:

$$\frac{\partial U_F}{\partial Z}+\frac{1}{R}\frac{\partial}{\partial R}\left(RV\right)=0 \tag{8}$$

$$\left(U_F+U_{fd}\right)\frac{\partial U_F}{\partial Z}+V\left[\frac{\partial U_F}{\partial R}-\left(\frac{3n+1}{n}R^{\frac{1}{n}}\right)\right]=-\frac{\partial P}{\partial Z}+\frac{1}{Re}\frac{1}{R}\frac{\partial}{\partial R}\left[\eta\,R\left(\frac{\partial U_F}{\partial R}+\frac{dU_{fd}}{dR}\right)\right] \tag{9}$$

where

$$\eta=\left\{\left[\frac{\partial U_F}{\partial R}-\left(\frac{3n+1}{n}\right)R^{\frac{1}{n}}\right]^{2}\right\}^{\frac{n-1}{2}} \tag{10}$$

and the inlet and boundary conditions become:

$$Z = 0: \quad U_F(R,Z) = 1 - U_{fd}(R)\ ; \quad V(R,Z) = 0 \tag{11.a,b}$$

$$R = 0: \quad \frac{\partial U_F(R,Z)}{\partial R} = 0\ ; \quad V(R,Z) = 0 \tag{11.c,d}$$

$$R = 1: \quad U_F(R,Z) = 0\ ; \quad V(R,Z) = 0 \tag{11.e,f}$$

The next step in the solution of Eqs. (8) to (10) is the elimination of the transversal velocity component, V(R,Z), and the pressure gradient, ($-\partial P/\partial Z$). First, the continuity equation (8) is integrated, to yield:

$$V(R,Z) = \frac{1}{R}\int_R^1 \xi \frac{\partial U_F}{\partial Z}\, d\xi \tag{12}$$

while the momentum equation is integrated over the channel cross-section to provide an expression for the pressure gradient:

$$-\frac{\partial P}{\partial Z} = 4\int_0^1 R\left(U_{fd} + U_F\right)\frac{\partial U_F}{\partial Z}\, dR - \frac{2}{Re}\left[\eta\left(\frac{dU_{fd}}{dR} + \frac{\partial U_F}{\partial R}\right)\right]\Bigg|_{R=1} \tag{13}$$

Equations (12) and (13) relate the transversal velocity and pressure gradient to the longitudinal velocity field, as required for completion of the integral transformation process. Following the formalism in the generalized integral transform technique, to construct the eingenfunctions expansions, an auxiliary eigenvalue problem is selected as:

$$\frac{d}{dR}\left(R\frac{d\psi_i(R)}{dR}\right) + \mu_i^2 R\psi_i(R) = 0\ ; \quad \text{em } 0 < R < 1 \tag{14a}$$

$$\left.\frac{d\psi_i(R)}{dR}\right|_{R=0} = 0\ ; \qquad \psi_i(R)\big|_{R=1} = 0 \tag{14b,c}$$

The eigenfunctions and the transcendental expression to calculate the eigenvalues are given, respectively, by:

$$\psi_i(R) = J_0(\mu_i R)\ ; \quad J_0(\mu_i) = 0 \tag{14d,e}$$

The eigenfunctions of this eigenvalue problem enjoys the following orthogonality property:

$$\int_0^1 R\tilde{\psi}_i(R)\tilde{\psi}_j(R)dR = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{14f}$$

and the normalization integral is defined as:

$$N_i = \int_0^1 R\psi_i^{\,2}(R)\, dR = \frac{1}{2}J_1^{\,2}(\mu_i) \tag{14g}$$

The problem given by Eqs. (14) allows the definition of the following integral-transform pair:

$$\overline{U}_i(Z) = \int_0^1 R\tilde{\psi}_i(R)U_F(R,Z)dR\ , \qquad \text{transform} \tag{15}$$

$$U_F(R,Z) = \sum_{i=1}^{\infty} \tilde{\psi}_i(R)\overline{U}_i(Z)\ , \qquad \text{inversion} \tag{16}$$

