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On Kaplun Limits and the Turbulent Boundary Layer Near the Leading Edge of a Flat Plate

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Abstract. *In the present work, some formal properties of singular perturbation equations are studied through the concept of "equivalent in the limit" of Kaplun, so that a proposition for the principal equations is derived. The proposition shows that if there is a principal equation at a point $(\eta, 1)$ of the $(\Xi \times \Sigma)$ product space, Ξ space of all positive continuous functions in $(0,1], \Sigma = (0,1]$, then there is also a principal equation at a point (η, ϵ) of $(\Xi, \times\Sigma)$, ϵ= first critical order. The converse is also true. The proposition is of great implication for it ensures that the asymptotic structure of a singular perturbation problem can be determined by a first order analysis of the formal domains of validity. The turbulent boundary layer asymptotic structure is then studied by application of Kaplun limits to the near region of the leading edge of a flat plate. As it turns out, a different asymptotic structures is found from those previously deduced by other authors; in fact the results show that a multi-layered structure exists near the leading edge which, however, is different from the classical structure commonly found in literature.*

Keywords: *Kaplun limits, asymptotic methods, leading edge.*

1. Introduction

In physics and mathematics many phenomena are modelled through intricate equations that present no analytical solution. That has forced engineers, physicists and applied mathematicians to develop techniques that yield approximated solutions to the problems they are faced with. These techniques often resort to sophisticated procedures which only in a very few cases are restrained to fully analytical frameworks. In many situations, analytical solutions have to be combined with numerical procedures to produce an approximate solution.

The Navier-Stokes equations are widely known for provinding formidable challenges to researchers seeking any form of closed analytical solutions. This is, primarily, a consequence of their non-linearity. Thus, as paradoxically as this may look, many methods have been developed in the past to extract as many qualitative information as can be possible before any solution strategy is attempted. We will discuss here one of these methods, the single limit process of Kaplun(1967).

The purpose of this work is, therefore, twofold: i) to consider more thoroughly some fundamental concepts and ideas used in solving perturbation problems, and ii) to study the turbulent boundary layer asymptotic structure near the leading edge of a flat plate by applying Kaplun limits to the Navier-Stokes equations.

Perturbation methods have evolved along the past forty years into a powerful tool for solving a large class of complex problems. They have, therefore, become a basic working tool of many engineers and applied mathematicians. In fact, a large number of papers can be found in literature which use perturbation methods as their primary solution procedure.

While some precise definitions can be enunciated, and exact results obtained to find uniform approximations and to perform the matching of functions, the determination of the domain of validity of an approximation is always difficult. Two important results in perturbation theory are the intermediate matching lemma and the extension theorem of Kaplun. These results are of fundamental importance for the construction of matched asymptotic expansions, but say nothing about the domain of validity of the approximations. To circumvent this difficulty, Kaplun(1967) applied the concept of limit-processes directly to the equations rather than to the solutions and enunciated an Ansatz, the Ansatz about domains of validity, which relates the domain of validity of solutions with the formal domain of validity of equations (a concept which is easily defined). Examples are known where Kaplun's ideas fail; however, for some difficult problems, e.g., the Stokes paradox of fluid mechanics, only consideration of these ideas can clarify the conceptual structure of the problem. Here, we study some formal properties of equations yielded by the definition of "equivalent in the limit" of Kaplun, and relate them to the actual problems of determining the overlap domain and of matching asymptotic expansions. The concept of "richer than" of Kaplun and Lagerstrom(1957) is given a more elaborated interpretation which leads to the derivation of a theorem for the principal equations. The theorem shows that if there is a principal equation at a point $(\eta, 1)$ of the $(\Xi \times \Sigma)$ product space, Ξ = space of all positive continuous functions on $(0, 1]$, $\Sigma = (0, 1]$, then there is also a principal equation at a point (η, ϵ) of $(\Xi \times \Sigma)$, ϵ = first critical order. The converse is also true. The consequence of this theorem is that, no matter to what order of magnitude we want an approximation to be accurate, it is always possible to find high-order solutions at points $(\eta, 1)$ of the $(\Xi \times \Sigma)$ space (η = point of the Ξ space obtained through passage of Kaplun's limit process, where a principal equation is located) which satisfy the required degree of accuracy, overlap and cover the entire domain.

In an attempt to make clearer Kaplun's ideas, Lagerstrom and Casten(1972) published a work where a survey of some ideas on perturbation methods was presented. Again, a heuristic approach was used. The work, however, presented some new definitions and results which were known to work for leading-order approximate solutions. Some of these results have recently been revisited in publications by Lagerstrom(1988) and by Silva Freire and Hirata(1990).

The formal properties of equations here studied are aimed at boundary layer problems. The theorem of the principal equations formalises the notion of distinguished limit so often used in literature, allowing Kaplun's ideas to be used in a systematic manner. The matched asymptotic expansions method, for example, depends on two crucial guesses for the determination of approximate solutions: the choice of the stretching function and the choice of the asymptotic expansions. These choices are normally guided by physical arguments, but are in the end always made by trial-and-error. In fact, the determination of the stretching function and of the asymptotic expansions has always been seen as an art. With the theorem of the principal equations, the stretching function can be immediately found, whereas the appropriate gauge functions for the asymptotic expansions can be obtained from Kaplun's concept of critical orders.