In terms of the transformed potentials defined by Eq. (15), the transversal velocity component and pressure gradient are rewritten as:

$$V(R,Z)=\sum_{i=1}^{\infty}\frac{A_i(R)}{R}\frac{d\overline{U}_i(Z)}{dZ} \tag{17}$$

$$-\frac{\partial P}{\partial Z}=4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_{ij}\overline{U}_j\frac{d\overline{U}_i}{dZ}+2\sum_{i=1}^{\infty}\left[B_{fdi}-\left(\frac{3n+1}{n}\right)C_i\right]\frac{d\overline{U}_i}{dZ}-\frac{2}{Re}\left\{\eta(1,Z)\left[\left(\sum_{i=1}^{\infty}\tilde{\psi}'_i(1)\overline{U}_i\right)-\left(\frac{3n+1}{n}\right)\right]\right\} \tag{18}$$

where

$$A_i(R)=\int_R^1\xi\,\tilde{\psi}_i(R)\,d\xi=\frac{1}{\mu_i N_i^{1/2}}\left[J_1(\mu_i)-R\,J_1(\mu_i R)\right] \tag{19a}$$

$$B_{fdi}=\int_0^1 R U_{fd}(R)\tilde{\psi}_i(R)\,dR \tag{19b}$$

$$C_i=\int_0^1 R^{\frac{1}{n}}A_i(R)\,dR \tag{19c}$$

$$\eta(1,Z)=\left\{\left[\left(\sum_{i=1}^{\infty}\tilde{\psi}'_i(1)\overline{U}_i(Z)\right)-\left(\frac{3n+1}{n}\right)\right]^2\right\}^{\frac{n-1}{2}} \tag{19d}$$

Equation (9) is now integral transformed through the operator $\int_0^1\psi_i(R)/N_i^{1/2}\,dR$, to yield the transformed ordinary differential equations:

$$\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{\infty}\left[D_{ijk}+E_{ijk}-4A_i(0)\delta_{jk}\right]\overline{U}_j+F_{fdik}-\left(\frac{3n+1}{n}\right)G_{ik}-2A_i(0)\left[B_{fdk}-\left(\frac{3n+1}{n}\right)C_k\right]\right\}\frac{d\overline{U}_k}{dZ}=H_i \tag{20}$$

The inlet condition, Eq. (11a), is similarly integral transformed to provide:

$$\overline{U}_i(0)=\bar{f}_i=\int_0^1 R\tilde{\psi}_i(R)\left[1-U_{fd}(R)\right]dR \tag{21}$$

where the various coefficients in Eq. (20) are given by:

$$D_{ijk}=\int_0^1 R\tilde{\psi}_i(R)\tilde{\psi}_j(R)\tilde{\psi}_k(R)\,dR \tag{22a}$$

$$E_{ijk}=\int_0^1\tilde{\psi}_i(R)\tilde{\psi}'_j(R)A_k(R)\,dR \tag{22b}$$

$$F_{fdik}=\int_0^1 R U_{fd}(R)\tilde{\psi}_i(R)\tilde{\psi}_k(R)\,dR \tag{22c}$$

$$G_{ik}=\int_0^1 R^{\frac{1}{n}}\tilde{\psi}_i(R)A_k(R)\,dR \tag{22d}$$

$$H_i=-\frac{2}{Re}\left\{\eta(1,Z)\left[\left(\sum_{k=1}^{\infty}\tilde{\psi}'_k(1)\overline{U}_k\right)-\left(\frac{3n+1}{n}\right)\right]\right\}A_i(0)-\frac{1}{Re}\int_0^1 R\tilde{\psi}'_i(R)\eta\left[\left(\sum_{k=1}^{\infty}\tilde{\psi}'_k(R)\overline{U}_k\right)-\left(\frac{3n+1}{n}\right)R^{\frac{1}{n}}\right]dR \tag{22e}$$

The inversion formula, Eq. (15), and Eq. (6) are recalled to construct the original potential for the longitudinal velocity component, in the form:

$$U(R,Z) = U_{fd}(R) + \sum_{i=1}^{NC} \tilde{\psi}_i(R)\overline{U}_i(Z) \quad (23)$$

where NC is the truncation order for the velocity eigenfunction expansion.