The asymptotic structure of the turbulent boundary layer has been extensively investigated by a number of authors in the past twenty years. Unlikely the laminar flow case, whose solution has been known since the sixties, the turbulent problem poses some questions which still have to be understood and answered. Of course, all difficulties stem from the introduction of the time-averaged equations. To make these equations a determined system, closure conditions must be introduced to relate the Reynolds stresses to the mean flow velocities. The Reynolds stresses, the time averages of the fluctuating velocities, describe the effect of turbulent fluctuations on the mean flow; if they could be determined, the mean flow equations could solved and the asymptotic structure unveiled. Many closure conditions have been proposed in literature but, unfortunately, none of them are generally valid.

Using only the hypothesis that the order of magnitude of the Reynolds stresses do not change throughout the boundary layer, some authors (Yajnik(1970), Mellor(1972)) have found the turbulent boundary layer to have a two-deck structure consisting of a wall region and a defect region. Other authors using closure conditions in terms of eddy-viscosity (Bush and Fendell(1972)) or $\kappa - \epsilon$ (Deriat and Guiraut(1986)) models have reached the same conclusion, making the two-deck asymptotic structure of the turbulent boundary layer the basis of most subsequent work.

Recently, however, there has been a claim that the turbulent boundary has instead a three-layered structure (Long and Chen(1981), Sychev and Sychev(1987), Melnik(1989)) and that this is the only structure that can possibly handle flows subject to pressure gradients.

In this work, the asymptotic structure of the turbulent boundary layer is investigated by applying Kaplun limits directly to the Navier-Stokes equation. As mentioned before, we will be specially concerned in studying the asymptotic structure of the turbulent boundary layer near the leading edge of a flat plate.

2. The Fundamentals of the Theory

We shall consider perturbation methods to find approximate solutions to differential equations of the form

$$\epsilon\, E_1(x, y, ..., y^{(n)}) + E_2(x, y, ..., y^{(n-1)}, \epsilon) + ... = 0, \qquad (1)$$

that is, equations where the small parameter ϵ multiplies the highest derivative term. E_i is a given function of the variables $x, y, ..., y^{(n)}, \epsilon$. Here $y^{(n)}$ is used to denote $d^n y/dx^n$.

The method to be studied here aims at developing a procedure to find approximate solutions to equations with form of Eq.(1) which are valid in different parts of the domain. This can be achieved by the introduction of a limit process that determines the terms of Eq.(1) which have a dominant effect in the various regions of the domain.

To define this limit process some basic concepts need to be introduced. The following topology is introduced on the collection of order classes (Meyer, 1967).

For positive, continuous functions of a single variable ϵ defined on $(0,1]$, let $ord\,\eta$ denote the class of equivalence.

$$ord\,\eta = \{\theta(\epsilon)\ /\ lim\ \theta(\epsilon)/\eta(\epsilon), \quad \epsilon \to 0, \quad \text{exists and is } \neq 0\}. \tag{2}$$

A partial ordering is constructed on these functions by defining

$$ord\ \eta_1 < ord\ \eta_2 \quad \Leftrightarrow \quad lim\,\frac{\eta_1}{\eta_2} = 0, \quad \epsilon \to 0. \tag{3}$$

A set D of order classes is said to be convex if $ord\ \delta_1, ord\ \delta_2 \in D$ and $ord\ \delta_1 < ord\ \theta < ord\ \delta_2$ together imply $ord\ \theta \in D$. A set D is said to be open if it is convex and if $ord\,\theta \in D$ implies the existence of functions γ, δ such that $ord\,\theta > ord\,\gamma \in D$ and $ord\ \theta < ord\ \delta \in D$. A set D, on the other hand, is said to be closed if it is convex and has particular elements $ord\ \delta_1, ord\ \delta_2$ such that $ord\ \delta_1 \leq ord\ \theta \leq ord\ \delta_2$ for every $ord\ \theta \in D$. Two order sets, D and D' are said adjacent if: i) $D' > D$ and ii) $\eta < D'$ and $\eta' > D \to \eta' > \eta$. We may refer to D' as being the upper adjacent region of D. Analogously, D is said to be the lower adjacent region of D'.

Definition (Lagerstrom, 1988). We say that $f(x,\epsilon)$ is an approximation to $g(x,\epsilon)$ uniformly valid to order $\delta(\epsilon)$ in a convex set D (f is a δ-approximation to g), if

$$lim\,\frac{f(x,y) - g(x,y)}{\delta(\epsilon)} = 0, \quad \epsilon \to 0, \text{ uniformly for } x \text{ in } D. \tag{4}$$

The function $\delta(\epsilon)$ is called a gauge function.

The essential idea of η-limit process is to study the limit as $\epsilon \to 0$ not for fixed x near the singularity point x_d, but for x tending to x_d in a definite relationship to ϵ specified by a stretching function $\eta(\epsilon)$. Taking without any loss of generality $x_d = 0$, we define

$$x_\eta = \frac{x}{\eta(\epsilon)}, \quad G(x_\eta;\epsilon) = F(x;\epsilon), \tag{5}$$

with $\eta(\epsilon)$ a function defined in Ξ.

Definition (Meyer, 1967). If the function $G(x_\eta;+0) = \ lim\,G(x_\eta;\epsilon)$, $\epsilon \to 0$, exists uniformly on $\{x_\eta/|x_\eta| > 0\}$; then we define $lim_\eta\ F(x;\epsilon) = G(x_\eta;+0)$.

Thus, if $\eta \to 0$ as $\epsilon \to 0$, then, in the limit process, $x \to 0$ also with the same speed of η, so that x/η tends to a non-zero limit value.

One of the central results of Kaplun's work is the extension theorem, which is here presented in the following version (Meyer, 1967).

Kaplun's extension theorem. If $f(x;\epsilon)$ is a $\xi(\epsilon)$-approximation to $g(x;\epsilon)$ uniformly in a closed interval D_0, then it is so also in an open set $D \supset D_0$.

The above theorem was firstly published in Kaplun and Lagerstrom(1957) in connection with the Stokes paradox for flow at low Reynolds number. It needs to be complemented by an Axiom and by an Ansatz to relate the formal domain of validity of an equation with the actual domain of validity of its solution. The idea of Kaplun was to shift the emphasis to applying limit-processes directly to the equations rather than to the solutions, establishing some rules to determine the domain of validity of solutions from the formal domain of validity of an equation.

The set of equations that will result from passage of the limit is referred to by Kaplun as the "splitting" of the differential equations. The splitting must be seen as a formal property of the equation obtained through a "formal passage of the η-limit process". To every order of η a correspondence is induced, $lim_\eta \to$ associated equation, on that subset of Ξ for which the associated equation exists.

Definition. The formal limit domain of an associated equation E is the set of orders η such that the η-limit process applied to the original equation yields E.

Passage of the η-limit will give equations that are distinguished in two ways: i) they are determined by specific choices of η, and ii) they are more complete, or in Kaplun's words, "richer" than the others, in the sense that, application of the η-limit process to them will result in other associated equations, but neither of them can be obtained from any of the other equations.

Limit-processes which yield "rich" equations are called principal limit-processes. The significance of principal limit-processes is that the resulting equations are expected to be satisfied by the corresponding limits of the exact solution. The notion of principal equation will be formalised below.

The above concepts and ideas can be given a more rigorous interpretation if we introduce Kaplun's concept of equivalent in the limit for a given set of equations for a given point (η, δ) of the (Ξ, Σ) product space.

Given any two associated equations E_1 and E_2, we define the remainder of E_1 with relation to E_2 as

$$\mathbb{R}(x_\eta; \epsilon) = E_1(x_\eta; \epsilon) - E_2(x_\eta; \epsilon), \tag{6}$$

where ϵ denotes a small parameter.

According to Kaplun(1967), $\mathbb{R}$ should be interpreted as an operator giving the "apparent force" that must be added to E_2 to yield E_1.

Definition (of equivalence in the limit) (Kaplun, 1967). Two equations E_1 and E_2 are said to be *equivalent in the limit* for a given limit-process, lim_η, and to a given order, δ, if

$$\frac{\mathbb{R}(x_\eta; \epsilon)}{\delta} \to 0, \ as \ \epsilon \to 0, \ x_\eta \text{ fixed.} \tag{7}$$

The following propositions are important; they can be found in Kaplun(1967). The symbol $\sim$ is used to indicate equivalent in the limit whereas $\not\sim$ indicates not equivalent in the limit.

Proposition 1: If $E \sim E'$ for the point (η', δ') of the product space $\Xi \times \Sigma$, then $E \sim E'$ for all points (η, δ) such that $\eta = \eta'$ and $\delta \gg \delta'$. Conversely, if $E \not\sim E'$ for the point (η', δ'), then $E \not\sim E'$ for all points (η, δ) such that $\eta = \eta'$ and $ord\, \delta \ll ord\, \delta'$.

Proposition 2: If $E \sim E'$ for the point (η, δ) of the product space $\Xi \times \Sigma$, and if associated equations for that point exist for E, then they exist also for E' and are identical for both.

Proposition 3: If associated equations exist for E and E' respectively, corresponding to $\eta = \eta'$ and the sequence $\delta = \delta'_0$, δ'_1, ..., δ'_n, δ' where $\delta'_n > \delta' > \delta'_{n+1}$, and are identical for both, then $E \sim E'$ for the point (η', δ').

We can make the following definition.

Definition (of formal domain of validity). The formal domain of validity to order δ of an equation E of formal limit domain D is the set $D_e = D \cup D'_i s$, where $D'_i s$ are the formal limit domains of all equations E'_i such that E and E'_i are equivalent in D'_i to order δ.

Definition (of principal equation). An equation E of formal limit domain D, is said to be principal to order δ if:
i) one can find another equation E', of formal limit domain D', such that E and E' are equivalent in D' to order δ;
ii) E is not equivalent to order δ to any other equation in D.

An equation which is not principal is said to be intermediate.

To relate the formal properties of equations to the actual problem of determining the uniform domain of validity of solutions, Kaplun(1967) advanced two assertions, the Axiom of Existence and the Ansatz about domains of validity. These assertions constitute primitive and unverifiable assumptions of perturbation theory.

Axiom (of existence) (Kaplun, 1967). If equations E and E' are equivalent in the limit to the order δ for a certain region, then given a solution S of E which lies in the region of equivalence of E and E', there exists a solution S' of E' such that as $\epsilon \to 0, |S - S'|/\delta \to 0$, in the region of equivalence of E and E'.

In other words, the axiom states that there exists a solution S' of E' such that the "distance" between S and S' is of the same order of magnitude of that between E and E'.

In using perturbation methods, the common approach is to consider the existence of certain limits of the exact solution or expansions of a certain form. This is normally a sufficient condition to find the associated equations and to assure that the axiom is satisfied (Kaplun(1967). Equivalence in the limit, however, is a necessary condition as shown by propositions (1) to (3).