Quantities of practical interest can be analytically evaluated from their usual definitions, such as the local Fanning friction factor:

$$f = -\frac{2}{Re}\left[\left[\left(\frac{\partial U}{\partial R}\right)^2\right]^{\frac{n-1}{2}} \frac{\partial U}{\partial R}\right]_{R=1} \quad ; \quad \left.\frac{\partial U}{\partial R}\right|_{R=1} = \left(\sum_{i=1}^{\infty} \tilde{\psi}_i'(1)\overline{U}_i(Z)\right) - \left(\frac{3n+1}{n}\right) \quad (24,25)$$

3. Results and discussion

Numerical results for the velocity profiles and Fanning friction factors were produced for different values of power-law indices, namely n = = 0.5; 0.75; 1.0; 1.25 and 1.5, at entrance region of a circular tube. The computational code was developed in FORTRAN 90/95 programming language and implemented on a PENTIUM-IV 1.3 GHz computer.

First, the numerical code was validated for the case n = 1.0 (newtonian situation) against those results presented by Hornbeck (1964) and Liu (1974), which employed the Finite Difference Method for the solution of the same problem. The routine DIVPAG from IMSL Library (1991) was used to numerically handled the truncated version of the system of ordinary differential equation (20), with a relative error target of 10^{-8} prescribed by the user, for the transformed potentials. These results were produced for Re = 2000, but it should be noted that the dimensionless axial coordinate X^+ makes the results independent of the apparent Reynolds number. The definition of X^+ is written as:

$$X^+ = \frac{Z}{Re} \quad (26)$$

Table (1) shows the convergence behavior of the longitudinal velocity component at the centerline of the circular tube, as well as its comparison with the results presented Hornbeck (1964) and Liu (1974) demonstrating a good agreement, which provides a direct validation of the numerical code here developed. This same analysis is also shown in Fig. (1), where it is observed a monotonic convergence for the longitudinal velocity component at the centerline of the circular tube, U_c.

Table 1. Convergence behavior of the longitudinal velocity component at the centerline of the circular tube for power-law index n = 1.0.

X^+	NC = 20	NC = 40	NC = 60	NC = 80	Hornbeck (1964)	Liu (1974)
0.0002116	1.015	1.033	1.067	1.113	-	1.100
0.0005000	1.096	1.111	1.127	1.145	1.150	-
0.001058	1.169	1.183	1.192	1.200	-	1.210
0.001250	1.187	1.201	1.209	1.215	1.227	-
0.005000	1.393	1.408	1.414	1.415	1.433	-
0.005288	1.405	1.420	1.429	1.427	-	1.439
0.01204	1.607	1.625	1.631	1.631	-	1.644
0.01250	1.618	1.636	1.642	1.642	1.660	-
0.04924	1.906	1.933	1.945	1.946	-	1.971
0.05000	1.907	1.935	1.946	1.947	1.970	-
0.06250	1.923	1.949	1.960	1.961	1.986	-
0.06281	1.923	1.949	1.960	1.961	-	1.989
0.07634	1.933	1.957	1.968	1.969	-	1.996
0.08993	1.941	1.962	1.972	1.972	-	1.999
1.0	2.000	2.000	2.000	2.000	-	-
5.0	2.000	2.000	2.000	2.000	-	-