To the axiom of existence there corresponds an Ansatz; namely that there exists a solution S of E which lies in the region of equivalence of E and E'. More explicitly, we write.

Ansatz (about domains of validity) (Kaplun, 1967). An equation with a given formal domain of validity D has a solution whose actual domain of validity corresponds to D.

The word "corresponds to" in the Ansatz was assumed by Kaplun to actually mean "is equal to"; this establishes the link we needed between the "formal" properties of the equation and the actual properties of the solution.

The Ansatz can always be subjected to a *canonical test* which consists in exhibiting a solution S' of E' which lies in the region of equivalence of E and E' and is determined by the boundary conditions that correspond to S.

Because the heuristic nature of the Axiom and of the Ansatz, comparison to experiments will always be important for validation purposes. The theory, however, as implemented through the above procedure, is always helpful in understanding the matching process and in constructing the appropriate asymptotic expansions.

3. The Proposition of the Principal Equations

The "splitting" of the equations obtained through the definition of equivalent in the limit may be extended to higher orders by introducing a fictitious perturbation of an arbitrary order δ. Thus, according to Kaplun(1967), for higher orders the splitting of the equations corresponding to arbitrary limit processes becomes more complicated and less significant; the operation of splitting is then merely reduced to exhibit some of the typical associated equations and some of the sufficient conditions under which they are associated. In fact, Kaplun lists three reasons why the splitting for higher orders should not be considered in detail: i) the equations associated with a given point (η, δ) depend on the choice of the δ'_n for the corresponding limit process and may depend on the amount of information used in connection with the preceding terms, ii) the δ'_n depend to greater extent on boundary conditions and hence are difficult to determine a priori, and iii) many trivial splitting of the associated equations arise, corresponding to expansions of the preceding terms by different limit processes.

Here, we want to further extend the above notions. In what follows we will show that, for certain points of the (Ξ, Σ) product space, the determination of the associated equations will depend on the choice of some discrete values of δ'_n. It results that the order of validity of an approximation is defined by open intervals determined by the discrete δ'_n's. Furthermore, no trivial splitting will result in these certain points.

To extend the previous results to higher orders, we consider solutions of the form

$$f = f_0 + \Delta(\epsilon) f_1, \tag{8}$$

where $\Delta(\epsilon) \in \Xi$.

Some questions are now in order. Which function is $\Delta(\epsilon)$ for a given differential equation? Is $\Delta(\epsilon)$ the same for all regions of the domain?

The first question is complex and involves speculating on the existence and uniqueness of solutions. Of course, uniqueness of $\Delta(\epsilon)$ can never be assured since given any $\Delta(\epsilon)$, one can always present another $\Delta'(\epsilon)$ such that $\Delta'(\epsilon)$ is exponentially close to $\Delta(\epsilon)$. Thus, according to Kaplun, there will always be a "question of choice" for the determination of the appropriate asymptotic expansions which must be solved relying on intuition and physical insight. An adequate $\Delta(\epsilon)$ can however be determined in a very natural way. We require $\Delta(\epsilon)$ to be such that the resulting equation for f_1 does not provide a trivial solution. A $\Delta(\epsilon)$ satisfying this condition is said to be a critical $\Delta(\epsilon)$. Analogously, its order, $ord\,\Delta(\epsilon)$, is called critical order. More precisely:

Definition (of critical order) (Kaplun(1967)). An order $ord\,\Delta(\epsilon)$ is said to be critical if:
i) the corrections to f_0 to any order ζ in $D, D = \{\zeta / ord\,\Delta(\epsilon) < ord\,\zeta < 1\}$, are trivial;
ii) the corrections to f_0 to any order ζ in the complement of D are not trivial.

The above definition suggests that approximate solutions for different regions of the domain should not in general have the same $\Delta(\epsilon)$. Of course, equal $\Delta's$ might happen as a mere coincidence; however, it is important to give emphasis to that, normally this is not the case.

To find the several order approximate equations we substitute Eq.(8) into the original equations and perform elementary operations such as addition, multiplication, subtraction, differentiation and so on. If these operations are justified, that is, if they do not lead to any non-uniformity, we then collect the terms of same order of magnitude and construct a set of approximate equations. Thus, it is clear that in the process of collecting terms, to each term E_1 of order, say ν there will always correspond another term E_2 of order $\nu\Delta(\epsilon)$.

Consider now an equation E where E_1 and E_2 denote the first two critical order terms. We call the operator $\Pi_1(E) = E_1$ the first order projection of E onto E_1. Analogously, the operator $\Pi_2(E) = E_2$ is called the second order projection of E onto E_2.

We can then enunciate the following proposition.

Proposition (of the principal equations). If there is a principal equation, E_1, at a point $(\eta, 1)$ of the (Ξ, Σ) product space, then there is also a principal equation, E, at a point (η, ϵ) of (Ξ, Σ) with $E_1 = \Pi_1(E)$.

Proof: Suppose E_1 is a principal equation at a point $(\eta, 1)$ of the (Ξ, Σ) product space. Then one can find a term R_{1l} such that R_{1l} is order unity in D (the domain of E_1) but $ord\, 1 < ord R_{1l} < ord\, \epsilon$ in D_u, the upper adjacent domain of D. Here ϵ denotes the first critical order. Define $E_1' = E_1$ - R_{1l}.

Let now E_2 and E_2' denote the first order associated equations in D and D_u respectively. Then, there is a term R_{2l} such that R_{2l} is order ϵ in D but $ord\, R_{2l} < ord\, \epsilon$ in D_u. Define $E_2' = E_2$ - R_{2l}.