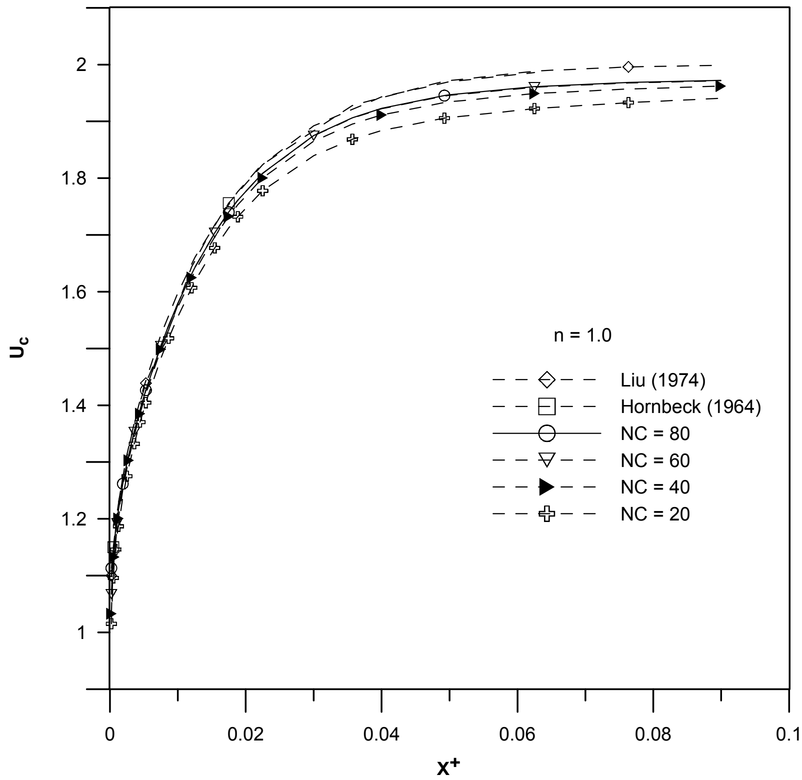

Figure 1. Convergence behavior of the U_c component velocity for power-law index n = 1.0.

Figures (2) and (3) bring a convergence behavior of the U_c component velocity for the cases of power-law indices n = 0.5 and 1.25, and the same observations are verified as for the case of n = 1.0, i.e., a monotonic convergence for this component velocity.

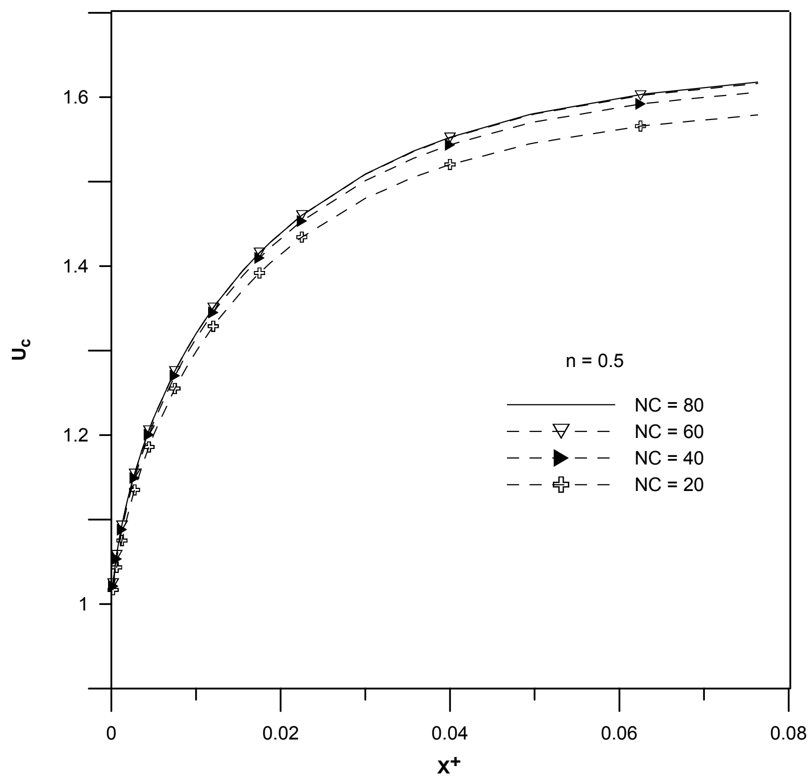

Figure 2. Convergence behavior of the U_c component velocity for power-law index n = 0.5.