It results that the structure of the lower adjacent region is

$$ord\ R_{1l} < ord\ E_1' < ord\ R_{2l} < ord\ E_2' \tag{9}$$

This yields that no other equation is equivalent to order ϵ to equation $E = E_1' + R_{1l} + E_2' + R_{2l}$ in D. However, E and $E'(= E_1' + R_{1l} + E_2')$ are equivalent in D_u to $ord\, \epsilon$. We conclude E is a principal equation at a point (η, ϵ) of the (Ξ, Σ) product space.

The converse of the above proposition is obviously true, that is: if E is a principal equation at a point (η, ϵ) of the (Ξ, Σ) space, where η denotes the formal limit domain of E and ϵ the first critical order, then $\Pi_1(E)$ is a principal equation at a point $(\eta, 1)$ of the (Ξ, Σ) space.

What the above proposition clearly states is that the position in the (Ξ,Σ) product space where the principal equations are located can be searched by looking only at the lowest order associated equations. Furthermore, it says that these lowest order approximations are good up to the first critical order and that no trivial splitting will arise. This fact is only valid for the particular point in (Ξ,Σ) space where the principal equation holds. In the upper and lower adjacent domains trivial splitting will occur.

It results that higher order splitting should not, in fact, be considered. The principal equations of the problem, those that retain most of the information about the problem solution, can have their position determined only through an analysis of the lowest order terms. Then the concept of critical order can be applied to the solution to find the appropriate asymptotic expansions for the problem.

4. The Asymptotic Structure of the Turbulent Boundary Layer

Boundary layer problems played a central role in the development of singular perturbation methods. In fact, the basic ideas of singular perturbation methods remount to Prandtl's boundary layer theory of a laminar flow. Prandtl's matching principle for laminar boundary layers was systematically discussed and generalised in the fifties yielding well established procedures and solutions which have rendered the laminar flow problem solved.

In regard to turbulent flows, two approaches were used: in the first, asymptotic techniques were applied to the averaged equations without appealing to any closure model (Yajnik(1970), Mellor(1972)); in the second, eddy-viscosity (Bush and Fendell(1972)) or $\kappa - \epsilon$ (Deriat and Guiraud(1986)) models were used to find high order approximations. Most theories divide the turbulent boundary layer into two regions. Other authors, Long and Chen(1981), Sychev and Sychev(1987), Melnik(1989), however, have recently claimed that the turbulent boundary layer has instead a three-layered structure. This structure considers a new region in which a balance of inertia forces, and pressure and turbulent friction forces occurs. The formulation of Melnik is based on a two-parameter expansion of the boundary layer equations, the new additional small parameter resulting from the particular turbulence closure model he uses.

The discussions that have led to the development of the three-layered asymptotic model for the turbulent boundary layer result from the recognition that two-layered models cannot deal with large flow disturbances in the stream-wise direction. When a turbulent boundary layer is subjected to a large longitudinal adverse pressure gradient, the velocity deficit is large and the mean momentum equation is non-linear; this makes the classical matching arguments which result in a log-law and in a two deck structure, not valid anymore. The classical wall characteristic velocity, the friction velocity, may become an inappropriate scaling parameter so that new formulations will have to be developed for the problem at hand.

Here, we will investigate the turbulent boundary layer from the point of view of Kaplun's single limits. The purpose is to formally arrive at a three-layered structure which is compatible with the class of problem to be studied: the turbulent boundary layer near the leading edge of a flat plate.

For an incompressible two-dimensional turbulent flow over a smooth surface in a prescribed pressure distribution, the time-averaged motion equations; i.e., the continuity equation and the Navier-Stokes equation can be written as

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{10}$$

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_j} - \epsilon^2 \frac{\partial}{\partial x_j}\left(\overline{u'_j u'_i}\right) + \frac{1}{R}\frac{\partial^2 u_i}{\partial x_j^2}, \tag{11}$$

where the notation is classical. Thus $(x_1, x_2) = (x, y)$ stand for the co-ordinates, $(u_1, u_2) = (u, v)$ for the velocities, p for pressure and R for the Reynolds number. The dashes are used to indicate a fluctuating quantity. In the fluctuation terms, an overbar is used to indicate a time-average.

All mean variables are referred to some characteristic quantity of the external flow. The velocity fluctuations, on the other hand, are referred to a characteristic velocity u_τ.

The correct assessment of the characteristic velocity is fundamental for the determination of the boundary layer asymptotic structure.Here we consider

$$ord\,(u'_i) = ord\,(u_\tau). \tag{12}$$

This result is valid for incompressible flows as well as for compressible flows.

The small parameter ϵ is, therefore, defined by

$$\epsilon = \frac{u_R}{U_\infty} = \frac{u_\tau}{U_\infty}. \tag{13}$$

The asymptotic expansions for the flow parameters are written as

$$u(x, y) = u_1(x, y) + \epsilon u_2(x, y), \tag{14}$$

$$v(x, y) = \eta[v_1(x, y) + \epsilon v_2(x, y)], \tag{15}$$

$$p(x, y) = p_1(x, y) + \epsilon p_2(x, y), \tag{16}$$

$$u'_i(x, y) = \epsilon u'_{i1}(x, y) + \epsilon^2 u'_{i2}(x, y). \tag{17}$$

$$\overline{u'v'}(x, y) = \epsilon^2\, \overline{u'v'_1}(x, y) + \epsilon^3\, \overline{u'v'_2}(x, y). \tag{18}$$

To find the asymptotic structure of the boundary layer we consider the following stretching transformation

$$\hat{y} = y_\eta = \frac{y}{\eta(\epsilon)}, \quad \hat{u}_i(x, y_\eta) = u_i(x, y). \tag{19}$$

with $\eta(\epsilon)$ defined on Ξ.