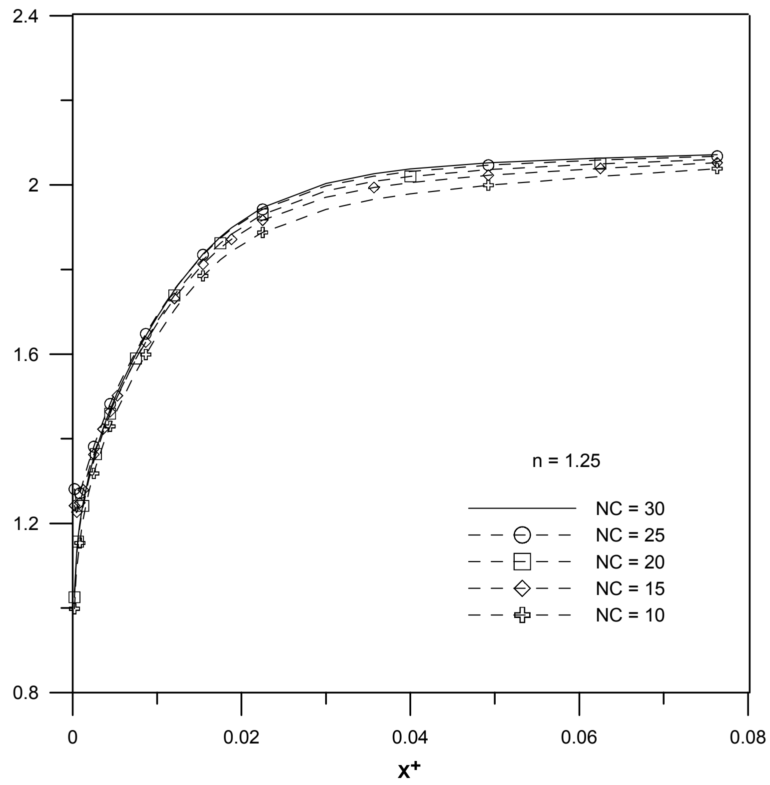

Figure 3. Convergence behavior of the U_c component velocity for power-law index n = 1.25.

Figure (4) illustrates the development of the longitudinal velocity profiles for different power-law indices, namely, n = 0.5; 0.75; 1.0; 1.25 and 1.5, as function of the transversal coordinate R, at specific axial positions X^+. From this figure it can be noticed that when the power-law index increases there is an increase in the value of the centerline velocity. In regions near the tube wall, it is verified that the velocity gradient diminishes as n increases, this is due to an increase of the apparent fluid viscosity, and consequently an increase of the wall stress. For practical engineering considerations, this effect leads to an undesirable increase of the pumping power to promote the flow of this type of fluids inside circular tubes.

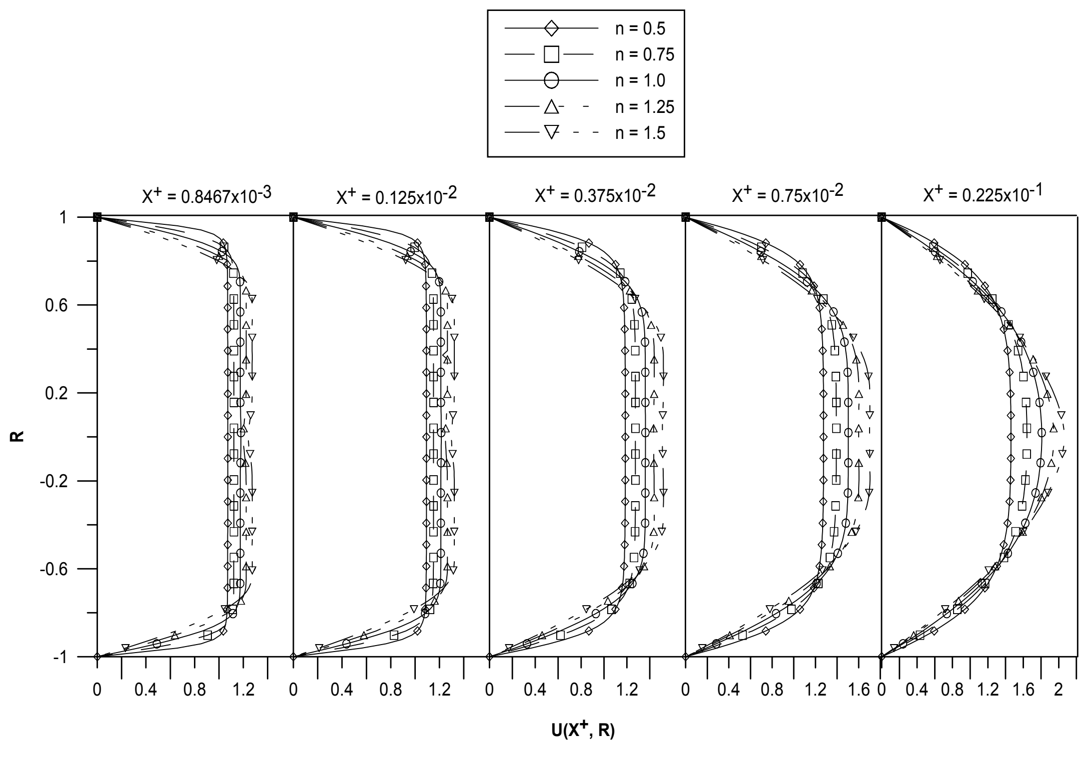

Figure 4. Development of the longitudinal velocity component along the entrance region of the circular tube for different power-law indices.

Results for the product Fanning friction factor-apparent Reynolds number are shown in Fig. (5) as function of the axial positions X^+, for n = 0.5; 0.75; 1.0; 1.25 and 1,5. It can be observed that the product fRe diminishes until to be reached the fully developed region, in which this parameter assumes a constant value. For higher power-law indices, fRe increases, and this fact can be explained by an increase of the apparent fluid viscosity in regions near to the tube wall as n increases. Also, it is noted that the product fRe presents higher values in the entrance region of the circular tube due to higher velocity gradients experimented by these fluids in this region.

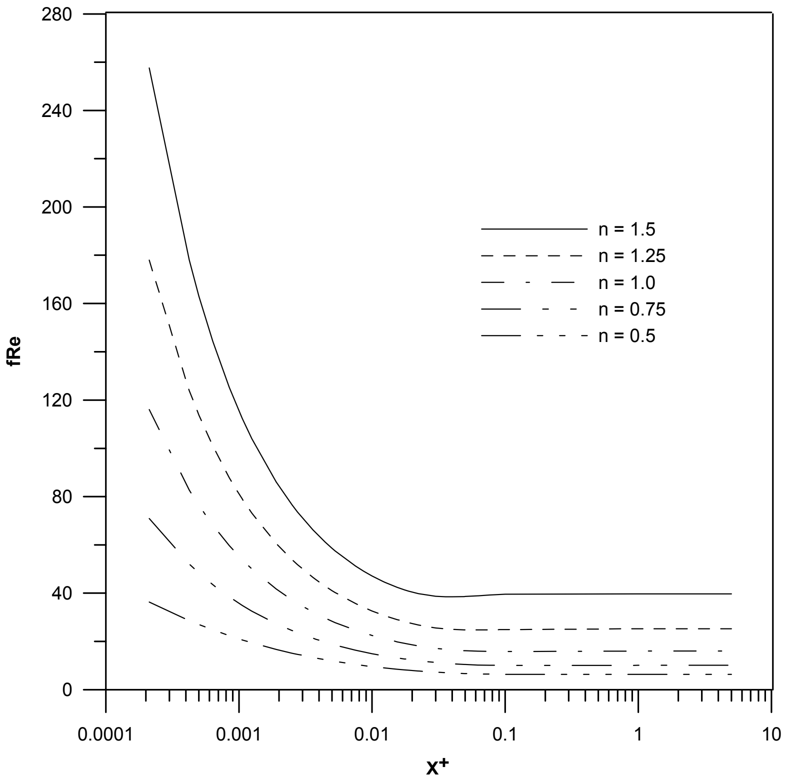

Figure 5. Development of the product Fanning friction factor-apparent Reynolds number in the entrance region for different power-law indices.

Finally, Tab. (2) shows a comparison of the present results for the product fRe in the fully developed region against those of Quaresma and Macêdo (1998). It is verified an excellent agreement between the two sets of results, once again validating the numerical codes developed here.