Upon substitution of Eq.(19) into Eqs.(14) to 17) and upon passage of the η-limit process onto the resulting equation we get:

x-momentum equation:

$$
\begin{array}{rllll}
 & ord(\delta) = 1 & & ord(\delta) = ord(\epsilon) & \\
\eta = 1: & D_{11} = P_1, & & D_{12} + D_{21} = P_2, & (20) \\
\epsilon < \eta < 1: & D_{11} = P_1, & & D_{12} + D_{21} = P_2, & (21) \\
\eta = \epsilon: & D_{11} = P_1, & & D_{12} + D_{21} = P_2 - (\overline{u'v'_1})_{\hat{y}}, & (22) \\
\epsilon^2 < \eta < \epsilon: & D_{11} = P_1, & (\overline{u'v'_1})_{\hat{y}} = 0, & D_{12} + D_{21} = P_2, & (23) \\
\eta = \epsilon^2: & D_{11} = P_1 - (\overline{u'v'_1})_{\hat{y}}, & & D_{12} + D_{21} = P_2 - (\overline{u'v'_2})_{\hat{y}}, & (24) \\
\epsilon^3 < \eta < \epsilon^2: & (\overline{u'v'_1})_{\hat{y}} = 0, & D_{11} = P_1, & (\overline{u'v'_2})_{\hat{y}} = 0, & (25) \\
\eta = \epsilon^3: & (\overline{u'v'_1})_{\hat{y}} = 0, & & D_{11} = P_1 - (\overline{u'v'_2})_{\hat{y}}, & (26) \\
1/\epsilon^2 R < \eta < \epsilon^3: & (\overline{u'v'_1})_{\hat{y}} = 0, & & (\overline{u'v'_2})_{\hat{y}} = 0, & (27) \\
\eta = 1/\epsilon^2 R: & (\overline{u'v'_1})_{\hat{y}} = 0, & & (\overline{u'v'_2})_{\hat{y}} = (u_2)_{\hat{y}\hat{y}}, & (28) \\
1/\epsilon R < \eta < 1/\epsilon^2 R: & (\overline{u'v'_1})_{\hat{y}} = 0, & (u_2)_{\hat{y}\hat{y}} = 0, & (\overline{u'v'_2})_{\hat{y}} = 0, & (29) \\
\eta = 1/\epsilon R: & (\overline{u'v'_1})_{\hat{y}} = (\hat{u}_2)_{\hat{y}\hat{y}}, & & (\overline{u'v'_2})_{\hat{y}} = (\hat{u}_3)_{\hat{y}\hat{y}}. & (30)
\end{array}
$$

where the following operators were used

$$D_{ij} = \hat{u}_i \frac{\partial \hat{u}_j}{\partial x} + \hat{v}_i \frac{\partial \hat{u}_j}{\partial y_\eta}, \qquad P_i = -\frac{1}{\rho}\frac{\partial \hat{p}_i}{\partial x}. \tag{31}$$

The above equations were arranged in three columns according to their respective order of approximation. The first column corresponds to the first order of approximation; the third one to the second order of approximation. The middle column corresponds to orders between the first and the second critical order. The extreme left of the lines indicates the point in the domain where the η-limit process was applied.

Passage of the η-limit process onto the y-momentum equation does not give any relevant information. In fact, we will find that for $ord\, \eta < ord\, \epsilon$ the first and second order pressure terms will dominate all the other terms. All information regarding the asymptotic structure of the boundary layer is, therefore, contained in the x-momentum equation.

The term $\hat{u}_1(x, y_\eta)$ is missing from equations (29) and (30) since from the no-slip condition $\hat{u}_1 = 0$ near the wall.

Equations (24) and (30) are distinguished in two ways: i) they are determined by specific choices of η, and ii) they are "richer" than the others in the sense that, application of the limit process to them yields some of the other equations, but neither of them can be obtained from passage of the limit process to any of the other equations. Thus, according to the definitions introduced in the previous sections, these equations are the principal equations. We have seen that principal equations are important since they are expected to be satisfied by the corresponding limits of the exact solution.

A complete solution to the problem should then according to the Axiom of Existence and Kaplun's Ansatz, be obtained from the principal equations located at points $ord\ \eta = ord\ \epsilon^2$ and $ord\ \eta = ord\ (1/\epsilon R)$. The formal domains of validity of these equations cover the entire domain and overlap in a region determined according to the definition of equivalent in limit.