Table 2. Comparison of the product fRe in the fully developed region for different power-law indices.

n	fRe	
	Present work	Quaresma and Macêdo (1998)
0.5	6.3246	6.32455
0.75	10.102	10.1023
1.0	16.000	16.0000
1.25	25.238	-
1.50	39.718	39.7175

4. Conclusions

Numerical results for the velocity field and product Fanning friction factor-apparent Reynolds number were produced by using the GITT approach in the solution of the momentum equations for the flow of non-Newtonian power-law fluids in circular tubes. Results for velocity profiles indicate that an increase of the power-law index promotes an increase of the centerline velocity in order to obey the mass conservation principle, this way demonstrating the strong influence of the viscous effects on the characteristics of the fluid flow. It was also observed that the product fRe is higher for the cases of dilatant fluids (n > 1) than for those of pseudoplastic ones (n < 1) due to an increase of the apparent fluid viscosity in regions near to the tube wall.

5. References

Chaves, C.L., 2001, "Integral Transformation of the Momentum and Energy Equations in the Flow of Non-Newtonian Fluids in Irregular Ducts", M.Sc. Thesis (in Portuguese), Chemical Engineering Department, Universidade Federal do Pará, Belém, Brazil.

Cotta, R.M. and Serfaty, R., 1991, "Integral Transform Algorithm for Parabolic Diffusion Problems with Nonlinear Boundary Layer and Equation Source Terms", Proceedings of the 7th Int. Conf. Num. Meth. For Thermal Problems, Stanford, Vol. II, pp. 916-926.

Cotta, R.M. and Carvalho, T.M.B., 1991, "Hybrid Analysis of Boundary Layer Equations for Internal Flow Problems", Proceedings of the 7th Int. Conf. Num. Meth. in Laminar and Turbulent Flow, Stanford, Vol. I, pp. 106-115.

Cotta, R.M., 1993, "Integral Transforms in Computational Heat and Fluid Flow", CRC Press, Boca Raton, USA.

Cotta, R.M. and Mikhailov, M.D., 1997, "Heat Conduction: - Lumped Analysis, Integral Transforms, Symbolic Computation", John Wiley, New York, USA.

Cotta, R.M., 1998, "The Integral Transforms Method in Thermal and Fluids Science and Engineering", Begell House, New York, USA.

Cuccurullo, G and Berardi, P.G., 1998, "Developing of Velocity and Temperature in Entrance Pipe Flow for Power Law Fluids", Proceedings of the11th International Heat Transfer Conference, Vol. 3, pp. 15-20.

Hornbeck, R.W., 1964, "Laminar Flow in the Entrance Region of a Pipe", Applied Scientific Research, Section A, Vol. 13, pp. 224-232.

IMSL Library, 1991, MATH/LIB, Houston, TX.

Lin, T. and Shah, V.L., 1978, "Numerical Solution of Heat Transfer to Yield Power Law Fluids Flowing in the Entrance Region", Proceedings of the 6th International Heat Transfer Conference, Vol. 5, pp. 317-322.

Liu, J., 1974, "Flow of a Bingham Fluid in the Entrance Region of an Annular Tube", M. S. Thesis, University of Wisconsin – Milwaukee, USA.

Machado, H.A. and Cotta, R.M., 1995, "Integral Transform Method for Boundary Layer Equations in Simultaneous Heat and Fluid Flow Problems", International Journal of Numerical Methods for Heat and Fluid Flow, Vol. 5, pp. 225-237.

Magno, R.N.O., 1998, "Solutions for the Boundary Layer Equations for a Non-Newtonian Fluid via Generalized Integral Transform Technique", M.Sc. Thesis (in Portuguese), Chemical Engineering Department, Universidade Federal do Pará, Belém, Brazil.

Nascimento, U.C.S., 2000, "Study of the Thermal Entry Region in the Laminar Flow of Bingham Plastics in Concentric Annular Ducts", M.Sc. Thesis (in Portuguese), Chemical Engineering Department, Universidade Federal do Pará, Belém, Brazil.

Quaresma, J.N.N., 1997, "Integral Transformation of the Navier-Stokes Equations in Three-Dimensional Laminar Flows", D.Sc. Thesis (in Portuguese), Mechanical Engineering Department, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.

Quaresma, J.N.N. and Macêdo, E.N., 1998, "Integral Transform Solution for the Forced Convection of Herschel-Bulkley Fluids in Circular Tubes and Parallel-Plates Ducts", Brazilian Journal of Chemical Engineering, Vol. 15, pp. 77-89.