To find the overlap region of equations (24) and (30), we must show these equations to have a common domain where they are equivalent. A direct application of the definition of equivalence in the limit to equations (24) and (30) yields

$$\mathbb{R} = \frac{D(\hat{u}_1) - P(\hat{p}_1) + D(\hat{u}_2) - P(\hat{p}_2) - (\hat{u}_2)_{\hat{y}\hat{y}} - (\hat{u}_3)_{\hat{y}\hat{y}}}{\epsilon^\alpha}. \tag{32}$$

Noting that the leading order term in region $ord\ (1/\epsilon R) < ord\ \eta < ord\ \epsilon^2$ is the turbulent term, of $ord\ (\epsilon^2/\eta)$, we normalise the above equation to order unity to find

$$\bar{\mathbb{R}} = \frac{\eta}{\epsilon^2}\mathbb{R}. \tag{33}$$

The overlap domain is the set of orders such that the η-limit process applied to $\bar{\mathbb{R}}$ tends to zero for a given α. Then since $ord\ (\partial/\partial y) = \epsilon$ and $ord\ (\partial/\partial x) = 1$, the formal overlap domain is given by

$$D_{overlap} = \{\eta/ \quad ord\ (\epsilon^{1+\alpha} R)^{-1} < ord\ \eta < ord\ (\epsilon^{2+\alpha})\}. \tag{34}$$

According to Kaplun's Ansatz about domains of validity, the approximate equations, Eqs. (24) and (25), only overlap if set (34) is a non-empty set, that is, if

$$0 \leq \alpha \leq -\frac{1}{2}\left(\frac{ln\, R}{ln\, \epsilon} + 3\right). \tag{35}$$

The implication is that the two-deck turbulent boundary layer structure given by the two principal equations, equations (24) and (30), provides approximate solutions which are accurate to the order of $\epsilon^{\alpha_{max}}$, where α_{max} is the least upper bound of the interval (35). This fundamental result can only be reached through the application of Kaplun's concepts and ideas to the problem.

In particular, the overlap domain of the first and second order of approximation are given respectively by

$$D_1^o \cap D_1^i = \{ord\eta/ \quad ord(1/\epsilon R) < ord(\eta) < ord(\epsilon^2)\} \tag{36}$$

and,

$$D_2^o \cap D_2^i = \{ord\eta/ \quad ord(1/\epsilon^2 R) < ord(\eta) < ord(\epsilon^3)\}. \tag{37}$$

We conclude that the turbulent boundary layer has a two-deck structure very much like the one derived by Sychev and Sychev. This structure, however, must change as the leading edge is approached. We shall see this next.

Before we move forward, however, some comments about the intermediate equations will be made.

For the formal limit domains which are not adjacent to the principal equations two approximated equations are always defined, separated by the first two critical orders. In this case the interpretation is simple and the local approximated equations and solutions well defined. For the regions adjacent to the principal equations, however, a correction with order between the first two critical orders is found. The interpretation of these equations is more complex and must be made in an individual basis. For example, in the turbulent boundary layer problem under consideration, the solution in the upper adjacent region must take into consideration, as the first two order of approximation equations, the leading order equation and the intermediate order equation; these equations will provide non-trivial solutions with physical information. For the lower adjacent region, however, the intermediate order equation provides a trivial solution; thus, no extra information is obtained from this equation except that the overlap domain for the first two order of approximation is not given by equation (37) but by

$$D_2^o \cap D_2^i = \{ord\eta/ \quad ord(1/\epsilon^2 R) < ord(\eta) < ord(\epsilon^2)\}. \tag{38}$$

5. The Flow Near the Leading Edge of a Flat Plate

The above asymptotic structure must undergo modifications if the flow near to the leading edge of a flat plate is to be considered.

A major difficulty for a direct translation of the classical boundary layer model into a model that applies for the leading edge is the fact that the assumption

$$\left|\frac{\partial^2 u}{\partial x^2}\right| \ll \left|\frac{\partial^2 u}{\partial y^2}\right|$$

does not apply any more.

In addition, when the friction velocity, u_τ, is used to develop the asymptotic structure of the boundary layer, a non-uniformity will occur near the leading edge point where $u_\tau = 0$. The result is that any theory advanced for the problem should explain in asymptotic terms how the far downstream two-deck structure reduces to an alternative structure near the leading edge.

To find the asymptotic structure of the boundary layer near a leading edge point we apply the following stretching transformation to the equations of the previous section

$$x_\Delta = \frac{x}{\Delta(\epsilon)}, \tag{39}$$

with $\Delta(\epsilon)$ defined on Ξ.

The resulting flow structure is given by:

x-momentum equation:

$$ord\ \Delta = ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial x_\Delta} = 0, \tag{40}$$

$$ord\ \epsilon^2 < ord\ \Delta < ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial x_\Delta} = 0, \tag{41}$$

$$ord\ \epsilon^2 = ord\ \Delta: \quad \hat{u}_1 \frac{\partial \hat{u}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial x_\Delta} = -\frac{\partial \overline{\hat{u'}_1^2}}{\partial x_\Delta} - \frac{\partial \overline{\hat{u'}_1 \hat{v'}_1}}{\partial y_\eta}, \tag{42}$$

$$ord\ 1/\epsilon R < ord\ \Delta < ord\ \epsilon^2: \quad 0 = -\frac{\partial \overline{\hat{u'}_1^2}}{\partial x_\Delta} - \frac{\partial \overline{\hat{u'}_1 \hat{v'}_1}}{\partial y_\eta}, \tag{43}$$

$$ord\ 1/\epsilon R = ord\ \Delta: \quad 0 = \frac{\partial^2 \hat{u}_1}{\partial x_\Delta^2} + \frac{\partial^2 \hat{u}_1}{\partial y_\eta^2} - \frac{\partial \overline{\hat{u'}_1^2}}{\partial x_\Delta} - \frac{\partial \overline{\hat{u'}_1 \hat{v'}_1}}{\partial y_\eta}, \tag{44}$$

$$ord\ \Delta < ord\ 1/\epsilon R: \quad 0 = \frac{\partial^2 \hat{u}_1}{\partial x_\Delta^2} + \frac{\partial^2 \hat{u}_1}{\partial y_\eta^2}. \tag{45}$$

y-momentum equation:

$$ord\ \Delta = ord\ 1: \quad \hat{u}_1 \frac{\partial \hat{v}_1}{\partial x_\Delta} + \hat{v}_1 \frac{\partial \hat{v}_1}{\partial y_\eta} + \frac{\partial \hat{p}_1}{\partial y_\eta} = 0, \tag{46}$$

$$ord\ \Delta < ord\ 1: \quad \frac{\partial \hat{p}_1}{\partial y_\eta} = 0. \tag{47}$$

Note that in region $(\Delta, \eta) = (\epsilon^2, \epsilon^2)$ the boundary layer formulation has to be modified so as to include an extra Reynolds stress term in the x-momentum equation. The y-momentum equation remains unchanged, with the pressure term dominating the leading order solution. Further close to the leading edge the complete viscous terms are recovered in the x-momentum equation; the pressure term still dominates the y-momentum equation.

The two principal equations in the x-direction are Eqs. 42 and 44. They cover the whole domain and overlap in $ord\, 1/\epsilon R < ord\, \Delta < ord\, \epsilon^2$. In the y-direction the motion is dominated by the single principal equation, Eq. 46. The result is that near the leading edge the motion governing equations are not the boundary layer equation. Rather, alternative equations need to be considered which include extra Reynolds and viscous stress terms.

Thus, the region of approximate validity of these equations are as follows:

1. Potential solution equation: $ord\ \epsilon^2 < ord\ \Delta, ord\ \eta < ord\ 1$.
2. Fully turbulent flow: $ord\ 1/\epsilon R < ord\ \Delta, ord\ \eta < ord\ \epsilon^2$.

3. Stokes equation: $ord\ \Delta, ord\ \eta < ord\ 1/\epsilon R$.

In addition, the y-momentum equation has to be considered in the analysis in the form of Eq. 46.

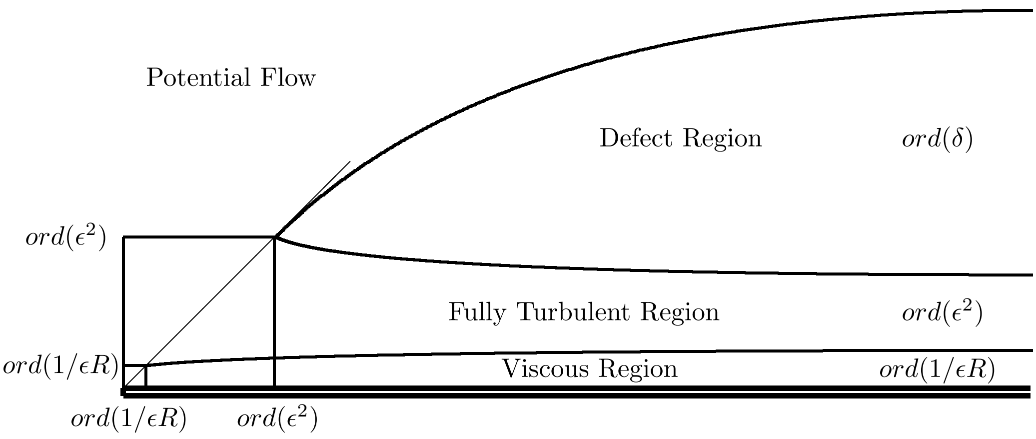

Figura 1. Asymptotic structure of the turbulent boundary layer near the leading edge of a flat plate.

Some experimental evidence (Sreenivasan(1989)) has shown that

$$\delta \propto x^{4/5}, \tag{48}$$

$$\epsilon \propto x^{-1/10}, \tag{49}$$

$$y_p \propto x^{1/2}, \tag{50}$$

where y_p denotes the point of maximum turbulent stress.

The structure presented in Fig. 1 is consistent with these results.

6. Final Remarks

In the first part of the paper, some ideas of Kaplun concerning limit processes have been extended to higher orders through the proposition of the principal equations. This result is central to our work, for it ensures that the asymptotic structure of a singular perturbation problem can be uniquely determined by a first order analysis of the formal domains of validity. The resulting principal equations are expected to be satisfied by the corresponding limits of the exact solution, so providing approximate solutions that overlap and cover the entire domain of validity.

In the second part of the paper, application of Kaplun limits to the equations of motion has shown the zero-pressure turbulent boundary layer to have a two deck structure, the principal equations being located at points $(\epsilon^2, 1)$ and $(1/\epsilon R, 1)$ of the product space $(\Xi \times \Sigma)$. The present results are very much in accordance with the earlier works of Yajnik, of Mellor and of Bush and Fendell. They seem to corroborate the idea that a one-parameter theory can correctly describe the flow structure and, furthermore, do not give any evidence to suggest the contrary.

The present analysis has also shown how the two-deck turbulent boundary layer structure develops into a one-deck structure near the leading edge of a flat plate. Despite the assertion by many authors that the full Navier-Stokes must be recovered by the leading edge point, we have shown here that consideration of the principal equations, Eqs. 42, 44 and 46, furnishes a complete set of approximate equations for the problem.

The proposed asymptotic structure is now being tested against some experimental data that the author have collected in the Laboratory of Turbulence Mechanics of COPPE/UFRJ. These results will be published in due time.

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